# Successive minima of lattice trajectories and topological games to compute fractal dimensions

*A variational principle in the parametric geometry of numbers*

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"I have said that Texas is a state of mind, but I think it is more than that. It is a mystique closely approximating a religion." – John Steinbeck

## My brilliant collaborators ...



Lior Fishman, David Simmons and Mariusz Urbański

# A variational principle in the parametric geometry of numbers

https://arxiv.org/abs/1901.06602

A variational principle in the parametric geometry of numbers, with applications to metric Diophantine approximation. Comptes Rendus **355**(8), 2017, 835–846.

https://arxiv.org/abs/1704.05277

*Acknowledgments: American Institute of Mathematics, SQuaREs (Structured Quartet Research Ensembles) program, UW-La Crosse, Simons Foundation, EPSRC, and the NSF.*

# Diophantine Approximation

$\mathbb{Q}$ being dense in $\mathbb{R}$ means that if $x$ is fixed, then the **error** $|x - p/q|$ can be made arbitrarily small by varying $p/q$ …at the cost of the **height** $q$ tending to infinity (unless $x$ is rational).

In fact, a more precise relation holds:

For every real number $x$ and for every $Q > 1$, there exists a rational $p/q$ such that

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{qQ} \quad \text{and} \quad 0 < q < Q$$

or equivalently

$$|xq - p| \leq Q^{-1} \quad \text{and} \quad 0 < q < Q.$$

First proven by Legendre in 1808 using the theory of continued fractions.

# Dirichlet's Theorem

Peter Gustav Lejeune Dirichlet

(1805 – 1859)

**Theorem (Dirichlet, 1842)**

*For every real $m \times n$ matrix $A$ and every $Q > 1$, there exists an integer vector $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n}$ such that*

$$\|A\mathbf{q} - \mathbf{p}\| \leq Q^{-n/m} \quad \text{and} \quad 0 < \|\mathbf{q}\| < Q.$$

Dirichlet's Schubfachprinzip!
…and later via Minkowski's Geometry of Numbers

(Legendre, 1808) For every real number $x$ and every $Q > 1$, there exists an integer vector $(p, q) \in \mathbb{Z}^2$ such that

$$|xq - p| \leq Q^{-1} \quad \text{and} \quad 0 < q < Q.$$

Matrix approx. framework a.k.a. Diophantine approx. for *systems of linear forms*

The system of $m$ linear forms in $n$ variables $L_1, \ldots, L_m : \mathbb{R}^n \to \mathbb{R}$, $L_i(x_1, \ldots, x_n) = \sum_j A_{ij} x_j$ corresponds to an $m \times n$ matrix $A$.

Fundamental contributions by Minkowski, Khintchine, Jarník, Cassels, Davenport, Baker, Sprindžuk, Schmidt…Margulis, Dani, Shah, Kleinbock, Breuillard, De Saxcé

# What is a *singular* matrix?

…in the sense of Diophantine approximation!

## Khintchine's idea (1926/1937/1948)

An $m \times n$ matrix $A$ is called **singular** if for every $\varepsilon > 0$, and for $Q$ sufficiently large, there exist integer vectors $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q} - \mathbf{p}\| \leq \varepsilon Q^{-n/m} \quad \text{and} \quad 0 < \|\mathbf{q}\| \leq Q.$$

- Singular matrices are "infinitely Dirichlet improvable"
- Singular numbers ($m = n = 1$) are rational numbers.
- Khintchine discovered irrational singular vectors.
  $\exists$ singular matrices (when $(m, n) \neq (1, 1)$) with entries lin. ind. over $\mathbb{Q}$.
- Khintchine proved $\mathrm{Sing}_{m,n}$ is Lebesgue null.
- …
- *Kleinbock–Moshchevitin–Weiss, Singular vectors on manifolds and fractals*
  *arXiv:1912.13070 (2020)*

# Dani correspondence principle

Fix $m, n \in \mathbb{N}$ and let $d = m + n$.
$G = \mathrm{SL}_d(\mathbb{R})$, $\Gamma = \mathrm{SL}_d(\mathbb{Z})$, $X = G/\Gamma$, and $x_0 = \mathrm{I}_d/\Gamma \in X$.
For each $t \in \mathbb{R}$ and for each $m \times n$ matrix $A$, let

$$g_t \overset{\text{def}}{=} \begin{bmatrix} e^{t/m}\,\mathrm{I}_m & \\ & e^{-t/n}\,\mathrm{I}_n \end{bmatrix}, \quad u_A \overset{\text{def}}{=} \begin{bmatrix} \mathrm{I}_m & -A \\ & \mathrm{I}_n \end{bmatrix}.$$

$$\|g_t u_A(\mathbf{p}, \mathbf{q})\| = \max(e^{t/m}\|A\mathbf{q} - \mathbf{p}\|, e^{-t/n}\|\mathbf{q}\|)$$

| **Diophantine properties of $A$** | **Dynamical properties of $(g_t u_A x_0)_{t\geq 0}$** |
|---|---|
| $A$ is badly approximable | $(g_t u_A x_0)_{t\geq 0}$ is bounded |
| $A$ is very well approximable | $\limsup_{t\to\infty} \frac{1}{t} d(x_0, g_t u_A x_0) > 0$ |
| $A$ is singular i.e. $\forall \varepsilon > 0\ \exists Q_0 > 1\ \forall Q \geq Q_\varepsilon$ $\exists (\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n}\ 0 < \|\mathbf{q}\| \leq Q,$ $\|A\mathbf{q} - \mathbf{p}\| \leq \varepsilon Q^{-n/m}$ | $(g_t u_A x_0)_{t\geq 0}$ is divergent |

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| $A$ is singular i.e. $\forall \varepsilon > 0\ \exists Q_0 > 1\ \forall Q \geq Q_\varepsilon$ $\exists (\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n}\ 0 < \|\mathbf{q}\| \leq Q$, $\|A\mathbf{q} - \mathbf{p}\| \leq \varepsilon Q^{-n/m}$ | $g_t u_A x_0 \to \infty$ as $t \to \infty$ i.e. $(g_t u_A x_0)_{t\geq 0}$ eventually leaves every compact set |

# Dani correspondence principle

BA/Sing/VWA are all Lebesgue null sets.

