What can be the limit of ergodic averages?

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3. Limit measures
4. Recurrence
5. Main question
6. Furstenberg's conjecture
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Goals, motivation

**Goal:** Determine all possible limits in the mean ergodic theorem along subsequences of times and weights.

**Why?** Besides its intrinsic interest, identification of the limit plays a role in recurrence, almost sure convergence and is the starting point of the Hardy-Littlewood circle method (Wahring-Goldbach, Roth...).
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**Why?** Besides its intrinsic interest, identification of the limit plays a role in recurrence, almost sure convergence and is the starting point of the Hardy-Littlewood circle method (Wahring-Goldbach, Roth...).
**Notations**

\[ A_t f. \] For a finite set \( t \) and function \( f \) defined on \( t \), we define the **arithmetic average** \( A_t f \) of \( f \) on \( t \) by

\[
A_t f = \frac{1}{\# t} \sum_{t \in t} f(t)
\]

\( e(\theta). \) We use Weyl’s notation, \( e(\theta) := e^{2\pi i \theta} \).

\([N]. \) We borrow from combinatorics \([N] := \{1, 2, \ldots, N\}\).

\( \nu f. \) We use the functional notation for integral:

\[
\nu f = \int f \, d\nu \quad \nu_{x \in X} f(x) = \int_X f(x) \, d\nu(x)
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\( \nu e^n. \) The \( n \)th Fourier coefficient of the measure \( \nu \) on the torus \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) is

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\nu e^n = \hat{\nu}(n).
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Good sequences

**Good times** A sequence $t = (t_n)_{n \in \mathbb{N}}$ is called **good times** if in any probability measure systems $(X, \mu, T), f \in L^2(X)$, the limit $\lim_{N \to \infty} \sum_{n \in [N]} f(T^n x)$ exists in $L^2$-norm.

**Good weights** A sequence $w = (w(n))_{n \in \mathbb{N}}$ is called a **good weight** if in any probability measure systems $(X, \mu, T), f \in L^2(X)$, the limit $\lim_{N \to \infty} \sum_{n \in [N]} w(n)f(T^n x)$ exists in $L^2$-norm.
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Equivalent formulations

By the spectral theorem, we have the following reformulations of good sequences.

**Good times**  For every real number $\alpha$, the limit $\lim_{N} A_{n \in [N]} e(t_n \alpha)$ exists.
For every real number $\alpha$, the limit $\lim_{N} A_{n \in [N]} \delta_{t_n \alpha}$ exists.

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**Limit measures**

For a good time $t = (t_n)$, we define the limit measure $\Lambda_{t, \alpha}$ by

$$\Lambda_{t, \alpha} = \lim_{N} \sum_{n \in \mathbb{N}} \delta_{t_n \alpha}$$

For a good weight $w = (w(n))$, we define the limit measure $\Lambda_{w, \alpha}$ by

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Identification of the limit will be done in terms of limit measures: Which Borel probability measure can be a limit measure?
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Identification of the limit will be done in terms of limit measures: Which Borel probability measure can be a limit measure?
Examples

We denote by $\lambda$ the Lebesgue probability measure on the torus $\mathbb{T}$.

\[
\Lambda_{N,\alpha} = \lim_{N \to \infty} \prod_{n \in [N]} \delta_{n \alpha} = \begin{cases} 
A_{b \in [q]} \delta_{b/q} & \text{if } \alpha = \frac{a}{q}, \gcd(a, q) = 1 \\
\lambda & \text{if } \alpha \text{ is irrational}
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\Lambda_{S,\alpha} = \lim_{N \to \infty} \prod_{n \in [N]} \delta_{n^2 \alpha} = \begin{cases} 
A_{b \in [q]} \delta_{a(b^2/q)} & \text{if } \alpha = \frac{a}{q}, \gcd(a, q) = 1 \\
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Recurrence

Let $t = (t_n)$ be a good time. In a probability measure preserving system $(X, m, T)$ let $E \subset X$ be a measurable set with $m(E) > 0$.

By the spectral theorem, there is a Borel measure $\mu = \mu_E$ on $\mathbb{T}$ so that

$$\lim_{N} \mathbb{A}_{n \in [N]} m(E \cap T^{-tn}E) = \lim_{N} \mathbb{A}_{n \in [N]} \mu_{\alpha \in \mathbb{T}} e^{tn}(\alpha)$$

$$= \mu_{\alpha \in \mathbb{T}} \left( \lim_{N} \mathbb{A}_{n \in [N]} e(t_n \alpha) \right)$$

$$= \mu(\Lambda \alpha e)$$

with introducing $N := \{ \alpha \in \mathbb{T} : \Lambda \alpha e \neq 0 \}$

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Note that $\lambda(N) = 0$, and, more generally, $\nu(N) = 0$ for every Rajchman measure $\nu$ ($\lim_{n \rightarrow \infty} \nu e^n = 0$).
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\lim_N \sum_{n \in [N]} m(E \cap T^{-t_n} E) = \lim_N \sum_{n \in [N]} \mu_{\alpha} e^{t_n}(\alpha) \\
= \mu_{\alpha} \left( \lim_N \sum_{n \in [N]} e(t_n \alpha) \right) \\
= \mu(\Lambda_{\alpha} e)
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with introducing \( \mathcal{N} := \{ \alpha \in \mathbb{T} : \Lambda_{\alpha} e \neq 0 \} \)

