Limit theorems for some type dependent expanding dynamical systems

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Introduction

Let $(X, \mathcal{F}, \mu, T)$ be an ergodic p.p.s. and $g : X \to \mathbb{R}$ be a zero mean integrable function. Let $S_n g = \sum_{j=0}^{n-1} g \circ T^j$. Then by the ergodic theorem

$$\lim_{n\to\infty} \frac{1}{n} S_n g = 0, \ \mu\text{-a.s.}$$

From a probabilistic point of view this is the law of large numbers. For expanding and hyperbolic maps several other limit theorems were derived for "sufficiently regular" observables g (e.g. Hölder or with bounded variation).

The Central Limit Theorem.

1. **The CLT.** Let $\hat{S}_n g = n^{-1/2} S_n g$. Then $\forall r \in \mathbb{R}$,

$$\mu\{x : n^{-1/2} S_n g(x) \le r\} \to \Phi_\sigma(r) := \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{r} e^{-t^2/2\sigma^2} dt$$

where $\sigma^2 = \lim_{n\to\infty} \mu[(\hat{S}_n g)^2]$ and $\Phi_0(r) := \mathbb{I}(r \ge 0)$.

Remark

$\sigma^2 = 0$ if and only if $g = h - h \circ T$ for some $h \in L^2(\mu)$.

2. **The Berry-Esseen Theorem-optimal convergence rate in the CLT.** If $\sigma^2 > 0$, there is a constant $C > 0$ so that for any $n \geq 1$,

$$\sup_{r \in \mathbb{R}} |F_n(r) - \Phi_\sigma(r)| \leq Cn^{-1/2}$$

where $F_n(r) := \mu\{x : n^{-1/2} S_n g(x) \leq r\}$.

3. **The local CLT (LLT).** If g is "non-arithmetic" (aperiodic) then for any continuous function G with compact support (or an indicator of a closed interval),

$$\mathbb{E}[G(S_n g - r)] = \phi_{n\sigma^2}(r) \int G(t)dt + o(n^{-1/2})$$

uniformly in $r \in \mathbb{R}$, where $\phi_a(r) = \frac{e^{-r^2/2a}}{\sqrt{2\pi a}}$ is the density of a normal random variable with zero mean whose variance is a.

Proof by spectral methods. The results follow from analysis of $\Phi_n(t) = \mu(e^{itS_n g})$ on appropriate domains. When $X_j = g_j \circ T^j$ are independent this analysis relies on

$$\Phi_n(t) = \left(\phi_X(t)\right)^n, \ \phi_X(t) = \mathbb{E}(e^{itX_1}).$$

For instance the CLT follows from Taylor expansion and writing

$$(\phi_X(t/\sqrt{n}))^n = \left(1 - t^2 v_X/2n + O(n^{-3/2})\right)^n \to e^{-v_X t^2/2}, \ v_X = \mathrm{Var}(X_1).$$

For expanding maps, there are (Quasi-Compact) operators $\mathcal{L}_t$ continuous in t s.t.

$$\Phi_N(t) = \mu(\mathcal{L}_t^n \mathbf{1}), \ \ \mathcal{L}_t^n = \mathcal{L}_t \circ \cdots \circ \mathcal{L}_t.$$

Now the results follow from the spectral properties of $\mathcal{L}_t$. The idea is that

$$\mathcal{L}_t^n \mathbf{1} \approx e^{n\Pi(t)} h_t, \ e^{\Pi(t)} = \text{Largest eigenvalue of } \mathcal{L}_t.$$

Large deviations type results ("convergence rate in the ergodic theorem").