The ergodicity of the $(g_t)$-action implies that it suffices to show that any trajectory that equidistributes is not in BA/Sing/VWA.

An equidistributed trajectory

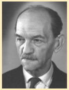
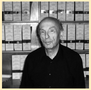
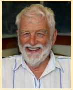
- …isn't bounded as the orbit must be dense.
- …isn't divergent as then we'd have escape of mass.
- …doesn't escape to infinity at a linear rate as then it would spend proportionally long times in cusps infinitely often, also implying escape of mass (along a subsequence).

The next natural challenge, tackled by several mathematicians including Jarník, Besicovitch, Schmidt, Patterson, Dodson, Dani, Kleinbock, Margulis, was the computation or estimation of the fractal dimensions of such sets.

# Measuring exceptional sets

**Hausdorff dimension** is an important tool for studying such intricate sets that are invisible to Lebesgue measure. It is morally a method of measuring the "size" of such sets, such that the Hausdorff dimension of a smooth manifold is equal to the dimension of the manifold.

# Quantifying the size of BA and VWA matrices

## Theorem (Jarník–Schmidt)

*The Hausdorff dimension of the set of badly approximable matrices is mn.*

1D case by Jarník (1928), and matrix case by Schmidt (1969).

## Theorem (Jarník–Besicovitch–Dodson)

*The Hausdorff dimension of the set of $\omega$-approximable matrices is*

$$mn\left(1-\frac{1}{n}\frac{\omega-\frac{n}{m}}{\omega+1}\right).$$

*In particular, the Hausdorff dimension of the set of very well approximable matrices is mn.*

1D case by Jarník (1929) and Besicovitch (1934) ind., matrix case by Dodson (1992).

# Quantifying the size of singular matrices


**Yitwah Cheung**
(arxiv 2007, Annals 2011)

$$\dim_H(\mathrm{Sing}(2,1)) = 4/3$$

**Yitwah Cheung & Nicolas Chevallier**
(arxiv 2014, Duke 2016)

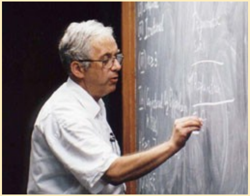
$$\dim_H(\mathrm{Sing}(m,1)) = m^2/(m+1)$$

# Quantifying the size of singular matrices

**Shirali Kadyrov**
**Dmitry Kleinbock**
**Elon Lindenstrauss**
**Grigoriĭ Margulis**
(arxiv 2014, Journal d'Analyse 2017)

$$\dim_H(\mathrm{Sing}(m,n)) \leq mn\left(1 - \frac{1}{m+n}\right)$$



# Their ingredients ...

Let $F_{\leq N}$ denote the set of all numbers in $[0,1]$ whose continued fraction expansions have partial quotients all $\leq N$. In his seminal 1928 paper Jarník established that for every $N \geq 8$

$$1 - \frac{4}{N\log(2)} \leq \dim_H(F_{\leq N}) \leq 1 - \frac{1}{8N\log(N)}$$

from which it follows that BA, whose elements have bounded partial quotients, has full dimension.

Schmidt developed (1966) his now eponymous topological game to extend Jarník's result to the matrix framework in 1969.

Jarník's dimension computation of $\omega$-approximable numbers was arithmetic in nature and involved continued fractions, as was his proof for BA. Besicovitch's proof followed principles more natural to geometric measure theory, and his methods led to Dodson's regular and ubiquitous systems.

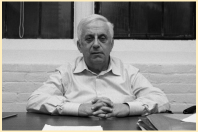
# Their ingredients …

CC's result for singular **vectors** was an equality and they needed to develop separate tools to deal with upper and lower bounds. They developed the notion of *best approximation vectors*, a multidimensional extension of Legendre's theorem on convergents of real continued fraction expansions, as well as the notion of *self-similar coverings* that construct Cantor sets with "inhomogeneous" tree structures.

Though KKLM were only able to prove an upper bound rather than an equality, they extend CC's upper bound to the **matrix** framework. They leveraged the hi-tech of integral inequalities developed by Alex Eskin, Grigoriĭ Margulis and Shahar Mozes: *Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture*, (Annals, 1998).

# ...between bewilderment, bafflement, and bemusement




## KKLM Conjecture

$$\boxed{\dim_H(\mathrm{Sing}(m,n)) \stackrel{?}{=} mn\left(1 - \frac{1}{m+n}\right)}$$

Recall KKLM had proved $\dim_H(\mathrm{Sing}(m,n)) \leq mn\left(1 - \frac{1}{m+n}\right)$

## Bugeaud–Cheung–Chevallier Question

$$\boxed{\dim_H(\mathrm{Sing}(2,1)) \stackrel{?}{=} \dim_P(\mathrm{Sing}(2,1))}$$

Recall that the Hausdorff dimension of a set is less than or equal to its packing dimension.

# …lies a sliver of progress

## Theorem (DFSU)

*For all* $(m,n) \neq (1,1)$*, we have*

$$\dim_H(\mathrm{Sing}(m,n)) = \dim_P(\mathrm{Sing}(m,n)) = \delta_{m,n} \stackrel{\text{def}}{=} mn\left(1 - \tfrac{1}{m+n}\right),$$

*where* $\dim_H(S)$ *and* $\dim_P(S)$ *denote Hausdorff and packing dimensions of a set* $S$*, respectively.*

## Translation via Dani's correspondence principle

The divergent trajectories of any one-parameter diagonal action on the space of unimodular lattices with exactly two Lyapunov exponents of opposite signs has equal Hausdorff and packing dimensions. In fact, the Hausdorff and packing codimensions of the divergent trajectories in $G/\Gamma$ are both $\delta_{m,n}$.