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Note that \( \lambda(\mathcal{N}) = 0 \), and, more generally, \( \nu(\mathcal{N}) = 0 \) for every Rajchman measure \( \nu \)
\((\lim_{|n| \to \infty} \nu e^n = 0)\).
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Results

Let $\mathbb{T}_q = \left\{ \frac{b}{q} : b \in [q] \right\}$, the set of $q$th roots of unity.

**Theorem** (Lesigne-Quas-Rosenblatt-Wierdl (2024)).

1. Suppose the probability measure $\nu$ is supported on $\mathbb{T}_q$, and $\alpha = \frac{a}{q}$, $\gcd(a, q) = 1$. Then there is a good time $t = (t_n)$ so that $\nu = \Lambda_{t, a/q}$.

2. If $t = (t_n)$ is a good time and $\alpha$ is irrational then $\Lambda_{t, \alpha}$ is a continuous measure.
Main question

Let $t = (t_n)$ be a good time. What measure can $\Lambda_{t,\alpha}$ be?

▶ We have seen that if $\alpha$ is rational then $\Lambda_{t,\alpha}$ can be any probability measure.
▶ If $\alpha$ is irrational, can it be any continuous measure?
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- We have seen that if $\alpha$ is rational then $\Lambda_{t,\alpha}$ can be any probability measure.
- If $\alpha$ is irrational, can it be any continuous measure?
**Limit measure can be any Rajchman measure**

**Theorem (Lesigne-Wierdl (2024)).**
Suppose $\nu$ is a Rajchman probability measure, that is, $\lim_{|n| \to \infty} \nu e^n = 0$, and let $\alpha$ be irrational.
Then there is a good time $t = (t_n)$ so that $\nu = \Lambda_{t,\alpha}$.
If $\nu$ is absolutely continuous with respect to $\lambda$ then $t = (t_n)$ can be pointwise good.

**Definition (Representation of a measure).**
Let $\nu$ be a Borel probability measure on $\mathbb{T}$ and $\alpha \in \mathbb{R}$.
We say $\nu$ can be represented at $\alpha$, if $\nu = \Lambda_{t,\alpha}$ for a good time $t = (t_n)$.
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Suppose \( \nu \) is a Rajchman probability measure, that is, \( \lim_{|n| \to \infty} \nu e^n = 0 \), and let \( \alpha \) be irrational.
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**Definition** (Representation of a measure).
Let \( \nu \) be a Borel probability measure on \( \mathbb{T} \) and \( \alpha \in \mathbb{R} \).
We say \( \nu \) can be represented at \( \alpha \), if \( \nu = \Lambda_{t,\alpha} \) for a good time \( t = (t_n) \).

▶ If \( \nu \) is not Rajchman, so \( \limsup_{|n| \to \infty} |\nu e^n| > 0 \), then
\( \lambda\{ \alpha \in \mathbb{T} : \nu \text{ cannot be represented at } \alpha \} = 1 \). (Lesigne-Quas-Rosenblatt-Wierdl (2024))
▶ There is a continuous, non-Rajchman measure which can be represented at an irrational \( \alpha \). (Cuny-Parreau (2024))
**Limit measure can be any Rajchman measure**

**Theorem** (Lesigne-Wierdl (2024)).
Suppose \( \nu \) is a Rajchman probability measure, that is, \( \lim_{|n| \to \infty} \nu e^n = 0 \), and let \( \alpha \) be irrational.
Then there is a good time \( t = (t_n) \) so that \( \nu = \Lambda_{t, \alpha} \).
If \( \nu \) is **absolutely** continuous with respect to \( \lambda \) then \( t = (t_n) \) can be **pointwise** good.

**Definition** (Representation of a measure).
Let \( \nu \) be a Borel probability measure on \( \mathbb{T} \) and \( \alpha \in \mathbb{R} \).
We say \( \nu \) can be **represented at** \( \alpha \), if \( \nu = \Lambda_{t, \alpha} \) for a good time \( t = (t_n) \).

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**Theorem** (Lesigne-Wierdl (2024)).