1. **Exponential concentration inequalities.** $\exists\ c, C > 0$ s.t. for any $\epsilon > 0$ and $n \geq C/\epsilon$ we have

$$\mu\{x : |S_n g(x)| \geq \epsilon n\} \leq 2e^{-c\epsilon^2 n}.$$

2. **Local large deviations principle.** If $\sigma^2 > 0$ then $\exists\ \delta_0 > 0$ s.t. for any interval $J \subset [-\delta_0, \delta_0]$,

$$\lim_{n\to\infty} \frac{1}{n} \ln \mu(S_n g/n \in J) = -\inf_{x\in J} I(x)$$

where I is the Fenchel-Legendre transform of

$$\Lambda(t) = \lim \frac{1}{n} \ln \mu(e^{tS_n g}),\ t \in [-\delta_0, \delta_0].$$

3. **Global moderate deviations principle.** Let (a_n) be s.t. $a_n/\sqrt{n} \to \infty$ and $a_n/n \to 0$. If $\sigma^2 > 0$ then for any interval $J \subset \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{c_n} \ln \mu(S_n g/a_n \in J) = -\inf_{x\in J} I(x), \;\; I(x) = \frac{x^2}{2\sigma^2}$$

where $c_n = a_n^2/n$. For instance we can take $a_n = n/\ln^{\epsilon} n$ and then $c_n = n/\ln^{2\epsilon} n$.

Sketch of proof.

By the Gärtner-Ellis theorem, it is enough to show that

$$(a_n^2/n) \ln \mu(e^{tS_n/a_n}) \to t^2\sigma^2/2.$$

Now, there are Q.C. operators $\mathcal{R}_t$ and $t_0, c_1, c_2 > 0$ s.t. $\forall |t| \leq t_0$,

$$\mu(e^{tS_n}) = \mu(\mathcal{R}_t^n \mathbf{1}) \approx e^{nP(t)} H(t)$$

where $0 < c_1 \leq H(t) \leq c_2$. We have $P(0) = 0$, $P'(0) = \mu(g) = 0$ and $P''(0) = \sigma^2$. Thus, using Taylor approximation of order 2,

$$(a_n^2/n) \ln \mu(e^{tS_n/a_n}) \approx t^2\sigma^2/2 + O(1/a_n) + O(n/a_n^2) \to t^2\sigma^2/2.$$

Time dependent expanding dynamical systems.

Let $(\mathcal{E}_j, d_j)$, $j \in \mathbb{Z}$ be a two sided sequence of compact metric spaces so that $\mathrm{diam}(\mathcal{E}_j) \leq 1$. Let m_j be a sequence of probability ("reference") measures on $\mathcal{E}_j$. Let $T_j : \mathcal{E}_j \to \mathcal{E}_{j+1}$ be a sequence of maps s.t. $(T_j)_* m_j \ll m_{j+1}$. Set

$$T_j^n = T_{j+n-1} \circ \cdots \circ T_{j+1} \circ T_j,\ n \geq 0.$$

Simplified model (Expanding maps on the interval)

Let $\mathcal{E}_j = [0, 1)$ (mod 1), μ_j = Lebesgue and let $\mathcal{I}_j$ be an at most countable partition of $[0, 1)$ into intervals of the form $[a, b)$ so that for all $I \in \mathcal{I}_j$:

(i) the map $T_j|I : I \to [0, 1)$ is differentiable;

(ii) $T_j^{n_0}(I) = [0, 1)$ for some constant n_0;

(iii) $|T_j'| \geq \gamma > 1$ on I for some constant γ;

(iv) the maps $\ln(T_j'|I)$ are α uniformly Hölder continuous for some $\alpha \in (0, 1]$.

Remark

Weaker conditions can be imposed which, for instance, include the case when all T_j are the Gauss map. In particular we can replace the Hölder continuity of $\ln(T_j'|I)$ with a weaker condition.

We can also consider the following more complicated model.

Assumption (Local covering maps)

$\exists\ \xi > 0, \alpha \in (0,1], \gamma > 1,\ L, n_0 \in \mathbb{N}$, *and* $D > 0$ *s.t.* $\forall\ j \in \mathbb{Z}$:

(i) [Sequential topological exactness] For any $x \in \mathcal{E}_j$,

$$T_j^{n_0} B_j(x,\xi) = \mathcal{E}_{j+n_0},\ \ B_j(x,\xi) := \{w \in \mathcal{E}_j : d_j(w,x) < \xi\}. \tag{1}$$

(ii) [Pairing property] $\forall\ x_1, x_2 \in \mathcal{E}_{j+1}$ *s.t.* $d_{j+1}(x,x') < \xi$ *we can write* $T_j^{-1}\{x_i\} = \{y_{i,1}, ..., y_{i,k}\}$ *where* $k \leq D$ *and*

$$d_j(y_{1,s}, y_{2,s}) \leq \gamma^{-1} d_{j+1}(x_1, x_2),\ \ s = 1, 2, ..., k \tag{2}$$