# “One Ring to rule them all” &c.

All three theorems stated that compute exact dimensions are consequences of a single variational principle in the parametric geometry of numbers.

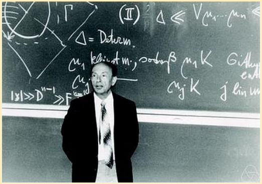
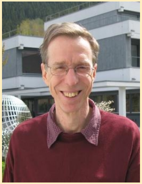
Our methods give a **unifying perspective** and **new proofs** of all these theorems. E.g. no dependence on the methods of KKLM, CC, and BCC!

These theorems are the tip of the iceberg. $\exists$ myriad applications of our variational principle, and plenty of open problems left in their wake.

See §1 Main Results in https://arxiv.org/pdf/1901.06602.pdf
See §3 Directions to further research in https://arxiv.org/pdf/1901.06602.pdf

# Our Ingredients …

- ⋆ Create flexible variants of Schmidt's game that compute dimensions
- ⋆ Extend Schmidt–Summerer–Roy's Parametric Geometry of Numbers

**Wolfgang M. Schmidt** *(Trans. AMS,1966)*
On badly approximable numbers and certain games

**Wolfgang M. Schmidt** *(Progress in Mathematics,1983)*
Open problems in Diophantine approximation (Luminy, 1982)

**Wolfgang M. Schmidt and Leonhard Summerer** *(Monatshefte, 2013)*
Diophantine approximation and parametric geometry of numbers

**Damien Roy** *(Annals, 2015)*
On Schmidt and Summerer parametric geometry of numbers

# Proof Strategy

Our proof strategy consists of four main steps:

1. Show that the Hausdorff and packing dimensions of any subset of $\mathbb{R}^d$ can be computed in terms of certain topological games.
2. Use the *Dani correspondence principle* to characterize the set of singular $m \times n$ matrices in terms of the *parametric geometry of numbers.*
3. Use the characterization of fractal dimensions in step 1 to derive a formula (the *variational principle*) for the fractal dimensions of any set defined in terms of the parametric geometry of numbers.
4. Apply the formula in step 3 to the characterization in step 2 to compute the fractal dimensions of the set of singular $m \times n$ matrices.

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- ⋆ The third step constitutes the bulk of the argument.
- ⋆ Varying second and fourth steps leads to several applications.

# Hausdorff and packing measures and dimensions

The $s$-dimensional Hausdorff measure of a set $A \subseteq \mathbb{R}^d$ is

$$\mathcal{H}^s(A) \stackrel{\text{def}}{=} \lim_{\varepsilon \searrow 0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam}(U_i))^s : \begin{array}{c} (U_i)_1^\infty \text{ is a countable cover of } A \\ \text{with } \operatorname{diam}(U_i) \le \varepsilon \ \ \forall i \end{array} \right\}.$$

The $s$-dimensional packing measure of a set $A \subseteq \mathbb{R}^d$ is

$$\mathcal{P}^s(A) \stackrel{\text{def}}{=} \inf \left\{ \sum_{i=1}^{\infty} \widetilde{\mathcal{P}}^s(A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i \right\}$$

where

$$\widetilde{\mathcal{P}}^s(A) \stackrel{\text{def}}{=} \lim_{\varepsilon \searrow 0} \sup \left\{ \sum_{j=1}^{\infty} (\operatorname{diam}(B_j))^s : \begin{array}{c} (B_j)_1^\infty \text{ is a countable disjoint collection of balls} \\ \text{with centers in } A \text{ and with } \operatorname{diam}(B_j) \le \varepsilon \ \ \forall j \end{array} \right\}$$

The Hausdorff dimension and packing dimension of a set $A \subseteq \mathbb{R}^d$ are:

$$\dim_H(A) \stackrel{\text{def}}{=} \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}$$

$$\dim_P(A) \stackrel{\text{def}}{=} \inf\{s : \mathcal{P}^s(A) = 0\} = \sup\{s : \mathcal{P}^s(A) = \infty\}.$$

# Rogers–Taylor–Tricot Density Theorems

*Computing Hausdorff and packing dimensions via local geometric-measure-theoretic information*

For any $\mathbf{x} \in \mathbb{R}^d$, the lower and upper pointwise dimensions of $\mu$ at $\mathbf{x}$ are

$$\underline{\dim}_{\mathbf{x}}(\mu) \stackrel{\text{def}}{=} \liminf_{\rho\to 0} \frac{\log \mu(B(\mathbf{x},\rho))}{\log \rho} \quad \text{and} \quad \overline{\dim}_{\mathbf{x}}(\mu) \stackrel{\text{def}}{=} \limsup_{\rho\to 0} \frac{\log \mu(B(\mathbf{x},\rho))}{\log \rho}.$$

**Theorem (Rogers–Taylor–Tricot (classical, all known at least by 1980s))**

*Fix $d \in \mathbb{N}$ and let $\mu$ be a locally finite Borel measure on $\mathbb{R}^d$. Then for every Borel set $A \subseteq \mathbb{R}^d$,*

- *If $\underline{\dim}_{\mathbf{x}}(\mu) \geq s$ for all $x \in A$ and $\mu(A) > 0$, then $\dim_H(A) \geq s$.*
- *If $\underline{\dim}_{\mathbf{x}}(\mu) \leq s$ for all $x \in A$, then $\dim_H(A) \leq s$.*
- *If $\overline{\dim}_{\mathbf{x}}(\mu) \geq s$ for all $x \in A$ and $\mu(A) > 0$, then $\dim_P(A) \geq s$.*
- *If $\overline{\dim}_{\mathbf{x}}(\mu) \leq s$ for all $x \in A$, then $\dim_P(A) \leq s$.*

Ken Falconer's Techniques in Fractal Geometry, (Wiley, 1997)

# Games people play

Given $0 < \beta < 1$, Alice and Bob play the $\delta$-dimensional Hausdorff (resp. packing) $\beta$-game as follows:

- The turn order is alternating, with Alice playing first.
- Alice begins by choosing a starting radius $\rho_0 > 0$.
- On the $k$th turn, Alice chooses a nonempty $3\rho_k$-separated set $A_k \subseteq \mathbb{R}^d$, and Bob responds by choosing a ball $B_k = B(\mathbf{x}_k, \rho_k)$, with

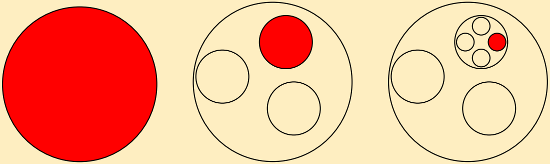
  $$\mathbf{x}_k \in A_k \quad \text{and} \quad \rho_k = \beta^k \rho_0$$

  *Alice's choice $A_k$ represents the collection of balls $\{B(\mathbf{x}, \rho_k) : \mathbf{x} \in A_k\}$ from which Bob chooses his ball.*
- On the first (0th) turn, Alice's choice $A_0$ can be any finite set, but on subsequent turns she must choose it to satisfy

  $$A_{k+1} \subseteq B(\mathbf{x}_k, (1-\beta)\rho_k). \tag{1}$$

  Note that this condition guarantees $B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$

# Games people play



Three consecutive rounds of the Hausdorff/packing game. On each round Alice presents Bob with a set of balls to choose from (represented by the set of centers of her balls), and Bob chooses one of the balls, which are colored/shaded above.

After infinitely many turns have passed, the point

$$\mathbf{x}_\infty = \lim_{k\to\infty} \mathbf{x}_k \in \bigcap_{k=0}^{\infty} B_k \tag{2}$$

called the outcome of the game is computed.

# Games people play

Let $\mathcal{A} = (A_k)_{k \in \mathbb{N}}$, and compute the numbers

$$\underline{\delta}(\mathcal{A}) \stackrel{\text{def}}{=} \liminf_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k} \frac{\log \#(A_i)}{-\log(\beta)} \quad \text{and} \quad \overline{\delta}(\mathcal{A}) \stackrel{\text{def}}{=} \limsup_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k} \frac{\log \#(A_i)}{-\log(\beta)}$$

that represent Alice's score in the Hausdorff and packing games, resp.

$S \subseteq \mathbb{R}^d$ is $\delta$-dimensionally Hausdorff (resp. packing) $\beta$-winning if Alice has a strategy to simultaneously ensure

- that the outcome $\mathbf{x}_\infty$ is in $S$, and
- that her score in the Hausdorff (resp. packing) game is at least $\delta$.

$S \subseteq \mathbb{R}^d$ is $\delta$-dimensionally Hausdorff (resp. packing) winning if for all sufficiently small $\beta > 0$ the set $S$ is $\delta$-dimensionally Hausdorff (resp. packing) $\beta$-winning.

# Games people play

## Theorem (DFSU)

*The Hausdorff (resp. packing) dimension of a Borel set $S \subseteq \mathbb{R}^d$ is the supremum of $\delta$ such that $S$ is $\delta$-dimensionally Hausdorff (resp. packing) winning.*

The same proof allows us to replace $\mathbb{R}^d$ with any **doubling metric space**.

A metric space is doubling if there exists constants $C, r_0$ such that every ball of radius $0 < r \leq r_0$ can be covered by at most $C$ balls of radius $r/2$.

Suppose $S$ is $\delta$-dimensionally Hausdorff winning. WTS: $\dim_H(S) \geq \delta$.

Fix $\beta > 0$ such that $S$ is $\delta$-dimensionally Hausdorff $\beta$-winning, and consider a strategy for Alice to win the $\delta$-dimensional Hausdorff $\beta$-game with target set $S$.

For each $k \geq 0$, let $E_k$ denote the union of all sets $A_k$ that Alice might choose in response to some possible sequence of moves that Bob could play.

Then the set

$$C \stackrel{\text{def}}{=} \bigcap_{k=0}^{\infty} \bigcup_{\mathbf{x}_k \in E_k} B(\mathbf{x}_k, \rho_k)$$

is the set of all possible outcomes of the game when Alice plays her winning strategy. It is a closed and totally disconnected set, contained entirely in $S$.

To bound $\dim_H(C)$, construct a probability $\mu$ on $C$ by considering the scenario where Alice plays according to her winning strategy and Bob plays randomly: on $k$th turn, Bob chooses point $\mathbf{x}_k \in A_k$ uniformly at random, independently of all previous choices. We have a random game whose outcome is distributed via $\mu$.

## Games people play: Proof sketch of lower bound for Hausdorff dimension

Fix $\mathbf{x} \in C$, and for every $k \geq 0$ let $\mathbf{x}_k \in E_k$ be chosen so that $\mathbf{x} \in B(\mathbf{x}_k, \rho_k)$. Then

$$B(\mathbf{x}, \rho_k) \cap C \subseteq B(\mathbf{x}_k, \rho_k)$$

and so

$$\mu(B(\mathbf{x}, \rho_k)) \leq \mu(B(\mathbf{x}_k, \rho_k)) = \left( \prod_{i=0}^{k} \#(A_i) \right)^{-1}$$

WTS: $\underline{\dim}_{\mathbf{x}}(\mu) \geq \delta$ (since then Rogers–Taylor–Tricot $\Rightarrow \dim_H(S) \geq \delta$.)

$$\begin{aligned}
\underline{\dim}_{\mathbf{x}}(\mu) &\stackrel{\text{def}}{=} \liminf_{\rho \to 0} \frac{\log \mu(B(\mathbf{x}, \rho))}{\log \rho} \\
&= \liminf_{k \to \infty} \frac{\log \mu(B(\mathbf{x}, \rho_k))}{\log \rho_k} && \text{(since } \rho_k = \beta^k \rho_0\text{)} \\
&\geq \liminf_{k \to \infty} \frac{\log \mu(B(\mathbf{x}_k, \rho_k))}{\log \rho_k} \\
&= \liminf_{k \to \infty} \frac{-\sum_{i=0}^{k} \log \#(A_i)}{k \log \beta + \log \rho_0} \\
&= \underline{\delta}(\mathcal{A}) \geq \delta && \text{(since Alice is using a winning strategy)}
\end{aligned}$$

$\square$

# The parametric geometry of numbers

The fundamental question:

Given a matrix $A$, what does its *successive minima function*

$$\mathbf{h} = \mathbf{h}_A = (h_1, \ldots, h_d) : [0, \infty) \to \mathbb{R}^d$$

$$h_i(t) \stackrel{\text{def}}{=} \log \lambda_i(g_t u_A \mathbb{Z}^d)$$

look like?

