THE OSU MATH DEPARTMENT DIRECTED READING PROGRAM ANNALS

The **OSU Mathemathics Department's Directed Reading Program** has started on the Spring semester of 2019. As the program grew and thrived through the years, we have decided it is time to start registering summaries of what the participants and their mentors have studied on their DRP's. We hope this file proves to be an useful resource for those interested in knowing how this program works and what to expect, as well as an interesting reading. You can reach us at ohiostatedrp@gmail.com.

Columbus, December of 2021

The DRP Committee

Last updated: August 19, 2024.

Contents

ABOUT THE DRP

1. What is a DRP?

   A DRP is a **"Directed Reading Program"** which pairs undergraduate math students with graduate mentors for a semester-long reading project. Projects are chosen based on mentee interest, with guidance from mentors, and range from working towards understanding a particular theorem to reading an interesting textbook.

2. What are the benefits of a DRP?

   For undergraduates:

   - Chance to learn math one-on-one and see topics outside the standard curriculum;
   - Learn fundamental skills: how to read proofs, how to give effective presentations;
   - Makes graduate school more approachable.

   For graduate students:

   - Get experience mentoring;
   - It's fun!

3. What are the expectations?

   For undergraduates:

   - About 4 hours per week working on the reading;
   - One meeting with your mentor each week;
   - Give a short presentation (between 10 and 15 minutes long) to the other participants at the end of the semester;
   - Submit a short summary/abstract (roughly between one paragraph and half a page) of what you have studied during the semester. This is a good chance to practice writing and to elaborate on things you might not have been able to cover on the presentation!

   For graduate students:

   - Meet with your mentee once a week;
   - Help the mentee come up with a reading project.

4. Want to know more?

   You can visit `www.drp-network.org`, the national organization for DRP's.

FALL 2021

Title: Speckled Butterflies: How a linear stability analysis predicts the survival of the fittest

Student: Athena Stamos & Elizabeth Arend

Mentor: Katelynn Huneycutt

Description: Throughout the semester, our group read chapters 1, 2.2, 3.1-3.4 and 4.3 in *A Course in Mathematical Biology*. Over the course of 10 weeks, we (1) were introduced to various modeling processes (such as the Beverton-Holt and Ricker models) and how different problems require different approaches; (2) studied discrete-time equations, which taught us how to use fixed points and cobwebbing to cleanly model population genetics; (3) applied our knowledge of ordinary differential equations and linear algebra (in particular, eigenvalues and eigenvectors) to create vector fields that beautifully illustrated predator-prey interactions; and (4) learned about a class of partial differential equations—reaction-diffusion equations—and how biologists use these equations to model populations where spacial spread is important. We chose to do our presentation on how a linear stability analysis predicts the "survival of the fittest." In short, our group enjoyed learning about the various tools biomathematicians employ to tackle real-world problems.

Reference(s):

- de Vries, G.; Hillen, T.; Lewis, M.; Müller, J.; Schönfisch, B.; **A Course in Mathematical Biology**, Quantitative modeling with mathematical and computational methods. Mathematical Modeling and Computation, 12. *Society for Industrial and Applied Mathematics (SIAM), Philadelphia*, PA, 2006. xii+309 pp. ISBN: 0-89871-612-8.

Title: Basic notions of topology and a topological proof of the infinitude of primes

Student: Conner Heald

Mentor: Caleb Dilsavor

Description: We spent the bulk of this semester discussing Topology, reading from the Munkres text. We also spent a couple weeks discussing Dynamical Systems, reading from the Brin & Stuck text. Our discussions of Topology were mostly concerned with the more basic concepts: notions of topologies and open/closed sets, types of topologies, continuous functions, metric spaces, etc. We have also studied Fürstenberg's proof of the infinitude or primes.

Reference(s):

- Munkres, J. R.; **Topology**. Second edition. *Prentice Hall, Inc., Upper Saddle River, NJ*, 2000. xvi+537 pp. ISBN: 0-13-181629-254-01.