(iii) [Finite covering] There are $x_{1,j}, x_{2,j}, ..., x_{L_j,j}$, $L_j \leq L$ *in* $\mathcal{E}_j$ *so that*

$$\mathcal{E}_j = \bigcup_{s=1}^{L_j} B_j(x_{s,j}, \xi). \tag{3}$$

(iv) For any j, *let* $S_{j,y}$ *be an inverse branch of* T_j *around* $T_j(y)$. *Then* $\ln\left(d(S_{j,y})_* m_{j+1}/dm_j\right)$ *are uniformly* α *Hölder continuous.*

Example

We can consider random (or sequential) Sierpiński gaskets and other maps on random fractal, see *Distance Expanding Random Mappings, Thermodynamic Formalism, Gibbs Measures and Fractal Geometry* by Mayer, Skorulski and Urbańsky.

Theorem (Absolutely continuous mixing equivariant measures)

There exists a family of probability measures μ_j on $\mathcal{E}_j$ so that

$$\mu_j \ll m_j, \;\; (T_j)_*\mu_j = \mu_{j+1},$$

and $h_j := d\mu_j/dm_j$ is α-Hölder continuous uniformly and $\exists c, C > 0$ s.t. $c \leq h_j \leq C$.
Moreover, $\exists A > 0$ and $\delta \in (0,1)$ s.t. for any $n \geq 1$, α-Hölder continuous function $g : \mathcal{E}_j \to \mathbb{C}$ and a bounded function $f : \mathcal{E}_{j+n} \to \mathbb{C}$

$$\left|\mu_j(g \cdot f \circ T_j^n) - \mu_j(g)\mu_{j+n}(f)\right| \leq A\|g\|_{Hold}\mu_{j+n}(|f|)\delta^n. \qquad (4)$$

We are interested in obtaining limit theorems for random variables of the form

$$S_n g(x) = \sum_{j=0}^{n-1} g_j \circ T_0^j(x)$$

where x is distributed according to either μ_0 (and in some cases m_0), and $\{g_j\}$ is a sequence of bounded Hölder continuous function with exponent α (uniformly in j).

Henceforth, for $g_j : \mathcal{E}_j \to \mathbb{C}$, we set

$$\|g\|_{H,\alpha} = \|g\|_H = \|g\| = \sup|g| + v_\alpha(g)$$

where

$$v_\alpha(g) = \sup\left\{|g(x) - g(y)|/(d_j(x,y))^\alpha : \ x \neq y\right\}.$$

We also define

$$\mathcal{H}_{j,\alpha} = \{g : \mathcal{E}_j \to \mathbb{C} : \|g\| < \infty\}$$

and $\mathcal{H}_{j,\alpha,+} = \{g \in \mathcal{H}_{j,\alpha} : g \geq 0\}$.

The variance

Theorem

(1) The variance of $S_n g$ with respect to μ_0 does not tend to ∞ as $n \to \infty$ if and only if

$$g_j - \mu_j(g_j) = Y_j - Y_{j+1} \circ T_j$$

for a sequence $\{Y_j\}$ or uniformly bounded α-Hölder continuous functions (uniformly in j).

(2) In the interval map case, $\exists c, n_1 > 0$ s.t.

$$Var_{\mu_0}(S_n g) \geq cn, \, \forall n \geq n_1$$

if all T_j are sufficiently close to one interval map T in the sense of Keller-Liverani, and all the functions g_j are sufficiently close in the norm $\| \cdot \|_{H,\alpha}$ to a Hölder function g which is not a coboundary with respect to T (related condition can be imposed for more general T_j).

Remark

The CLT follows from Limit theorems for sequential expanding dynamical systems by Conze and Raugi (2007).

