Behavior of ergodic averages along a subsequence and the grid method.

Ergodic Theory Seminar at OSU

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Plan

1. Preliminaries
2. Motivation
3. Main Results
4. Strong sweeping out property
5. Idea of the proof
6. Open problems

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Preliminaries

Let (X, Σ, μ) be a non-atomic probability space, and (T^t) be a measure-preserving flow on (X, Σ, μ). We will call the quadruple (X, Σ, μ, T^t) a dynamical system.

Definition: By a flow $\{T^t : t \in \mathbb{R}\}$ we mean a group of measurable transformations $T^t : X \to X$ with $T^0(x) = x$, $T^{t+s} = T^t \circ T^s$, $s, t \in \mathbb{R}$.

Example(i): For a fixed $r \in \mathbb{N}$, $(\mathbb{T}, \Sigma, \lambda, T^t)$ is a dynamical system, where $\mathbb{T} = [0, 1) (\text{mod } 1)$ and $T^t(x) := x + tr$.

Example (ii): For a fixed vector $\boldsymbol{r} = (r_1, r_2, \ldots, r_K) \in \mathbb{N}^K$, $(\mathbb{T}^K, \Sigma^K, \lambda^{(K)}, \boldsymbol{T}^t)$ is a dynamical system where $\boldsymbol{T}^t(\boldsymbol{x}) := \boldsymbol{x} + t\boldsymbol{r}$.

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Notation: $[N] = \{1, 2, \ldots, N\}$.

Pointwise ergodic theorem

For any $f \in L^1$, the averages $\dfrac{1}{N} \sum_{n \in [N]} f(T^n x)$ converge a.e.

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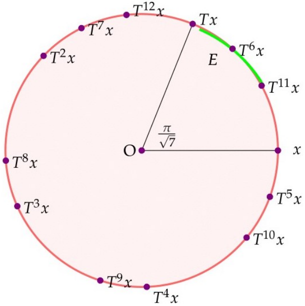

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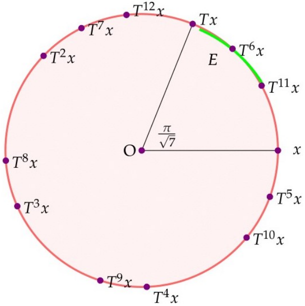

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Motivation

In 1971, it was proved by Krengel that for an arbitrary (a_n), the averages $\frac{1}{N}\sum_{n\in[N]} f(T^{a_n}x)$ may not converge for a.e. x.

Let $1 \le p \le \infty$. A sequence (a_n) of positive real numbers is said to be *pointwise good* for L^p if for every system (X, Σ, μ, T^t) and every $f \in L^p(X)$, $\lim_{N\to\infty} \frac{1}{N}\sum_{n\in[N]} f(T^{a_n}x)$ exists for almost every $x \in X$.

A sequence (a_n) of positive real numbers is said to be *pointwise bad* for L^p if for every aperiodic system (X, Σ, μ, T^t), there is an element $f \in L^p(X)$ such that $\lim_{N\to\infty} \frac{1}{N}\sum_{n\in[N]} f(T^{a_n}x)$ does not exist a.e.

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- (2^n) is pointwise L^∞-bad. [Bellow, 1983]
- $(2^{\frac{n}{\log\log n}})$ are pointwise L^∞-bad.[S.M.-Roy-Wierdl, 2023]

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- $(\log n), (\log\log n)$ are pointwise L^∞-bad. [Jones-Wierdl, 1994]
- For $n \in \mathbb{N}$, let $\Omega(n)$ denote the number of prime factors of n, counted with multiplicity. For example, $\Omega(6) = 2, \Omega(27) = 3$. Then $\Omega(n)$ is pointwise L^∞-bad.[Loyd, 2022]

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If $\alpha \geq 2$ is a *positive integer*, then (n^{α}) is pointwise L^p-good for $p > 1$. [Bourgain, 1988 & 1989]

Primes are pointwise L^p-good for $p > 1$. [Wierdl, 1988].

Question: Is it true that (n^{α}) is pointwise L^p-good for $p > 1$ when α is a positive non-integer real number?

It follows from the work of Fejér and Van der Corput that (n^{α}) is good for mean convergence, when α is a positive real number.

In 1994, it was proved by Bergelson-Boshernitzan-Bourgain (BBB) that if α is a *positive non-integer rational number*, then (n^{α}) is pointwise L^{∞}-bad. This means in every aperiodic dynamical system, we can find a L^{∞} function f such that the averages $\frac{1}{N}\sum_{n\in[N]} f(T^{n^{\alpha}}x)$ fail to converge a.e.