- Brin, M.; Stuck, G.; **Introduction to Dynamical Systems**, Corrected paper back edition of the 2002 original. *Cambridge University Press, Cambridge*, 2015. xii+247 pp. ISBN: 978-1-107-53894-8; 978-0-521-80841-5

Title: Matroids

Student: David Novikov

Mentor: Henry Tsang

Description: We started with the definition of matroids and how they relate to vectors and graphs. We explored multiple definitions/representations and how to move between them. We spent some time covering greedy algorithms and their proofs. We started exploring how to count regions in hyperplane arrangments and got reasonably far before it came time to finish the semester and generate a presentation.

Reference(s):

- Gordon, G.; McNulty, J.; **Matroids: a Geometric Introduction**, *Cambridge University Press, Cambridge*, 2012. xvi+393 pp. ISBN: 978-0-521-14568-8.

Title: Parrondo's Principle

Student: Gabby Krugman

Mentor: Amanda Pan

Description: The goal of this reading project was to study game theory and the applications of mathematical probability rules as they apply to game theory. We reviewed probability rules and theorems and learned about random walks. Then, we used these preliminary facts to guide our reading of the sections of the textbook that we selected. First, we studied the paradox delineated by Parrondo's Principle and saw how two losing games could be combined to create a winning game, and we reviewed examples of that. Then, we looked into the statistical properties of two-sided coins and studied some coin-play examples such as The St. Petersburg Paradox and The Two-Armed Bandit. Next, we reviewed the probability rules that dictate how card games are played and read about the statistical patterns behind card shuffling and the game of Blackjack. The next area of study we chose involved the stock market and horse racing, and how statistical "games" like those can be explained to produce strategies for winning guided by statistical logic. Overall, our joint study of *The Theory of Gambling and Statistical Logic* by Richard A. Epstein this semester was guided by the desire to learn about game theory and how probability rules can inform players on the optimal strategies for maximum profit return.

Reference(s):

- Epstein, R. A.; **The Theory of Gambling and Statistical Logic**. Second Special Edition. *Elsevier/Academic Press, Amsterdam*, 2013. xiv+451 pp. ISBN: 978-0-12-397857-8.

Title: Complex Analysis and Cauchy-Goursat Theorem

Student: Gene Tsubasa Harada

Mentor: Michael Lane

Description: We will mainly talk about complex integrals, especially using the Cauchy-Goursat theorem. We will go over a summary of what complex analysis is, and after that, we will introduce the Cauchy-Goursat Theorem. By using one example,

$$\oint_C \left(\frac{1}{z} + \frac{1}{z-1}\right) \mathrm{d}z,$$

we want to show a very unique approach of using the theorem to solve a line integral. The process is so unique that I was really amazed when I saw it for the first time: it was something that I have never seen in real analysis.

Reference(s):

- Churchill, R. V.; Brown, J. W.; **Complex Variables and Applications**, Fourth edition. *McGraw-Hill Book Co., New York*, 1984. x+339 pp. ISBN: 0-07-010873-0.

Title: Non-standard analysis

Student: Kabir Belgikar

Mentor: Ivo Terek

Description: Non-standard analysis is one of those topics that is always whispered about or hinted at but never discussed. This is partly because the topic is somewhat obscure and partly because there are quite a few misconceptions (or even outright falsehoods) floating around. All of this gave a somewhat unique, irresistible appeal to the subject and this was the primary motivation for picking this topic. In accordance with our chosen book, we started by learning how to construct a number system ${}^*\mathbb{R}$, called the hyperreals. This then led to an exploration of ultrafilters and a few misadventures relating to Zorn's lemma. As a byproduct of this, we were able to show that repeating this construction starting with the hyperreals led "nowhere" (in the sense that ${}^{**}\mathbb{R}$ doesn't carry more infinitesimal information than ${}^*\mathbb{R}$). After toying around with this new object, we were well-poised to study basic analysis through a new lens. Since limits, derivatives, and integrals were familiar territory, studying this posed little challenge. However, we did see some interesting (and arguably more intuitive) proofs of fundamental results like the Intermediate Value and Bolzano-Weierstrass theorems. Along the way, we also came up with a proof for why there are as many hyperintegers as there are (ordinary) reals, by exhibiting an injection $\mathbb{R} \hookrightarrow {}^*\mathbb{Z}$ and establishing that $|\mathbb{R}| \geq |{}^*\mathbb{Z}|$ with cardinal arithmetic. This culminated in a discussion about topologies on the hyperreals, in which we saw new characterizations for fundamental topological objects in the reals, in hyperreal terms.