Limit theorems

Theorem (A self-normalize Berry-Esseen theorem)

Set $\sigma_n = \sqrt{Var_{\mu_0}(S_n g)}$. *Suppose that*

$$\lim_{n\to\infty} \sigma_n n^{-\frac{1}{3}} = \infty$$

and set $\bar{S}_n g = S_n g - \mu_0(S_n g)$. *Then* $\exists C > 0$ *s.t.* $\forall n \geq 1$ *and* $r \in \mathbb{R}$,

$$\left| \mu_0\{x \in \mathcal{E}_0 : \bar{S}_n u(x) \leq r\sigma_n\} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r} e^{-\frac{1}{2}t^2} dt \right| \leq C n \sigma_n^{-3}. \quad (5)$$

In particular, when σ_n^2 *grows linearly fast in* n *the above left hand side does not exceed* $C_1 n^{-\frac{1}{2}}$ *for some constant* C_1.

The local limit theorem.

As usual, before presenting the LLT we first distinguish between two cases.

1. **Lattice case**: The functions g_j take values at $h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}$ for some $h > 0$. In this case we set $I_h = (-\frac{2\pi}{h}, \frac{2\pi}{h}) \setminus \{0\}$, $R_h = h\mathbb{Z}$. Let also κ_h be the measure assigning unit mass to each one of the elements of R_h.

2. **Non-Lattice case** There exists no h as above. In this case we set $h = 0$ and $I_h = I_0 = \mathbb{R} \setminus \{0\}$. We also set $\kappa_0 =$ Lebesgue (on $\mathbb{R}$) and $R_h = R_0 = \mathbb{R}$.

Theorem (A local limit theorem)

Suppose that for any compact interval $J \subset I_h$ we have

$$\lim_{n\to\infty} \sqrt{n} \sup_{t\in J} |\mu_0(e^{itS_n g})| = 0 \tag{6}$$

and that $\exists c_0 > 0$ s.t. $\sigma_n^2 \geq c_0 n$ for all n large enough. Then for any continuous function $G : \mathbb{R} \to \mathbb{R}$ with compact support (or an indicator of a bounded interval) we have

$$\int G(S_n g(x) - r) d\mu_0(x) =$$

$$\left(\int_{-\infty}^{\infty} G(t) d\kappa_h(t)\right) \frac{\exp(-\frac{(r-\mu_0(S_n g))^2}{2\sigma_n^2})}{\sqrt{2\pi}\sigma_n} + o(\sigma_n^{-1/2})$$

uniformly in $r \in R_h$. When the limit $\sigma^2 = \lim_{n\to\infty} \frac{1}{n}\sigma_n^2$ exists then we can replace σ_n with $\sigma\sqrt{n}$.

A generalization of the De-Moivre Laplace theorem.

Remark

The classical De-Moivre Laplace theorem states that for iid $X_1, X_2, \ldots$ *s.t.* $P(X_j = 1) = p$, $P(X_j = 0) = q := 1 - p$ $(p \in (0,1))$ *we have*

$$P(S_n = k) = \binom{n}{k} p^k q^{n-k} \approx \frac{e^{-(k-np)^2/2npq}}{\sqrt{2\pi npq}}, \ S_n = \sum_{j=1}^{n} X_j.$$

Note that $\mathbb{E}(S_n) = np$ *and* $Var(S_n) = npq$. *The LLT generalizes this to more general sequences. Indeed, when* g_j *are integer-valued with* $h = 1$ *the LLT reads*

$$\mu_0(S_n g = k) = \frac{e^{-(k-\mu_0(S_n g)))^2/2\sigma_n}}{\sqrt{2\pi}\sigma_n} + o(\sigma_n^{-1/2}).$$

Remark

We are able to verify (6) for several maps T_j and functions which are random. For instance, we can consider the case when $T_j = T_{\zeta_j}$ and $g_j = g_{\zeta_j}$ arise as realizations of several classes of stationary or mixing (but not stationary) sequences $\{\zeta_j\}$ of random variable. A very simplified case is when ζ_j is a sequence of independent but not identically distributed discrete random variables so that $\sum_{j=1}^{n} P(\zeta_j = 0) \gg \sqrt{n \ln n}$ and that (T_0, g_0) satisfy certain a periodicity conditions (or a maximal lattice condition).

Large deviation type results.