The $j^{th}$ successive minimum of a lattice $\Lambda \subseteq \mathbb{R}^d$ w.r.t. the unit ball is the number

$$\lambda_j(\Lambda) = \inf \left\{ \lambda > 0 : \begin{array}{c} \Lambda \cap \lambda B(0,1) \text{ contains at least } j \\ \text{linearly independent vectors} \end{array} \right\}.$$

# The parametric geometry of numbers

Dani correspondence principle shows that Diophantine questions about a matrix are equivalent to questions about its successive minima function.

Thus our dictionary translates as:

| **Diophantine properties of a matrix $A$** | **Asymptotic properties of $h_1 = h_{A,1}$** |
|---|---|
| $A$ is badly approximable | $\limsup_{t\to\infty} -h_1(t) < \infty$ |
| $A$ is very well approximable | $\limsup_{t\to\infty} \frac{-1}{t} h_1(t) > 0$ |
| $A$ is singular | $\liminf_{t\to\infty} -h_1(t) = \infty$ |

# Templates

**Q:** When can a given function $\mathbf{f} : [0, \infty) \to \mathbb{R}^d$ be approximated to within a constant by the successive minima function of some $m \times n$ matrix?

**A:** When it can be approximated up to a constant by an $m \times n$ *template.*

---

A $m \times n$ **template** is a piecewise linear map $\mathbf{f} : [0, \infty) \to \mathbb{R}^d$ such that:

(I) $\boxed{f_1 \leq \cdots \leq f_d}$

(II) $\boxed{-\frac{1}{n} \leq f_j' \leq \frac{1}{m}}$ for all $j = 1, \ldots, d$.

(III) For all $j = 0, \ldots, d$ and for every interval $I$ such that $f_j < f_{j+1}$ on $I$, the function

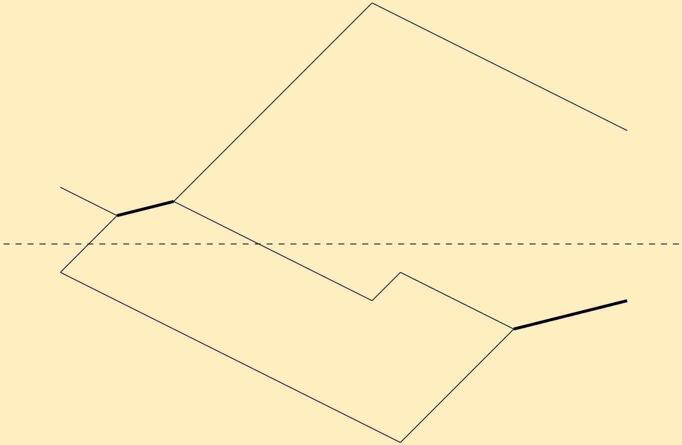
$$F_j \stackrel{\text{def}}{=} \sum_{0<i\leq j} f_i$$

is convex on $I$ with slopes in

$$Z(j) \stackrel{\text{def}}{=} \left\{ \frac{L_+}{m} - \frac{L_-}{n} : \begin{array}{c} L_\pm \in \mathbb{Z}, \quad L_+ + L_- = j, \\ 0 \leq L_+ \leq m, \quad 0 \leq L_- \leq n \end{array} \right\}.$$

Exercise: (II) and (III) imply that each $f_i$ takes on at most finitely many slopes, all of which are multiples of *mnd*!

# Templates



Joint graph of $f_1, f_2, f_3$ for a $1 \times 2$ template $\mathbf{f}$.

# Templates

## Theorem (DFSU)

(i) *For every $m \times n$ matrix $A$, there exists an $m \times n$ template $\mathbf{f}$ such that*

$$|\mathbf{h}_A - \mathbf{f}| \leq C.$$

(ii) *For every $m \times n$ template $\mathbf{f}$, there exists an $m \times n$ matrix $A$ such that*

$$|\mathbf{h}_A - \mathbf{f}| \leq C.$$

Special case when $m = 1$ due to Roy (Annals, 2015).

# A variational principle

Suppose we want to compute the Hausdorff dimension of

$$\mathrm{Sing}_{m,n} = \left\{ A : \liminf_{t\to\infty} -h_{A,1}(t) = \infty \right\}.$$

We can write

$$\mathrm{Sing}_{m,n} = \{A : \mathbf{h}_A \in \mathcal{S}\}$$

where

$$\mathcal{S} = \left\{ \mathbf{f} : [0,\infty) \to \mathbb{R}^d : \liminf_{t\to\infty} -f_1(t) = \infty \right\}.$$

Note that $\mathcal{S}$ is closed under finite perturbations:

$$\text{if } \mathbf{f} \in \mathcal{S} \text{ and } |\mathbf{g} - \mathbf{f}| \leq C, \text{ then } \mathbf{g} \in \mathcal{S}.$$