Reference(s):

- Goldblatt, R.; **Lectures on the Hyperreals: An Introduction to Non-Standard Analysis**, Graduate Texts in Mathematics, 188. *Springer-Verlag, New York*, 1998. xiv+289 pp. ISBN: 0-387-98464-X.

Title: Generating Functions

Student: Nigel Krekeler

Mentor: Ian Cavey

Description: This semester we studied about 1/2-2/3 of Herbert Wilf's *generatingfunctionology*, a book on generating functions. The format each week essentially entailed me reading a few sections in Wilf, working on a handful of relevant problems from the text, and then discussing these problems and other questions I had from the reading during a roughly 1-1.5 hour meeting. The sections we covered in Wilf primarily dealt with how to find generating functions, general properties of types of generating functions, and applications of generating functions to various combinatorial questions. Occasionally our discussion veered off topic into other areas of math, mostly different things from combinatorics or a little analysis. Overall, the semester was certainly successful in exposing me to new topics of math I feel I would not have otherwise been exposed to (especially in undergrad), and I think I learned a lot from my participation in the DRP.

Reference(s):

- Wilf, H. S.; **generatingfunctionology**, Third edition. *A K Peters, Ltd., Wellesley, MA*, 2006. x+245 pp. ISBN: 978-1-56881-279-3; 1-56881-279-5.

Title: The Yoneda Lemma

Student: River Oxenreider

Mentor: Matthew Carr

Description: This semester, we read the first three chapters of *Category Theory in Context* by Emily Riehl. We learned about categories, functors, and natural transformations. We also learned about limits, colimits, and adjunctions in category theory. We were introduced to the method of diagram chasing and became familiar with using it in proofs. We studied the Yoneda Lemma, a cornerstone of category theory. We also explored applications of the Yoneda Lemma, such as the Yoneda Embedding and the Density Theorem.

Reference(s):

- Riehl, E.; **Category Theory in Context**. *Dover Publications, Inc., Mineola, New York*, 2016. 272 pp. ISBN: 0-48680-903-X.

Title: RSA Encryption in Cryptography

Student: Rylan Roberts

Mentor: Mario Gómez

Description: For this semester, we went over Hoffstein's *Introduction to Mathematical Cryptography*, Chapters 1-5. We did several exercises from each week's chapter, including ideas from number theory courses and beyond. This includes divisibility, modular arithmetic, GCD's, ciphers, discrete logarithm, Diffee-Hellman, public-key cryptography, and more. We discussed various encryption techniques throughout the semester, but our main concern was the level of security behind each encryption system, such as how easy is it to break given certain information.

Reference(s):

- Hoffstein, J.; Pipher, J.; Silverman, J.; **An Introduction to Mathematical Cryptography**, Second edition. Undergraduate Texts in Mathematics. *Springer, New York*, 2014. xviii+538 pp. ISBN: 978-1-4939-1710-5; 978-1-4939-1711-2.

Title: Quaternions for Computer Graphics

Student: Seth Blankenship

Mentor: Kyle Binder

Description: The book that we read for the directed reading program was *Quaternions for Computer Graphics* by John Vince. The focus of the book is on an introduction to quaternions for someone who may not have a background in mathematics. It begins with a refresher on the different numeric sets and groups and their properties. Then it continues to introduce the complex plane and expand that understanding on to quaternions. The book then closes with some advantages of why one would want to use quaternions as a coordinate system.

Reference(s):

- Vince, J.; **Quaternions for Computer Graphics**. Second edition. *Springer London, United Kingdom*, 2021. 181 pp. ISBN: 978-1-44717-508-7.

Title: Topological Data Analysis and The Premier League

Student: Umar Jara

Mentor: Scott Newton

Description: Over the course of the semester, we studied Topological Data Analysis (TDA) and its applications. TDA is founded upon algebraic topology methods used on data viewed as point clouds. Using homology and persistent homology, one can extract features from these point clouds. These features often give significant information about the shape of the data and can be further used in exploratory data analysis or even machine learning. Additionally, we used TDA methods to explore and analyze soccer (football) data from the 2018-2019 season.