Theorem (Exponential concentration inequalities)

There is a constant $c, C > 0$ so that for any $\epsilon > 0$ and $n \geq C/\epsilon$ we have

$$\mu_0\{x : |S_n g(x)| \geq \epsilon n\} \leq 2e^{-c\epsilon^2 n}.$$

Theorem (Self Normalized global moderate deviations principle)

Set $\sigma_n^2 := var_{\mu_0}(S_n g)$ *and suppose that*

$$\lim_{n\to\infty} \sigma_n^3/n = \infty.$$

Let (ε_n), $\varepsilon_n > 0$ *so that* $\lim \varepsilon_n = 0$ *and*

$$\lim_{n\to\infty} \frac{n}{\sqrt{\varepsilon_n}\sigma_n^3} = 0.$$

Set $W_n = \frac{S_n g - \mu_0(S_n g)}{\sigma_n(\varepsilon_n)^{-1/2}}$. *Then for any interval* $J \subset R$,

$$\lim_{n\to\infty} \varepsilon_n \ln \mu_0\{x : W_n(x) \in J\} = -\inf_{x\in J} I(x),\ \ I(x) = \frac{x^2}{2}.$$

Remark

The scaling sequence (ε_n) is optimal when σ_n^2 grows linearly fast in n in the sense that (ε_n) can be any sequence satisfying $\lim_{n\to\infty} \varepsilon_n n = \infty$. So the magnitude of $\sigma_n(\varepsilon_n)^{-1/2}$ could be $n/\ln^\delta n$ for $\delta > 0$.

Theorem (Large deviations principle for random maps and functions)

For a wide class of sequences of non-stationary mixing random variables $\{\zeta_n\}$ and maps $T_j = T_{\zeta_j}$ and functions $g_j = g_{\zeta_j}$ which depend on $\{\zeta_n\}$ the following holds true. There exists $\delta_0 > 0$ so that for any interval $J \subset [-\delta_0, \delta_0]$,

$$\lim_{n\to\infty} \frac{1}{n} \ln \mu_0(S_n g/n \in J) = -\inf_{x\in J} I(x)$$

where I is the Fenchel-Legendre transform of $\Lambda(t) = \lim \frac{1}{n} \ln \mu_0(e^{tS_n g}),\ t \in [-\delta_0, \delta_0]$.

Remark

The theorem also holds true for stationary ergodic sequences, but this is essentially known.

On the ides behind the proofs. For any $j \in \mathbb{Z}$ and $z \in \mathbb{C}$ we define a complex transfer operator by

$$(\mathcal{L}_z^{(j)} f)(x) := \sum_{y:\, T_j y = x} \phi_j(y) e^{z g_j(y)} f(y),\ x \in \mathcal{E}_{j+1}$$

where $f : \mathcal{E}_j \to \mathbb{C}$, in the interval map case $\phi_j(y) = 1/T_j'(y)$ while in the other case $\phi_j = d(T_{j,y}^{-1})_* m_{j+1}/dm_j$.

Remark

The following duality relation holds

$$\int f(T_j y)g(y)dm_j(y) = \int \mathcal{L}_0^{(j)} g(x) \cdot f(x)dm_{j+1}(x),$$

$$\text{for } g \in L^\infty(m_j),\ f \in L^\infty(m_{j+1}).$$

Hence, the operators $\mathcal{L}_z^{(j)}$ are perturbation of the dual $\mathcal{L}_0^{(j)}$ of the Koopman operator $g \to g \circ T_j$. Since we do not have a single operator there is no spectral theory to exploit.

Step 1: Operators and characteristic functions.

Let us set

$$\mathcal{L}_z^{j,n} = \mathcal{L}_z^{(j+n-1)} \circ \cdots \circ \mathcal{L}_z^{(j+1)} \circ \mathcal{L}_z^{(j)}.$$

Then

$$\mu_0(e^{zS_n g}) = \mu_n(\mathcal{L}_z^{0,n} h_0/h_n)$$

where we recall that $d\mu_j = h_j dm_j$ satisfy $(T_j)_*\mu_j = \mu_{j+1}$. In order for classical tools from probability to be applicable, we need to get a better understanding of the asymptotic behavior of $\mathcal{L}_z^{0,n}$.