Suppose $\nu$ is a Rajchman probability measure, that is, $\lim_{|n|\to\infty} \nu e^n = 0$, and let $\alpha$ be irrational.

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**Definition** (Representation of a measure).

Let $\nu$ be a Borel probability measure on $\mathbb{T}$ and $\alpha \in \mathbb{R}$.

We say $\nu$ can be represented at $\alpha$, if $\nu = \Lambda_{t,\alpha}$ for a good time $t = (t_n)$.

- If $\nu$ is not Rajchman, so $\limsup_{|n|\to\infty} |\nu e^n| > 0$, then
  $\lambda\{\alpha \in \mathbb{T} : \nu \text{ cannot be represented at } \alpha\} = 1$. (Lesigne-Quas-Rosenblatt-Wierdl (2024))

- There is a continuous, non-Rajchman measure which can be represented at an irrational $\alpha$. (Cuny-Parreau (2024))
Furstenberg’s conjecture

For a good time $t = (t_n)$ and $\varepsilon > 0$, define the level set $L_\varepsilon$ by $L_\varepsilon = \{ \alpha \in \mathbb{T} : |\Lambda_{t,\alpha} e| > \varepsilon \}$ and let $L = \bigcup_{\varepsilon > 0} L_\varepsilon$.

Lemma (Lesigne-Quas-Rosenblatt-Wierdl 2024).

1. For every $\varepsilon > 0$ the level set $L_\varepsilon$ must be nowhere dense.
2. The set $L$ is of first Baire category.

Let $\nu$ be a continuous Borel probability measure on $\mathbb{T}$ which is invariant with respect to multiplication by 2 and 3: for every $p \in \mathbb{Z}$, $\nu e^p = \nu e^{p2/3^k}$ for every $j, k \in \mathbb{N}$.

Suppose $\nu = \Lambda_{t,\alpha}$ for an irrational $\alpha$. We claim, $\nu = \lambda$. Suppose to the contrary: $\nu e^p = \Lambda_{t,\alpha} e^p \neq 0$ for some $p \in \mathbb{Z} \setminus \{ 0 \}$. Then

$$\nu e^p = \nu e^{p2/3^k} = \Lambda_{t,\alpha} e^{p2/3^k} = \Lambda_{t,\alpha p2/3^k} e$$

But by Furstenberg’s theorem, the set $\{ 2^j3^k (\rho \alpha) : j, k \in \mathbb{N} \}$ is dense in $\mathbb{T}$. $\Rightarrow$ with Lemma/1.
**Furstenberg’s conjecture**

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**Lemma** *(Lesigne-Quas-Rosenblatt-Wierdl 2024).*

1. For every $\varepsilon > 0$ the level set $L_\varepsilon$ must be nowhere dense.
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**Furstenberg’s conjecture**

For a good time \( t = (t_n) \) and \( \varepsilon > 0 \), define the **level set** \( L_{\varepsilon} \) by \( L_{\varepsilon} = \{ \alpha \in \mathbb{T} : |\Lambda_{t, \alpha} e| > \varepsilon \} \) and let \( L = \bigcup_{\varepsilon > 0} L_{\varepsilon} \).

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**Proof**

**Theorem** (Lesigne-Wierdl (2024)).

Suppose \( \nu \) is a Rajchman probability measure, that is, \( \lim_{|n| \to \infty} \nu e^n = 0 \), and let \( \alpha \) be irrational.

Then there is a good time \( t = (t_n) \) so that \( \nu = \Lambda_{t, \alpha} \).

If \( \nu \) is **absolutely** continuous with respect to \( \lambda \) then \( t = (t_n) \) can be **pointwise** good.

We first construct a good weight \( w \) with \( \Lambda_{w, \alpha} = \nu \): For an appropriately fast increasing \( N_1 < N_2 < \ldots \), if \( N_k \leq N < N_{k+1} \) then \( \Lambda_{w, \beta} e \) is approximated by

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\Lambda_{\alpha \in [N]} \rho_k(n \alpha) e(n \beta) = \Lambda_{\alpha \in [N]} \left( \sum_{h \in [-k, k]} \nu(e^h) \left( 1 - \frac{|h|}{k+1} \right) e(-hn \alpha) \right) e(n \beta)
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\sum_{h \in [-k, k]} \nu(e^h) \left( 1 - \frac{|h|}{k+1} \right) \Lambda_{\alpha \in [N]} e(n(\beta - h \alpha)) \quad \text{for} \quad \beta \in \mathbb{T}
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**Major arcs** centered at \( h \alpha, h \in [-k, k] \). Usual major arcs are centered at rational points.

We then randomly “construct” the good time \( t \) so that \( \Lambda_{t, \beta} e = \Lambda_{w, \beta} e \).
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