Reference(s):

- Needham, T.; **Introduction to Applied Topology**. OSU MATH4570 Lecture Notes, 2017.
- Bunch, E.; **Topological Data Analysis and Persistent Homology**. Blog article, 2018.

Title: One-Dimensional Nonlinear Dynamics and Bifurcations

Student: Viktor Giordano

Mentor: Kriti Sehgal

Description: Over the course of the semester, we have studied the nonlinear dynamical systems of one dimension. First, building an understanding of stability at individual equilibrium points of the system. This led to the study of bifurcations and qualitative changes that can occur within these systems. Examples from physics and mathematical biology were practiced, such as mass and spring motion and population growth. A Python program that computes bifurcation points and plots bifurcation diagrams for a given system was developed to display the comprehension of the aforementioned material. Two-dimensional systems were briefly introduced; however, the presentation will focus on the one-dimensional case.

Reference(s):

- Strogatz, S. H.; **Nonlinear Dynamics and Chaos**, With applications to physics, biology, chemistry, and engineering. Second edition. *Westview Press, Boulder*, CO, 2015. xiii+513 pp. ISBN: 978-0-8133-4910-7; 978-0-8133-4911-4.

Title: Local and global solutions to polynomials

Student: Yang Gao

Mentor: Bhawesh Mishra

Description: Let $f(\mathbf{x})$ be a polynomial in n variables, i.e., $\mathbf{x} = (x_1, \ldots, x_n)$, and k be an integer. Our project was geared towards ultimately learning about solutions of Diophantine equations $f(\mathbf{x}) = k$. Initially, we learned the basic techniques from the *Elementary Number Theory* book by Jones & Jones. For the first few weeks, we reviewed modular arithmetic, the Chinese remainder theorem, solutions to systems of linear equations etc. We especially focused on discovering new proofs for statements in the book. In the latter weeks, we studied some of the historically significant polynomial Diophantine equations. For instance, we discussed several elementary proofs to the infinitude of primes, local solvability of equations of the form $x^n = k$, and the Dedekind-Hasse principle for $\sum_{i=1}^{n} x_i^2 = k$. Finally, we started studying various classes of equations for their local solvability in the last third of the semester. Studying local solutions has implications for many famous mathematics problems such as the Waring's problem, the Schinzel's hypothesis, etc. We studied particular proofs of various examples of locally solvable equations. Our study was primarily based upon my mentor's recently published paper (2021). For instance, we proved that $x^3 + y^3 = k$ has local solutions if and only if $x^3 + y^3 \equiv k \pmod{63}$. Similarly, we also studied that the equation $x^7 + y^7 + z^7 + w^7 = k$ has local solutions for every integer k.

Reference(s):

- Mishra, B.; **Intersective polynomials arising from sums of powers**. Comm. Algebra 49 (2021), no. 6, 2633–2644. DOI: 10.1080/00927872.2021.1879827.

- Jones, G.; Jones, J.; **Elementary Number Theory**, Springer Undergraduate Mathematics Series. *Springer-Verlag London, Ltd., London*, 1998. xiv+301 pp. ISBN: 3-540-76197-7.

Title: The Maximum Flow Problem

Student: Yifan Zhu

Mentor: Shreeya Behera

Description: This semester, the first book we read was *Chromatic Graph Theory*. In this book, we studied the basic concepts of the graph theory. This includes order, size, degree, path, cycle, complete, bipartitions, cut, connectivity, etc. We also covered the proof of Menger's theorem. The second book we read was Introduction to Algorithms. This book helped us connect what I learned in my computer science classes with the corresponding algorithms in graph theory. We discussed Dijkstra's single source shortest path algorithm, Floyd Warshall's all pair shortest path algorithm, Kruskal and Prim's minimum spanning tree algorithm, and topological sorting. For the end-of-term presentation, I decided to dive deep into the maximum flow problem and give a presentation on it.