# A variational principle

## Theorem (DFSU)

*Let $\mathcal{S}$ be a class of functions closed under finite perturbations and let*

$$\mathcal{D}(\mathcal{S}) = \{A : \mathbf{h}_A \in \mathcal{S}\}.$$

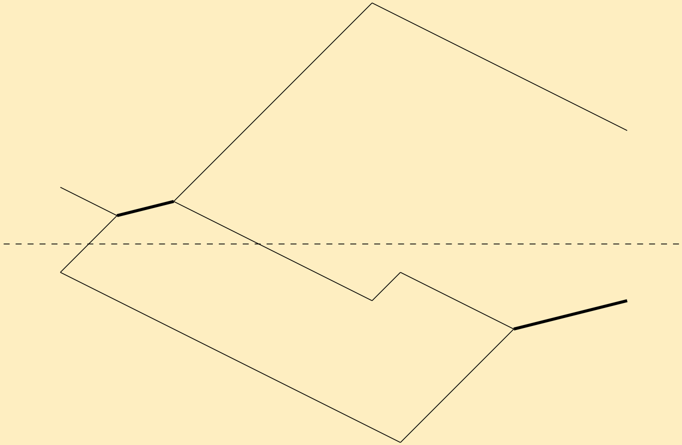
*Then*

$$\dim_H(\mathcal{D}(\mathcal{S})) = \sup_{\mathbf{f} \in \mathcal{S} \cap \mathcal{T}} \underline{\delta}(\mathbf{f}),$$
$$\dim_P(\mathcal{D}(\mathcal{S})) = \sup_{\mathbf{f} \in \mathcal{S} \cap \mathcal{T}} \overline{\delta}(\mathbf{f}),$$

*where $\mathcal{T}$ is the class of $m \times n$ templates, and the functions $\underline{\delta}, \overline{\delta} : \mathcal{T} \to [0, mn]$, called the lower and upper average contractivities, will be defined below.*

Notions of a template and its lower average contractivity do not require any machinery beyond elementary combinatorics and piecewise linear functions.

# Defining the lower and upper average contractivities



Joint graph of $f_1, f_2, f_3$ for a $1 \times 2$ template $\mathbf{f}$.

# Defining the lower and upper average contractivities

The contraction rate $\delta(\mathbf{f}, I)$ of a template $\mathbf{f}$ on an interval of linearity $I$:

- Interpret the motion occurring in $I$ as being the result of "collisions" between $m$ particles going up and $n$ particles going down.
- Count the number of particle pairs that are "moving towards" each other (including particles "colliding" with each other).

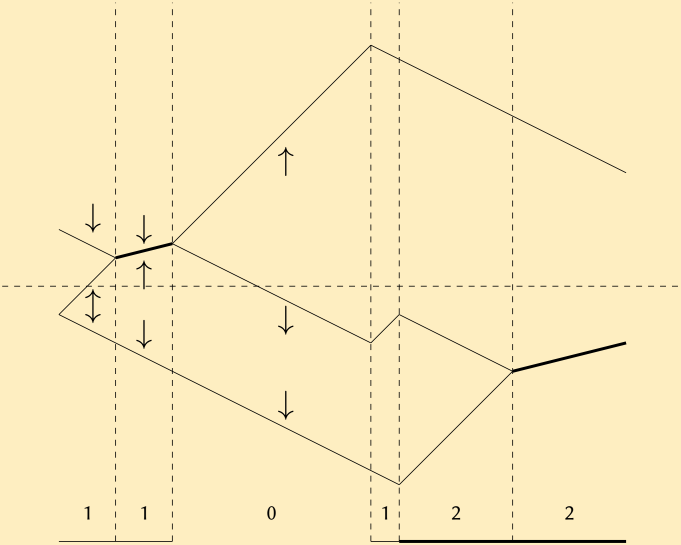
Write $\delta(\mathbf{f}, t) = \delta(\mathbf{f}, I)$ for all $t \in I$. Then let

$$\Delta(\mathbf{f}, T) \stackrel{\text{def}}{=} \frac{1}{T} \int_0^T \delta(\mathbf{f}, t)\, \mathrm{d}t$$

$$\underline{\delta}(\mathbf{f}) \stackrel{\text{def}}{=} \liminf_{T \to \infty} \Delta(\mathbf{f}, T) \qquad \overline{\delta}(\mathbf{f}) \stackrel{\text{def}}{=} \limsup_{T \to \infty} \Delta(\mathbf{f}, T)$$

lower avg contractivity of $\mathbf{f}$ upper avg contractivity of $\mathbf{f}$

# Defining the lower and upper average contractivities



# Proof sketch for HD computation

Recall $\dim_H(\mathrm{Sing}_{m,n})$ is equal to the sup of the lower average contractivities of $m \times n$ singular templates $\mathbf{f}$ (whose $f_1(t) \to -\infty$). Consider

$$\phi(t) = \frac{mn}{m+n} \max\left(m|f_1(t)|, n|f_{m+n}(t)|\right)$$

Then

$$\phi(t_2) - \phi(t_1) \le \int_{t_1}^{t_2} \left(mn - \frac{mn}{m+n} - \delta(t)\right) \,\mathrm{d}t,$$

where $\delta(t)$ denotes the contractivity at time $t$. Since $\mathbf{f}$ is singular, this inequality is valid as long as $t_1$ is sufficiently large. Dividing by $t_2$ and taking limsup as $t_2 \to \infty$, we see that the limsup of $\phi(t)/t$ is related to the lower average contractivity of $\mathbf{f}$:

$$\limsup \frac{\phi(t)}{t} \le mn - \frac{mn}{m+n} - \underline{\delta}(\mathbf{f}),$$

where $\underline{\delta}(\mathbf{f})$ denotes the lower average contractivity.

# Proof sketch for HD computation

On the other hand, since the potential energy $\phi(t)$ is always nonnegative, the left-hand side is $\geq 0$. Rearranging gives the inequality

$$\underline{\delta}(\mathbf{f}) \leq mn - \frac{mn}{m+n}$$

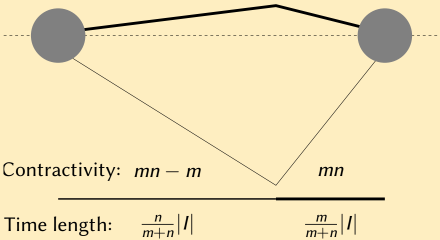
and taking the supremum over $\mathbf{f}$ shows that the Hausdorff dimension of the set of singular $m \times n$ matrices is at most $mn - \frac{mn}{m+n}$.

