Fluctuation of Ergodic Averages and Other Stochastic Processes

Ergodic Theory seminar at OSU

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Plan

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1. Main Questions
2. Usual ergodic averages
3. Lebesgue Differentiation
4. Dyadic Martingles
5. Open Problems
6. Idea of Proofs

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Main Questions

Suppose that $(X, \mathcal{B}, \mu)$ is a probability space, $T_N : L^1 \to L^1$ is a bounded, linear operator and $f \in L^1$ is a real valued function such that $(T_N f)$ converges to $\int f$ for almost every x.

Question I: For any function $f \in L^1$, is the convergence of $T_N f(x)$ non-monotone?

Question II: Can we say that for almost every $x \in X$, $T_N f(x)$ *fluctuates* around its limit infinitely often, that is, is $(T_N f(x) - \int f) > 0$ for infinitely many N, and $(T_N f(x) - \int f) < 0$ for infinitely many (different) N?

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Usual ergodic averages

Let T be an ergodic measure preserving transformation on the probability space (X, Σ, μ). For $f \in L^1$, define
$\mathbb{A}_{[N]}f(T^n x) = \frac{1}{N}\sum_{n=1}^{N} f(T^n x)$.

Example (Irrational rotation): Let θ be an irrational number, and f be a continuous function on $[0,1]$. Then for every $x \in [0,1)$,
$\mathbb{A}_{[N]}f(x+n\theta) = \frac{1}{N}\sum_{n=1}^{N} f(\{x+n\theta\})$ converge to $\int f$ as $N \to \infty$, where $\{y\}$ denotes the fractional part of a real number y.

$\mathbb{A}_{[N]}f(\{x+n\theta\})$ cannot be monotone on on any set of positive measure.

Proposition: The above conclusion is in fact true for any ergodic transformation T and any $f \in L^1$.

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Theorem (Halász)

For any $f \in L^1$, $\left(\mathbb{A}_{[N]}f(T^n x) - \int f\right)$ *almost everywhere changes sign infinitely often in the weaker sense that it cannot be eventually positive or negative.*

Theorem (S.M., Rosenblatt, Wierdl)

(*Informal*) *If* $\mathbb{A}_{[N]}f(T^n x) \geq \int f$ *eventually for a set of positive measure of* x, *then* $f(x) = \mathbb{1}_A(x) - \mathbb{1}_A(Tx)$.

Question: Let (N_k) be an increasing sequence and consider the operator

$$\mathbb{A}_{[N_k]}f(T^n x) = \frac{1}{N_k}\sum_{n=1}^{N_k} f(T^n x).$$

Can we say that $\mathbb{A}_{[N_k]}f(T^n x)$ have non-monotonicity/ fluctuation property for all bounded functions?

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Counter example.

For any given irrational α, one can construct an increasing sequence (N_k) such that for almost every $x \in \mathbb{T}$, the averages are $\left(\mathbb{A}_{[N_k]} f(x + n\alpha)\right)_k$ strictly monotone.

Theorem

Assume that T is an ergodic transformation on a non-atomic probability space $(X, \mathcal{B}, \mu)$ and (N_i) is sublacunary. Then there is a residual set $\beta \subset \mathcal{B}$ with the following property:
For any $E \in \beta$ there exists a constant $d = d(E, N_i) > 0$ such that for a.e. x, $\mathbb{A}_{[0,N_{i+1})} \mathbb{1}_E(T^n x) > \frac{d}{N_{i+1}} + \mathbb{A}_{[0,N_i)} \mathbb{1}_E(T^n x)$ infinitely often and for a.e. x, $\mathbb{A}_{[0,N_{i+1})} \mathbb{1}_E(T^n x) < -\frac{d}{N_{i+1}} + \mathbb{A}_{[0,N_i)} \mathbb{1}_E(T^n x)$ infinitely often.

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If T is ergodic, and (N_k) is an unbounded sequence, then for a typical $f \in L^1$ and for almost every $x \in X$, $\mathbb{A}_{[N_k]} f(T^n x)$ fluctuates infinitely often around its mean.

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Lebesgue Differentiation

Definition. An approximate identity is said to be *proper* if it is normalized and it has shrinking support.
Example. For $n \in \mathbb{N}$ and $x \in \mathbb{R}$ define $\phi_n(x) = n\mathbb{1}_{[-\frac{1}{n},\frac{1}{n}]}(x)$.

Theorem

Let $B = L^p(\mathbb{R})$ for some fixed $p, 1 \leq p \leq \infty$, or $B = CB(\mathbb{R})$. Suppose that (ϕ_n) is a proper approximate identity and (ϵ_n) is a sequence of positive numbers with $\lim_{n\to\infty} \epsilon_n = 0$. Then for a typical function f, we have both $\limsup_{n\to\infty} \frac{\phi_n\star f(x)-f(x)}{\epsilon_n} = \infty$ a.e. and $\liminf_{n\to\infty} \frac{\phi_n\star f(x)-f(x)}{\epsilon_n} = -\infty$ a.e.

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Dyadic Martingles

Take $\mathcal{D}_n$ to be the dyadic σ-algebra whose atoms are the intervals $D_{n,j} = [\frac{j}{2^n}, \frac{j+1}{2^n}], j = 0, \ldots, 2^n - 1$. The finite σ-algebras $\mathcal{D}_n$ are increasing, with the measure-theoretic completion of $\bigcup_{n \in \mathbb{N}} \mathcal{D}_n$ being the Lebesgue σ-algebra in $[0, 1]$.

Then for any $G \in L^1$ and for a.e. $x \in [0, 1]$, $E(G|\mathcal{D}_n)(x) \to G(x)$ as $n \to \infty$.

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For a dense G_δ set $\mathcal{O}$ of functions in $L^1[0, 1]$, if $G \in \mathcal{O}$, for a.e. $x \in [0, 1]$, $E(G|\mathcal{D}_n)(x) > G(x)$ for infinitely many $n \geq 1$, and $E(G|\mathcal{D}_n)(x) < G(x)$ for infinitely many (other) $n \geq 1$.

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Open Problems

Problem 1. For a given (N_k) unbounded, can we characterize those functions which $\mathbb{A}_{[N_k]}f(T^n x)$ fail to oscillate.

Problem 2. Can we describe completely the functions for which we do not have fluctuations of the dyadic martingles in the limit, on some set of positive measure?

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