Reference(s):

- Chartrand, G.; Zhang, P.; **Chromatic Graph Theory**, Discrete Mathematics and its Applications (Boca Raton). *CRC Press, Boca Raton, FL*, 2009. xiv+483 pp. ISBN: 978-1-58488-800-0
- Cormen, T. H.; Leiserson, C. E.; Rivest, R. L.; Stein, C.; **Introduction to Algorithms**, Third edition. *MIT Press, Cambridge, MA*, 2009. xx+1292 pp. ISBN: 978-0-262-03384-8.

Title: The Toric Code

Student: Ziyao Su

Mentor: David Green

Description: This semester we studied the topology of the toric code. We first discussed the basic idea and the concept of topology. Then we talked about the vertex and plaquette operators and learned about the functions and properties of these operators on a lattice surface. Next, we studied the braiding system and, finally, we discussed the Levin-Wen theorem, including notions of homology to do so. We also calculated the homology of several examples.

Reference(s):

- Pachos, J. K.; **Introduction to Topological Quantum Computation**, *Cambridge University Press, Cambridge*, 2012. xii+206 pp. ISBN: 978-1-107-00504-4.

SPRING 2022

Title: The Lagrangean Method

Student: Alexis Mennona

Mentor: Stefan Nikoloski

Description: During the semester we covered the first 5 chapters of *A First Course in Optimization Theory*. In the first one we reviewed some of the necessary mathematical background, like functions, topology and linear algebra. In the next chapter we saw how optimization methods can be used to solve some real-world problems. In Chapter 3, we went through some necessary conditions for a global optimum to exists, one such example being the Extreme Value Theorem. In the final two chapters we learned how to find optimal points in the interior and on the boundary of the feasible set.

Reference(s):

- Sundaram, R. K.; **A first course in optimization theory.** *Cambridge University Press, Cambridge*, 1996. xviii+357 pp. ISBN: 0-521-49719-1.

Title: Classification of Surfaces

Student: Alina Li

Mentor: Aziz Burak Gülen

Description: This semester, we decided to follow the book *Topology of Surfaces* by L.C. Kinsey. We started with a review of some basic definitions and theorems in point-set topology (Chapters 2 & 3). Our goal was to cover until the end of section 4.4 where the classification of surfaces is proved. We first studied the necessary tools in sections 4.1, 4.2, and 4.3, where complexes, surfaces, and triangulations are introduced. We finished the semester by reading the proof of the main theorem that we aimed at, the classification of surfaces, Theorem 4.14.

Reference(s):

- Kinsey, L. C.; **Topology of surfaces.** Undergraduate Texts in Mathematics. *Springer-Verlag, New York, 1993*. viii+262 pp. ISBN: 0-387-94102-9.

Title: Stereographic Projection

Student: David Kruzel

Mentor: Yilong Zhang

Description: This semester, we focused on stereographic projection, which is a bijective map from the unit sphere in $\mathbb{R}^3$ with the north point removed $\mathbb{S}^2 \setminus \{N\}$ to the $z = 0$ plane. The map is conformal and sends circles on the spheres to circles or the lines on the plane. We explored different proofs of conformality. The first proof verifies that the map preserves the angle formed by two intersecting curves directly by brute force; The second proof reduces the verification to a planar geometry argument [Shurman]; The third proof verifies an equivalent definition of conformality from the point of view of Riemannian geometry: The pullback Euclidean metric $|\mathrm{d}z|^2$ on the plane differs from the metric on sphere $g_{\mathbb{S}^2}$ by a smooth function, namely,

$$\frac{4|\mathrm{d}z|^2}{(1+|z|^2)^2} = g_{\mathbb{S}^2},$$

where $z = u + iv$ and we identify $\mathbb{R}^2$ with $\mathbb{C}$. A Möbius transformation is an automorphism of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and has the form $f(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$. The stereographic projection extends the north pole and is a homeomorphism $\mathbb{S}^2 \cong \hat{\mathbb{C}}$. By considering a rotation of the sphere conjugate by stereographic projection, it induces a Möbius transformation on $\hat{\mathbb{C}}$. We explored this relationship following the article [Arnold & Rogness] and the note [Warwick]. As preparation, we reviewed some basic linear algebra and explored basic group theory through examples of 2-by-2 matrix groups, e.g., O(2). We also solved some exercises in [Reid & Szendröi, Chapter 1] and [Gamelin, Section 1.2].