# Proof sketch for HD computation

For the reverse inequality, we need to find a singular template for which

$$\phi(t_2) - \phi(t_1) \leq \int_{t_1}^{t_2} \left( mn - \frac{mn}{m+n} - \delta(t) \right) \, \mathrm{d}t$$

is arbitrarily close to equality. One way to construct such a template is to glue long intervals of equality with short intervals of strict inequality.




The average contractivity over the time interval $I$ is approximately equal to

$$(mn - m)\frac{n}{m+n} + mn\frac{m}{m+n} = mn - \frac{mn}{m+n}.$$

# Return to parametric geometry of numbers

Recall that for a given matrix $A$, the *successive minima function* of $A$ was

$$\mathbf{h} = \mathbf{h}_A = (h_1, \ldots, h_d) : [0, \infty) \to \mathbb{R}^d$$

where

$$\boxed{\mathrm{h}_j(t) \stackrel{\text{def}}{=} \log \lambda_j(g_t u_A \mathbb{Z}^d)}$$

| | |
|---|---|
| $A$ is badly approximable | $\limsup_{t\to\infty} -h_1(t) < \infty$ |
| $A$ is very well approximable | $\limsup_{t\to\infty} \frac{-1}{t} h_1(t) > 0$ |
| $A$ is singular | $\liminf_{t\to\infty} -h_1(t) = \infty$ |
| $A$ is very singular | $\liminf_{t\to\infty} \frac{-1}{t} h_1(t) > 0$ |

# Uniform exponents

The uniform exponent of irrationality of an $m \times n$ matrix $A$, denoted $\widehat{\omega}(A)$, is the supremum of $\omega$ such that for all $Q$ sufficiently large, there exist integer vectors $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q} + \mathbf{p}\| \leq Q^{-\omega} \text{ and } 0 < \|\mathbf{q}\| \leq Q.$$

By Dirichlet's theorem, we have $\widehat{\omega}(A) \geq n/m$ for all $A$.

For each $\omega > \frac{n}{m}$ it is natural to consider the levelset

$$\boxed{\mathrm{Sing}_{m,n}(\omega) \stackrel{\text{def}}{=} \{A : \widehat{\omega}(A) = \omega\}}$$

# Uniform exponents: Sample Results

**Theorem (DFSU, Precise formulas for the case $(m, n) = (1, 2)$)**

*For all $\omega \in (2, \infty)$, i.e. for all $\tau \in (0, 1/2)$, we have*

$$\dim_H(\mathrm{Sing}_{1,2}(\omega)) = \begin{cases} \frac{4}{3} - \frac{4}{3}\sqrt{\tau - 6\tau^3 + 4\tau^4} - 2\tau + \frac{8}{3}\tau^2 & \text{if } \tau \le \tau_0 \\ \frac{1-2\tau}{1+\tau} & \text{if } \tau \ge \tau_0 \end{cases}$$

$$\dim_P(\mathrm{Sing}_{1,2}(\omega)) = \begin{cases} \frac{4-8\tau}{3} & \text{if } \tau \le \tau_1 \\ 1 & \text{if } \tau \ge \tau_1 \end{cases}$$

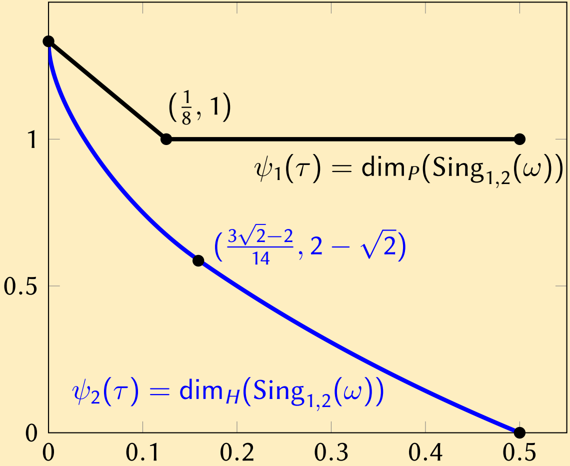
*where*

$$\tau_0 \stackrel{\text{def}}{=} \frac{3\sqrt{2} - 2}{14} \quad \text{and} \quad \tau_1 \stackrel{\text{def}}{=} \frac{1}{8}$$

The HD formula completes a cornucopia of bounds due to Baker, Bugeaud–Laurent, Laurent, Dodson, Yavid, Rynne, and Bugeaud–Cheung–Chevallier (1977-2017).

By Jarník's identity, we can solve the dual problem for $(m, n) = (2, 1)$.

# Precise formulas for the case $(m, n) = (1, 2)$



# Problems, questions, puzzles (I)

## Holy Grail

Precise formulas for HD/PD of $\mathrm{Sing}_{m,n}(\omega)$ in terms of $\omega$, $m$, $n$.

Though we have completely solved this problem in the cases $(m, n) = (1, 2)$ and $(m, n) = (2, 1)$, and for packing dimension in the case where $n \geq 2$, it is plausible that finding a closed form expression in all scenarios is hopeless.