Example

For instance, in order to prove the Berry-Esseen inequality, the idea is to show that, in an appropriate sense,

$$\mu_0(e^{itS_n g/\sigma_n}) = \mu_n(\mathcal{L}_{it/\sigma_n}^{0,n} h_0/h_n) \approx e^{-t^2/2}.$$

when $|t/\sigma_n| \le \delta$ for some $\delta > 0$ small enough. Note that $e^{-t^2/2}$ is the characteristic function of the standard normal distribution.

Step 2: Equivariant cones and their contraction properties.

For each $A > 0$ set

$$\mathcal{C}_{j,A,\mathbb{R}} =$$

$$\{g \in \mathcal{H}_{j,\alpha,+} : g(x) \leq e^{A_j d_j^\alpha(x,y)} g(y)\ \forall x, y \in \mathcal{E}_j \text{ s.t. } d_j(x,y) < \xi\}$$

where in the the interval maps case we set $\xi = 2$. Let

$$\mathcal{C}_{j,A} = \mathbb{C}'(\mathcal{C}_{j,A,\mathbb{R}} + i\mathcal{C}_{j,A,\mathbb{R}}),\ \mathbb{C}' = \mathbb{C} \setminus \{0\}$$

be the canonical complexification of $\mathcal{C}_{j,A,\mathbb{R}}$.

Proposition

There are $A > 0$, $m_0 \in \mathbb{N}$, δ_0 and $r_0 > 0$ so that for any $j \in \mathbb{Z}$, $z \in \mathbb{C}$ with $|z| \leq r_0$ and $n \geq m_0$ we have

$$\mathcal{L}_z^{j,n} \mathcal{C}_{j,A} \subset \mathcal{C}_{j+n,A}$$

and the complex projective diameter of the image does not exceed δ_0.

Step 3: Sequential "spectral" properties; main technical tool

Theorem (A sequential RPF theorem)

$\exists$ $r_0 > 0$, $A > 0$ and $\delta \in (0,1)$ so that $\forall$ $z \in \mathbb{C}$ with $|z| < r_0$ there are triplets $\lambda_j(z) \in \mathbb{C} \setminus \{0\}$, $h_j^{(z)} \in \mathcal{H}_{j,\alpha}$, $\nu_j^{(z)} \in \mathcal{H}_{j,\alpha}^*$ which are analytic in z and uniformly bounded s.t. for any $j \in \mathbb{Z}$, $g \in \mathcal{H}_{j,\alpha}$ and $n \geq 1$,

$$\left\| \frac{\mathcal{L}_z^{j,n} g}{\lambda_{j,n}(z)} - \nu_j^{(z)}(g) h_{j+n}^{(z)} \right\|_{H,\alpha} \leq A \|g\|_{H,\alpha} \delta^n \tag{7}$$

where $\lambda_{j,n}(z) = \lambda_j(z) \cdot \lambda_{j+1}(z) \cdots \lambda_{j+n-1}(z)$. Moreover,

$$\lambda_j(0) = 1, \nu_j^{(0)} = m_j \text{ and } \mu_j = h_j^{(0)} dm_j.$$

Furthermore, $\nu_j^{(z)}(h_j^{(z)}) = \nu_j^{(z)}(\mathbf{1}) = 1$,

$$\mathcal{L}_z^{(j)} h_j^{(z)} = \lambda_j(z) h_{j+1}^{(z)}, \; (\mathcal{L}_z^{(j)})^* \nu_{j+1}^{(z)} = \lambda_j(z) \nu_j^{(z)}.$$

Remark

When $X_j = g_j \circ T_0^j$ are independent then

$$\mu_0(e^{zS_ng}) = \prod_{j=0}^{n-1} \psi_j(z),\ \psi_j(z) = \mathbb{E}(e^{zX_j}).$$

In this case we have $\lambda_j(z) = \psi_j(z)$. In the more general case, the idea is that in some sense,

$$\mu_0(e^{zS_ng}) \approx \prod_{j=0}^{n-1} \lambda_j(z)$$

and so $\lambda_j(it)$ plays the role of the characteristic function in the independent case.

Stability.

The following theorem is needed in the proof of the stability of the variance. Its proof is based on the precise limiting expressions for $\lambda_j(z), h_j^{(z)}$ and $\nu_j^{(z)}$.