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Main Results

Theorem

If α is a positive non-integer rational number, then in every aperiodic system (X, Σ, μ, T^t) and for every $\epsilon > 0$, there exists a set $E \in \Sigma$ such that $\mu(E) < \epsilon$ and for a.e. $x \in X$,

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n\in[N]} \mathbb{1}_E(T^{n^\alpha} x) = 1 \text{ and } \liminf_{N\to\infty} \frac{1}{N} \sum_{n\in[N]} \mathbb{1}_E(T^{n^\alpha} x) = 0$$

Such oscillation behavior is known as the ‘strong sweeping property’.

Theorem

Let (a_n) be the sequence obtained by rearranging the elements of the set $\{m^{\frac{1}{2}} n^{\frac{1}{3}} : m, n \in \mathbb{N}\}$ in an increasing order. Then (a_n) is also strong sweeping out.

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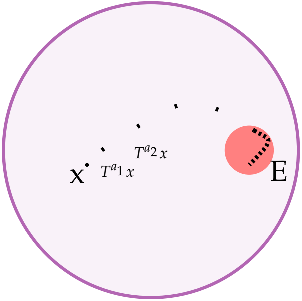
Strong sweeping out property

For (almost) every point $x \in X$, there is an $N = N(x)$, so that x is translated into the set E by T^{a_n} for many $n \in [N]$.



We have $T^{a_n}x \in E$ for many $n \in [N]$, that is, $\mathbb{1}_E(T^{a_n}x) = 1$ for many $n \in [N]$. Hence $\frac{1}{N} \sum_{n \in [N]} \mathbb{1}_E(T^{a_n}x) \approx 1$.

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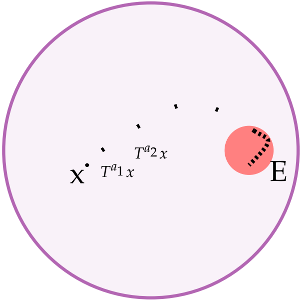

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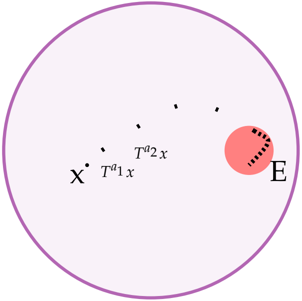

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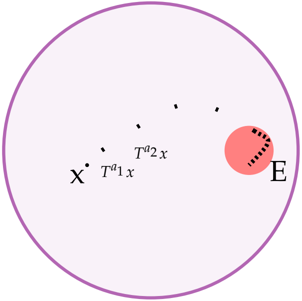

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Idea of the proof:

Theorem (Kronecker's diophantine theorem)

If $1, \theta_1, \theta_2, \ldots, \theta_n$ are real numbers, linearly independent over $\mathbb{Q}$, and if $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{T}$, then for $\epsilon > 0$, there exists $r \in \mathbb{N}$ such that $\mid r\theta_i - \alpha_i \mid < \epsilon$, where $\mathbb{T} = [0, 1)$ (mod 1).

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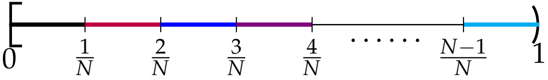

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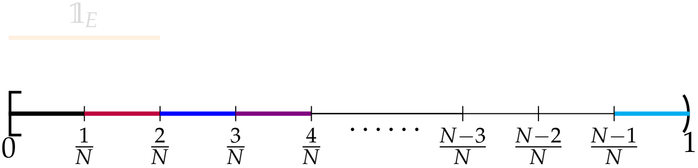

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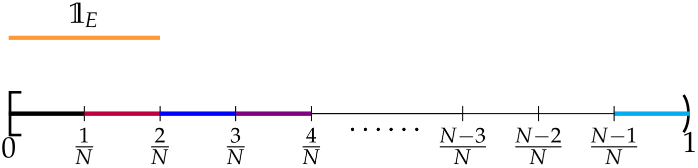

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The case when $S = (\sqrt{n})$

Lemma

The sequence $S = (\sqrt{n})$ can be partitioned as $S = \dot\cup_{k\in\mathbb{N}} S_k$ in such a way that for each k we have following:

1. $d_S(S_k) > 0.$
2. $\sum_{k\in\mathbb{N}} d_S(S_k) = 1.$
3. S_k *is linearly independent over* $\mathbb{Q}$.

