What can be the limit of ergodic averages?

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Plan

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1. Goals, motivation
2. Good sequences
3. Limit measures
4. Recurrence
5. Main question
6. Furstenberg's conjecture
7. Proof

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Goals, motivation

Goal: Determine **all possible limits** in the mean ergodic theorem along subsequences of times and weights.

Why? Besides its intrinsic interest, identification of the limit plays a role in **recurrence**, **almost sure convergence** and is the starting point of the **Hardy-Littlewood circle method** (Wahring-Goldbach, Roth…).

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Notations

$\mathbf{A}_{\mathbf{t}} f$. For a finite set $\mathbf{t}$ and function f defined on $\mathbf{t}$, we define the **arithmetic average** $\mathbf{A}_{\mathbf{t}} f$ of f on $\mathbf{t}$ by

$$\mathbf{A}_{\mathbf{t}} f = \mathbf{A}_{t \in \mathbf{t}} f(t) \coloneqq \frac{1}{\#\mathbf{t}} \sum_{t \in \mathbf{t}} f(t)$$

$\mathbf{e}(\theta)$. We use Weyl's notation, $\mathbf{e}(\theta) \coloneqq e^{2\pi i \theta}$.

$[N]$. We borrow from combinatorics $[N] \coloneqq \{1, 2, \ldots, N\}$.

νf. We use the functional notation for integral:

$$\nu f = \int f \, d\nu \quad \nu_{x \in X} f(x) = \int_X f(x) \, d\nu(x)$$

$\nu\, \mathbf{e}^n$. The ***n*th Fourier coefficient** of the measure ν on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is $\nu\, \mathbf{e}^n = \hat{\nu}(n)$.

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Good sequences

Good times A sequence $\mathbf{t} = (t_n)_{n\in\mathbb{N}}$ is called **good times** if in any probability measure systems $(X, \mathbf{m}, T)$, $f \in L^2(X)$, the limit $\lim_N \mathbf{A}_{n\in[N]} f(T^{t_n}x)$ exists in L^2-norm.

Good weights A sequence $w = (w(n))_{n\in\mathbb{N}}$ is called a **good weight** if in any probability measure systems $(X, \mathbf{m}, T)$, $f \in L^2(X)$, the limit $\lim_N \mathbf{A}_{n\in[N]} w(n)f(T^n x)$ exists in L^2-norm.

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Equivalent formulations

By the spectral theorem, we have the following reformulations of good sequences.

Good times For every real number α, the limit $\lim_N \mathbf{A}_{n\in[N]}\, \mathbf{e}(t_n\alpha)$ exsits.
For every real number α, the limit $\lim_N \mathbf{A}_{n\in[N]}\, \delta_{t_n\alpha}$ exists.

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Limit measures

For a good time $\mathbf{t} = (t_n)$, we define the limit measure $\Lambda_{\mathbf{t},\alpha}$ by

$$\Lambda_{\mathbf{t},\alpha} \coloneqq \lim_N \mathop{\mathbf{A}}_{n\in[N]} \delta_{t_n\alpha}$$

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Examples

We denote by λ the Lebesgue probability measure on the torus $\mathbb{T}$.

$$\Lambda_{\mathbb{N},\alpha} = \lim_N \mathop{\mathbf{A}}_{n\in[N]} \delta_{n\alpha} = \begin{cases} \mathop{\mathbf{A}}_{b\in[q]} \delta_{b/q} & \text{if } \alpha = \frac{a}{q}, \gcd(a,q) = 1 \\ \lambda & \text{if } \alpha \text{ is irrational} \end{cases}$$

$$\Lambda_{\mathbb{S},\alpha} = \lim_N \mathop{\mathbf{A}}_{n\in[N]} \delta_{n^2\alpha} = \begin{cases} \mathop{\mathbf{A}}_{b\in[q]} \delta_{a(b^2/q)} & \text{if } \alpha = \frac{a}{q}, \gcd(a,q) = 1 \\ \lambda & \text{if } \alpha \text{ is irrational} \end{cases}$$

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Recurrence

Let $\mathbf{t} = (t_n)$ be a good time. In a probability measure preserving system $(X, \mathbf{m}, T)$ let $E \subset X$ be a measurable set with $\mathbf{m}(E) > 0$.

By the spectral theorem, there is a Borel measure $\mu = \mu_E$ on $\mathbb{T}$ so that

$$\begin{aligned}\lim_N \mathbf{A}_{n\in[N]} \mathbf{m}(E \cap T^{-t_n} E) &= \lim_N \mathbf{A}_{n\in[N]} \mu_{\alpha\in\mathbb{T}} \mathbf{e}^{t_n}(\alpha) \\ &= \mu_{\alpha\in\mathbb{T}}\left(\lim_N \mathbf{A}_{n\in[N]} \mathbf{e}(t_n\alpha)\right) \\ &= \mu(\Lambda_\alpha \mathbf{e})\end{aligned}$$

with introducing $\mathcal{N} \coloneqq \{\, \alpha \in \mathbb{T} \;:\; \Lambda_\alpha \mathbf{e} \neq 0 \,\}$

$$= \mu_{\alpha\in\mathcal{N}}(\Lambda_\alpha \mathbf{e})$$

Note that $\lambda(\mathcal{N}) = 0$, and, more generally, $\nu(\mathcal{N}) = 0$ for every **Rajchman measure** ν ($\lim_{|n|\to\infty} \nu\, \mathbf{e}^n = 0$).