Reference(s):

- Arnold, D. N.; Rogness, J.; **Möbius transformations revealed.** *Notices Amer. Math. Soc.* 55 (2008), no. 10, 1226–1231.
- Gamelin, T. W. **Complex analysis.** Undergraduate Texts in Mathematics. *Springer-Verlag, New York*, 2001. xviii+478 pp. ISBN: 0-387-95093-1; 0-387-95069-9
- Reid, M.; Szendröi, B.; **Geometry and topology.** *Cambridge University Press, Cambridge, 2005.* xviii+196 pp. ISBN: 978-0-521-84889-3; 978-0-521-61325-5; 0-521-61325-6.
- Online notes. **Möbius Transformations**. http://www.warwickmaths.com/wp-content/uploads/2020/07/80_-Mbius-Transformations.pdf, 2021.
- Shurman, J. Online notes. **Stereographic projection is conformal.** https://people.reed.edu/~jerry/311/stereo.pdf, 2021.

Title: Two Proofs of the Prime Number Theorem

Student: Edward Liu

Mentor: Julian Mejia Cordero

Description: For this project, we read from Apostol's *Introduction to Analytic Number Theory* with the goal of covering the Prime Number Theorem and any prerequisites. We studied arithmetic functions, Dirichlet convolution, Dirichlet series, and properties of the Riemann zeta function. Additionally, we read chapters on miscellaneous topics such as quadratic reciprocity and Dirichlet's theorem on arithmetic progressions. Finally, we looked at two proofs of the Prime Number Theorem: the first proof used the Wiener-Ikehara Theorem and the second proof was given in the textbook.

Reference(s):

- Apostol, T. M.; **Introduction to analytic number theory.** Undergraduate Texts in Mathematics. *Springer-Verlag, New York-Heidelberg, 1976.* xii+338 pp.

Title: The Symmetries of Möbius Maps

Student: Haley Scott

Mentor: Amanda Pan

Description: This semester I jumped headfirst into exploring Möbius Transformations. Starting with no background in studying pure maths, this was a task that I could not have taken on without Amanda's patient help. We worked through the first three chapters of *Indra's Pearls* which set the framework for understanding the infinitude of fractals. I learned about the complex plane and its connection to the Riemann Sphere. We touched on some basics of groups, sets, and orbits in the name of defining the symmetries brought about from different mapping. I learned of the four classifications of Möbius maps depending on their number of fixed points and what type of translation, dilation, and rotation they are conjugations of. This was a fun semester, and I would like to again thank Amanda Pan for all her guidance!

Reference(s):

- Mumford, D.; Series, C.; Wright, D.; **Indra's Pearls: The vision of Felix Klein.** With cartoons by Larry Gonick. Paperback edition with corrections. *Cambridge University Press, Cambridge, 2015*. xix+395 pp. ISBN: 978-1-107-56474-9; 978-0-521-35253-6.

Title: On the Poincaré-Hopf Theorem

Student: Harrison Blake

Mentor: Qingsong Wang

Description: This semester we studied the Poincaré-Hopf index theorem, which states that the Euler characteristic number of a smooth manifold can be computed through a smooth vector field that only has isolated zeros. It is a fascinating result that exhibits local-to-global phenomena. We first learned some basics about the smooth manifold and vector fields. Then we studied the degree of the map, which will be used to define the index of isolated zeros. We then went over the proof of the Poincaré-Hopf index theorem and explored some applications, such as the Hairy Ball theorem.

Reference(s):

- Milnor, J. W.; **Topology from the differentiable viewpoint.** Revised reprint of the 1965 original. Princeton Landmarks in Mathematics. *Princeton University Press, Princeton, NJ*, 1997. xii+64 pp. ISBN: 0-691-04833-9.