To express the limit of our current understanding, note that we **do not have** conjectural formulas for Hausdorff dimension even for when

$$(m, n) \in \{(1, 3), (3, 1), (2, 2)\}.$$

## Problem

Waiting to be resolved by bright young dynamicists at Ohio State ☺

# Problems, questions, puzzles (II)

Let $\omega(A)$ and $\widehat{\omega}(A)$ denote the standard and uniform exponents of irrationality of a matrix $A$, respectively.

$$\widehat{\omega}(A) \stackrel{\text{def}}{=} \liminf_{Q\to\infty} \sup_{0<\|\mathbf{q}\|\leq Q} \sup_{\mathbf{p}} \frac{-\log\|A\mathbf{q}+\mathbf{p}\|}{\log Q}$$

$$\omega(A) \stackrel{\text{def}}{=} \limsup_{Q\to\infty} \sup_{0<\|\mathbf{q}\|\leq Q} \sup_{\mathbf{p}} \frac{-\log\|A\mathbf{q}+\mathbf{p}\|}{\log Q}$$

### Question

What is the behavior of the function

$$(\omega, \widehat{\omega}) \mapsto \dim_H(\{A : \omega(A) = \omega,\ \widehat{\omega}(A) = \widehat{\omega}\})?$$

# Problems, questions, puzzles (III)

Given $0 < \varepsilon < 1$, an $m \times n$ matrix $A$ is called $\varepsilon$-Dirichlet improvable if for all sufficiently large $Q$, there exists $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n}$ such that

$$\|A\mathbf{q} - \mathbf{p}\| \leq \varepsilon Q^{-n/m} \quad \text{and} \quad 0 < \|\mathbf{q}\| < Q.$$

$A$ is Dirichlet improvable if it is $\varepsilon$-Dirichlet improvable for some $0 < \varepsilon < 1$. Singular matrices are $\varepsilon$-Dirichlet improvable for *all* $0 < \varepsilon < 1$.

**Question**

How do the Hausdorff and packing dimensions of the set of $\varepsilon$-Dirichlet improvable $m \times n$ matrices vary as a function of $\varepsilon$?

**Problem**

Read Gugu Moreira's amazing paper – Geometric properties of the Markov and Lagrange spectra (Annals, 2018) – and dive into the $m = n = 1$ case ☺

# Problems, questions, puzzles (IV)

## Problem

Instead of the set of matrices $A$ such that $(g_t u_A \mathbb{Z}^{m+n})_{t\geq 0}$ is divergent (i.e. the set of singular matrices), consider the set of matrices $A$ such that $(h_t u_A \mathbb{Z}^{m+n})_{t\geq 0}$ is divergent, where $(h_t)_{t\geq 0}$ is some other diagonal flow, e.g.

$$h_t = \mathrm{diag}(e^{a_1 t}, \ldots, e^{a_m t}, e^{-b_1 t}, \ldots, e^{-b_n t}) \in \mathsf{SL}_{m+n}(\mathbb{R})$$

where $a_1, \ldots, a_m, b_1, \ldots, b_n$ are positive real numbers?

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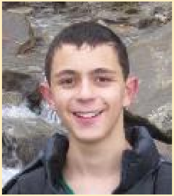
For example, is it possible to compute the Hausdorff and packing dimensions of the set of $m \times n$ matrices $A$ such that the trajectory $(h_t u_A \mathbb{Z}^{m+n})_{t\geq 0}$ is divergent as a function of $a_1, \ldots, a_m, b_1, \ldots, b_n$?

# Problems, questions, puzzles (IV)

When $m = 2$ and $n = 1$, this question has been resolved in case of the Hausdorff dimension!

✓ **Lingmin Liao, Ronggang Shi, Omri Solan & Nattalie Tamam**, *Hausdorff dimension of weighted singular vectors*, (JEMS, 2020).

# Problems, questions, puzzles (V)

## Problem (Bugeaud–Cheung–Chevallier)

What is the dimension of the set of vectors in $\mathrm{Sing}_{2,1}$ whose coordinates belong to Cantor's middle thirds set?

✓ **Osama Khalil (Utah)**, *Singular vectors on fractals and projections of self-similar measures*, (GAFA, 2020).

# Problems, questions, puzzles (VI)

**Conjecture (Cheung, 2011) $\to$ Theorem (Shi–Guan, 2020)**

*For any nonquasi-unipotent (a.k.a. partially hyperbolic) flow on a finite volume noncompact homogeneous space* $(G/\Gamma, g_t)$*, the set of points that lie on divergent trajectories of the flow has Hausdorff dim. strictly less than* $\dim G$.

✓ **Roggang Shi & Lifan Guan**, *Hausdorff dimension of divergent trajectories on homogeneous spaces*, (Compositio, 2020)

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Plenty more in
**A variational principle in the parametric geometry of numbers**
https://arxiv.org/abs/1901.06602

Pick your favorite problem in homogenous dynamics or metric Diophantine approximation and see what **templates** can do for you ☺

## Got any (yawn) questions/puzzles/problems other than these puhleeze?

- Self-affine fractals (DS)
  - §9 in Dimensions of self-affine sponges (Inventiones, 2017)
- Dimension Rigidty (DSU)
  - Dimension rigidity in conformal structures (Advances, 2017)
    Generalize Kleinian result to actions on CAT(-1) spaces
- (Semi)groups acting on Gromov hyperbolic metric spaces (DSU)
  - Appendix A in Geometry and dynamics in non-proper Gromov hyperbolic spaces (Math Surveys & Monographs, 2017)
- Diophantine approximation on fractals (DFSU)
  - §4 in BA on fractals from conformal dynamics (MRL, 2018)
  - §5 in BA on sponges and lower Assouad dimension (ETDS, 2019)
  - Almost any measure from dynamics and/or fractal geometry is quasi-decaying!
    - ★ Extremality & dynamically defined measures I (Selecta, 2018)
    - ★ Extremality & dynamically defined measures II (ETDS, 2020)

# …hmm these sound marginally more interesting – any zoominars on them?

- *Successive minima of lattice trajectories and topological games*
  Ohio State Online Ergodic Theory Seminar, October 2nd 2020
- *Is your favorite dynamically defined measure strongly extremal?*
  Resistência Dinâmica, October 9th 2020
- *Asymptotic expansions for fractal dimensions via thermodynamic formalism*
  Penn State 2020 Workshop in Dynamical Systems, October 30th 2020
- *New results in the dimension theory of continued fraction Cantor sets*
  Bristol ETDS Seminar, November 26th 2020
- *Thermodynamic formalism for coarse expanding dynamical systems*
  University of Maryland Dynamics Seminar, December 10th 2020
- …

☺
# Thank You

*Questions/Comments/Suggestions/Solutions/Criticism*

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