We will construct a 'bad set' E in the 2-dimensional torus $\mathbb{T}^2$ and find two integer r_1 and r_2 such that $\lambda(E)$ is small and for every $(x, y) \in \mathbb{T}^2$ we have
$\sup_N \frac{1}{N} \sum_{n\le N} \mathbb{1}_E\left(x + r_1\sqrt{n}, y + r_2\sqrt{n}\right) \ge d_S(S_1) + d_S(S_2)$.

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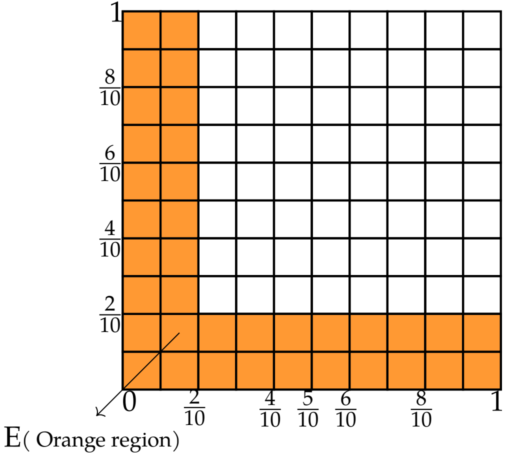
$$\sup_N \frac{1}{N} \sum_{n \le N} \mathbb{1}_E \left(x + r_1\sqrt{n}, y + r_2\sqrt{n} \right) \ge d_S(S_1) + d_S(S_2).$$

Goal: $\sup_N \frac{1}{N} \sum_{n \le N} \mathbb{1}_E\left(x + r_1\sqrt{n}, y + r_2\sqrt{n}\right) \ge d_S(S_1) + d_S(S_2).$



Choose r_1 and r_2 such that $\forall\, 1 \le i, j \le 10$, $\exists$ an 'interval' I such that $r_1\sqrt{n} \in \left[\frac{i-1}{10}, \frac{i}{10}\right]$ if $\sqrt{n} \in S_1 \cap I$ and $r_2\sqrt{n} \in \left[\frac{j-1}{10}, \frac{j}{10}\right]$ if $\sqrt{n} \in S_2 \cap I$.

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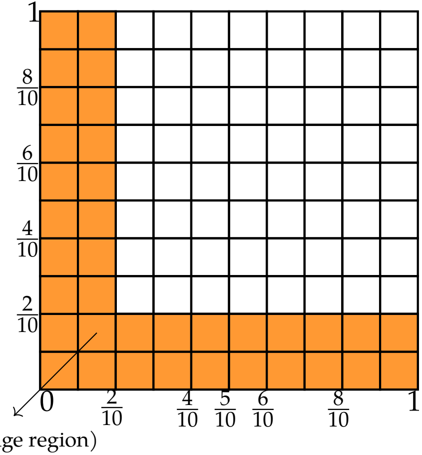

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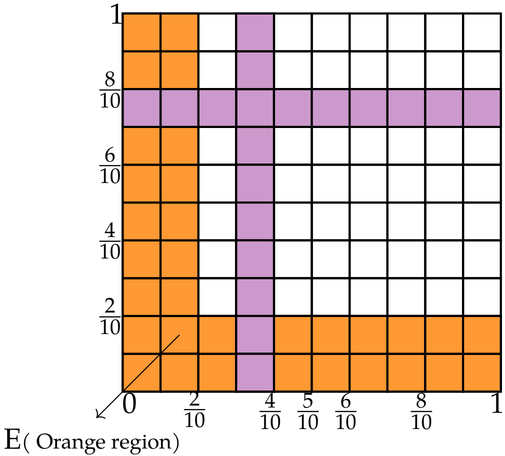

$r_{1}\sqrt{n} \in \left[\frac{3}{10}, \frac{4}{10}\right]$ if $\sqrt{n} \in S_{1} \cap I$ and $r_{2}\sqrt{n} \in \left[\frac{7}{10}, \frac{8}{10}\right]$ if $\sqrt{n} \in S_{2} \cap I$.