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Results

Let $\mathbb{T}_q = \left\{ \frac{b}{q} \,:\, b \in [q] \right\}$, the set of qth roots of unity.

Theorem *(Lesigne-Quas-Rosenblatt-Wierdl (2024)).*

1. Suppose the probability measure ν is supported on $\mathbb{T}_q$, and $\alpha = \frac{a}{q}$, $\gcd(a, q) = 1$. Then there is a good time $\mathbf{t} = (t_n)$ so that $\nu = \Lambda_{\mathbf{t},a/q}$.
2. If $\mathbf{t} = (t_n)$ is a good time and α is irrational then $\Lambda_{\mathbf{t},\alpha}$ is a continuous measure.

Main question

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Let $\mathbf{t} = (t_n)$ be a good time.
What measure can $\Lambda_{\mathbf{t},\alpha}$ be?

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Limit measure can be any Rajchman measure

Theorem *(Lesigne-Wierdl (2024)).*
Suppose ν is a Rajchman probability measure, that is, $\lim_{|n|\to\infty} \nu\, \mathbf{e}^n = 0$, and let α be irrational.
Then there is a good time $\mathbf{t} = (t_n)$ so that $\nu = \Lambda_{\mathbf{t},\alpha}$.
If ν is **absolutely** continuous with respect to λ then $\mathbf{t} = (t_n)$ can be **pointwise** good.

Definition *(Representation of a measure).*
Let ν be a Borel probability measure on $\mathbb{T}$ and $\alpha \in \mathbb{R}$.
We say ν **can be represented at** α, if $\nu = \Lambda_{\mathbf{t},\alpha}$ for a good time $\mathbf{t} = (t_n)$.

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- If ν is **not Rajchman**, so $\limsup_{|n|\to\infty} |\nu\, \mathbf{e}^n| > 0$, then $\lambda\{ \alpha \in \mathbb{T} : \nu \quad \text{cannot be represented at} \quad \alpha \} = 1$. (Lesigne-Quas-Rosenblatt-Wierdl (2024))
- There **is** a continuous, non-Rajchman measure which can be represented at an irrational α. (Cuny-Parreau (2024))

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Furstenberg's conjecture

For a good time $\mathbf{t} = (t_n)$ and $\varepsilon > 0$, define the **level set** L_ε by $L_\varepsilon = \{ \alpha \in \mathbb{T} : |\Lambda_{\mathbf{t},\alpha}\, \mathbf{e}| > \varepsilon \}$ and let $L = \cup_{\varepsilon>0} L_\varepsilon$.

Lemma *(Lesigne-Quas-Rosenblatt-Wierdl 2024).*

1. For every $\varepsilon > 0$ the level set L_ε must be nowhere dense.
2. The set L is of first Baire category.

Let ν be a continuous Borel probability measure on $\mathbb{T}$ which is invariant with respect to multiplication by 2 and 3: for every $p \in \mathbb{Z}$, $\nu\, \mathbf{e}^p = \nu\, \mathbf{e}^{p2^j3^k}$ for every $j, k \in \mathbb{N}$.

Suppose $\nu = \Lambda_{\mathbf{t},\alpha}$ for an irrational α. We claim, $\nu = \lambda$. Suppose to the contrary: $\nu\, \mathbf{e}^p = \Lambda_{\mathbf{t},\alpha}\, \mathbf{e}^p \neq 0$ for some $p \in \mathbb{Z} \setminus \{ 0 \}$. Then

$$\nu\, \mathbf{e}^p = \nu\, \mathbf{e}^{p2^j3^k} = \Lambda_{\mathbf{t},\alpha}\, \mathbf{e}^{p2^j3^k} = \Lambda_{\mathbf{t},p2^j3^k\alpha}\, \mathbf{e}$$

But by Furstenberg's theorem, the set $\{ 2^j3^k(p\alpha) : j, k \in \mathbb{N} \}$ is dense in $\mathbb{T}$. $\Rightarrow\Leftarrow$ with Lemma/1.

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Furstenberg's conjecture

For a good time $\mathbf{t} = (t_n)$ and $\varepsilon > 0$, define the **level set** L_ε by $L_\varepsilon = \{ \alpha \in \mathbb{T} : |\Lambda_{\mathbf{t},\alpha}\, \mathbf{e}| > \varepsilon \}$ and let $L = \cup_{\varepsilon>0} L_\varepsilon$.

Lemma *(Lesigne-Quas-Rosenblatt-Wierdl 2024).*

1. For every $\varepsilon > 0$ the level set L_ε must be nowhere dense.
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What can be the limit?

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Proof

Theorem *(Lesigne-Wierdl (2024)).*
Suppose ν is a Rajchman probability measure, that is, $\lim_{|n|\to\infty} \nu\,\mathbf{e}^n = 0$, and let α be irrational.
Then there is a good time $\mathbf{t} = (t_n)$ so that $\nu = \Lambda_{\mathbf{t},\alpha}$.
If ν is **absolutely** continuous with respect to λ then $\mathbf{t} = (t_n)$ can be **pointwise** good.

- We first construct a good **weight** w with $\Lambda_{w,\alpha} = \nu$: For an appropriately fast increasing $N_1 < N_2 < \dots$, if $N_k \le N < N_{k+1}$ then $\Lambda_{w,\beta}\,\mathbf{e}$ is approximated by
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