Theorem (Stability-the interval maps case)

Let $(\{T_j\},\{g_j\})$ and $(\{\tilde{T}_j\},\{\tilde{g}_j\})$ be two pairs sequences of interval maps and functions in $\mathcal{H}_{j,\alpha}$. Let $(\lambda_j(z), h_j^{(z)}, \nu_j^{(z)})$ and $(\tilde{\lambda}_j(z), \tilde{h}_j^{(z)}, \tilde{\nu}_j^{(z)})$ be the corresponding RPF triplets. Then $\exists r_1 > 0$ s.t. $\forall\ \varepsilon > 0$ there exists $\delta > 0$ so that if

$$\sup_j d_{KL}(T_j, \tilde{T}_j), \sup_j \|g_j - \tilde{g}_j\|_{H,\alpha} < \delta$$

then for any $z \in \mathbb{C}$ s.t. $|z| \le r_1$ we have

$$\max\left(|\lambda_j(z) - \tilde{\lambda}_j(z)|, \|h_j^{(z)} - \tilde{h}_j^{(z)}\|, \|\nu_j^{(z)} - \tilde{\nu}_j^{(z)}\|\right) < \varepsilon.$$

Sketch of proof

Stability of ν. Using the exponential convergence we have

$$\nu_j^{(z)}(g) = \Lambda_{j,n,z} + O(\delta^n),\ \Lambda_{j,n,z} = \frac{\mathcal{L}_z^{j,n} g}{\mathcal{L}_z^{j,n}\mathbf{1}}.$$

Given $\varepsilon > 0$ we can thus find n_0 s.t. $\nu_j^{(z)} = \Lambda_{j,n_0,z} + O(\varepsilon)$. Now if $T_j \approx \tilde{T}_j$ and $g_j \approx \tilde{g}_j$ then

$$\mathcal{L}_z^{j,n_0} \approx \mathbf{L}_z^{j,n_0}$$

where $\{\mathbf{L}_z^{(j)}\}$ is determined by $\{\tilde{T}_j\}$ and $\{z\tilde{g}_j\}$.

Stability of λ and h. The stability of $\lambda_j(z)$ follows from $\lambda_j(z) = \nu_{j+1}^{(z)}(\mathcal{L}_z^{(j)}\mathbf{1})$ and of $h_j^{(z)}$ from the latter two stabilities and the exponential convergence.

Corollary: stability of the variance. We first show that

$$\mathrm{Var}_{\mu_0}(S_n g) = \Pi_n''(0) + O(1)$$

where $\Pi_n(z) = \ln \lambda_{1,n}(z) = \sum_{j=0}^{n-1} \ln \lambda_j(z)$. Thus if for some T and g we have

$$\sigma_{T,g}^2 = \lim_{n\to\infty} \frac{1}{n}\mathrm{Var}_\mu \left(\sum_{j=0}^{n-1} g\circ T^j\right) > 0$$

then with $\tilde{T}_j = T$ and $\tilde{g}_j = g$, when $d_{KL}(T_j, T)$ and $\|g - g_j\|$ are small enough we get

$$\frac{1}{n}\mathrm{Var}_{\mu_0}\left(\sum_{j=0}^{n-1} g_j \circ T_0^j\right) \geq \frac{1}{n}\mathrm{Var}_\mu \left(\sum_{j=0}^{n-1} g\circ T^j\right) - \sigma_{T,g}^2/2 \approx \sigma_{T,g}^2/2.$$

Remark

If fact, we get the following "continuity" of the variance: $\forall \varepsilon > 0$ $\exists \delta > 0$ *s.t. if* $\sup_j \max(\|g_j - g\|, d_{KL}(T_j, T)) < \delta$ *then for any* $n \geq 1$*,*

$$\left| \frac{1}{n} Var_{\mu_0} \left(\sum_{j=0}^{n-1} g_j \circ T_0^j \right) - \sigma_{T,g}^2 \right| < \varepsilon.$$

Remark

We needed the analyticity of $\lambda_j(z)$ *in* z *to get the estimate on the derivatives* $P_n''(0)$*.*