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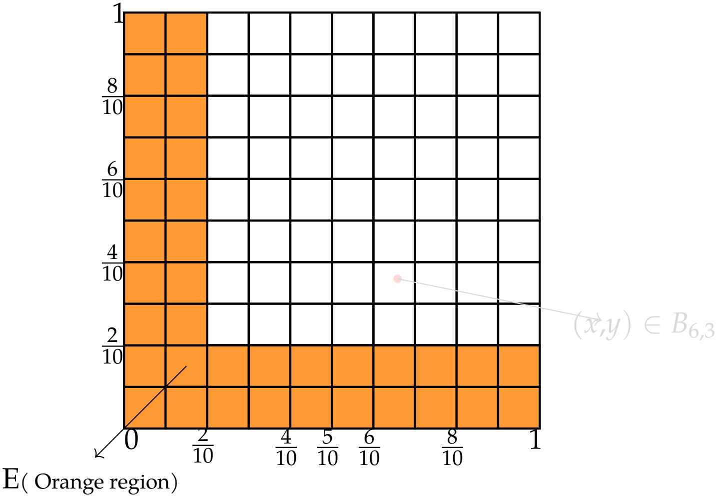

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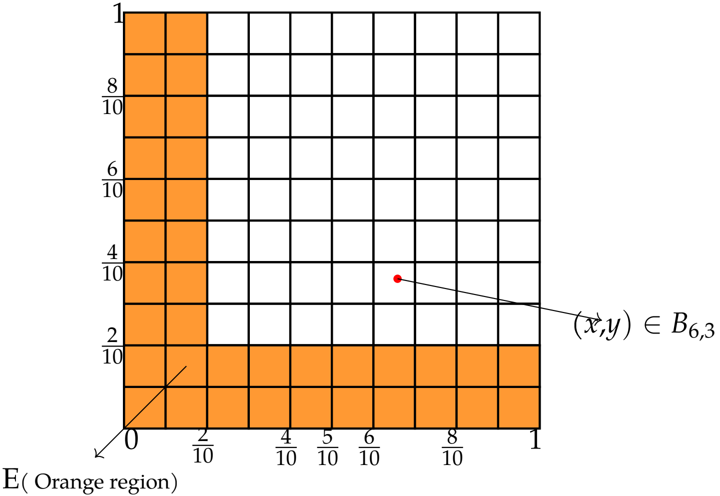

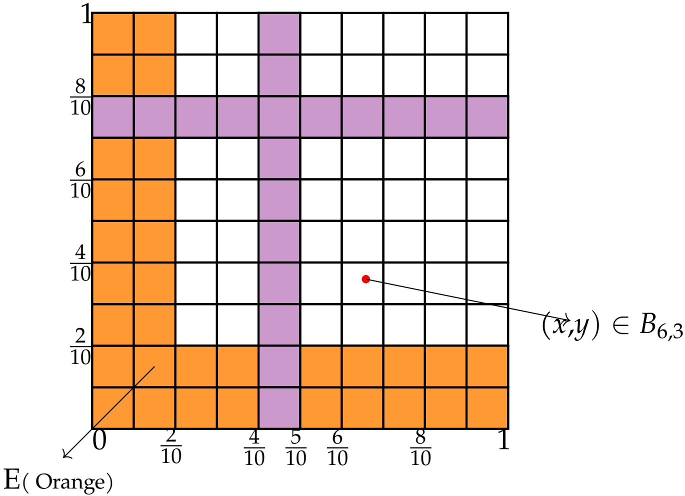

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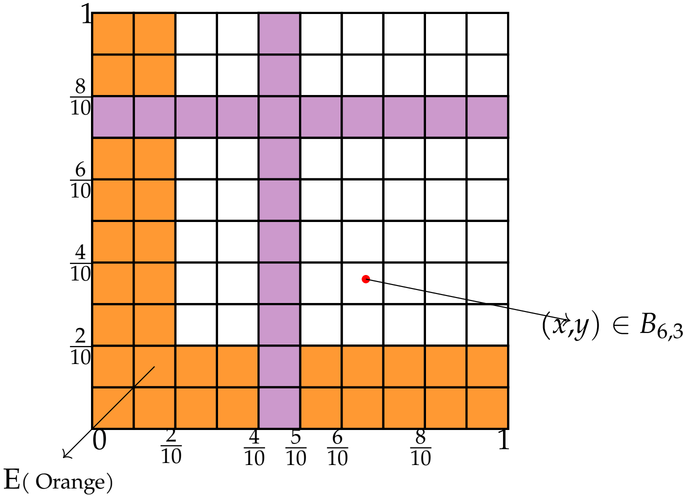

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Open problems

We saw that (n^α) is *strong sweeping out* when α is a *positive non-integer rational number*.

It can be proved that (n^α) is *strong sweeping out* for all but countably many α.

Problem I: Is it true that (n^α) is pointwise L^∞-bad for all positive irrational α?

Problem II: Let α be an irrational number. Is it true that (n^α) is linearly independent over the field of rationals?

If Problem II has an affirmative answer, then so does Problem I.

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Thank you!