Title: An Introduction to Category Theory through Brouwer's Fixed Point Theorem

Student: Jacob Monzel

Mentor: Amogh Parab

Description: Throughout this semester, we studied *Category Theory in Context* by Emily Riehl, and we were able to cover chapters one through four. Specifically, we studied 1.1. Abstract and concrete categories, 1.2. Duality, 1.3. Functoriality, 1.4. Naturality, 1.5. Equivalence of categories, 1.6. The art of the diagram chase, 1.7. The 2-category of categories, 2.1. Representable functors, 2.2. The Yoneda lemma, 2.3. Universal properties and universal elements, 2.4. The category of elements, 3.1. Limits and colimits as universal cones, 3.2. Limits in the category of sets, 3.3. Preservation, reflection, and creation of limits and colimits, 3.4. The representable nature of limits and colimits, 3.5. Complete and cocomplete categories, 3.6. Functoriality of limits and colimits, 4.1. Adjoint functors, and 4.2. The unit and counit as universal arrows. Usually, we covered a couple sections each week, and then we would meet every Friday and discuss what I had read and go off on tangents about related topics and Category Theory's uses in other areas of mathematics or sometimes Amogh's thesis topic dealing with Category Theory which I found very cool. In sum, we covered a very solid introduction to Category Theory that would be sufficient for most other unrelated fields of mathematics with a few random topics thrown in for fun. During this time, we were exposed to a wide variety of examples that related to a seemingly endless amount of topics— topology appeared in almost every context. Every section had a multitude of questions which I spent most of my time on after reading to practice the concepts I had learned. In all, I had an amazing experience and hopefully am able to do it again in the future!

Reference(s):

- Riehl, E.; **Category Theory in Context**. *Dover Publications, Inc., Mineola, New York*, 2016. 272 pp. ISBN: 0-48680-903-X.

Title: Category Theory

Student: Jason Weible

Mentor: Kyle Binder

Description: During this semester, I studied category theory out of the book *An Invitation to Applied Category Theory: Seven Sketches in Compositionality* by Brendan Fong and David I. Spivak. Starting out, the book discussed some definitions needed for understanding category theory such as Order, Meets, Joins, and Monoidal Preorders. Coming from a background in computer science, this was an uphill battle for me. However, by chapter 3, the book discussed an application of category theory to databases from computer science. This application of category theory was the highlight of my experience with the DRP this semester. Using category theory, one can mathematically determine if a data migration between two databases will work before conducting the migration. This aspect of category theory has me excited to continue learning category theory on my own. There are additional applications to category theory such as being able to determine if two wiring diagrams are the same or if two signal flow graphs are the same. Category theory seems to fit perfectly into the broad category of engineering system design, and in my opinion, should be taught in upper level engineering courses.

Reference(s):

- Fong, B.; Spivak, D. I.; **An invitation to applied category theory. Seven sketches in compositionality.** *Cambridge University Press, Cambridge*, 2019. xii+338 pp. ISBN: 978-1-108-71182-1; 978-1-108-48229-5.

Title: A key lemma from Morse Theory

Student: John White

Mentor: Matthew Carr

Description: We spent the first part of the semester working our way through Hirsch's *Differential Topology*. Subjects discussed include what Morse functions are, how we can topologize smooth maps between spaces, and transversality. In the second part of the semester, we read Audin and Damian's *Morse Theory and Floer Homology*. We covered the topic in more detail, including Morse homology. By the end of the semester, we had to read about Symplectic Geometry. We had the tools to state and parse the Arnold Conjecture, which deals with periodic solutions of a certain differential equation. Floer homology was developed as a tool to prove this conjecture. It involves taking the space of loops on a manifold, and treating it as an infinite dimensional manifold.

Reference(s):

- Audin, M.; Damian, M.; **Morse theory and Floer homology.** Translated from the 2010 French original by Reinie Erné. Universitext. *Springer, London; EDP Sciences, Les Ulis*, 2014. xiv+596 pp. ISBN: 978-1-4471-5495-2; 978-1-4471-5496-9; 978-2-7598-0704-8.
- Hirsch, M. W.; **Differential topology.** Corrected reprint of the 1976 original. Graduate Texts in Mathematics, 33. *Springer-Verlag, New York*, 1994. x+222 pp. ISBN: 0-387-90148-5

Title: The Metropolis-Hastings Algorithm

Student: John Wright

Mentor: Paul Duncan

Description: This semester, we covered a small range of topics from probability theory. We began by studying the measure-theoretic formulations of foundational concepts in probability, focusing on the first chapter of Ross and Peköz' textbook. We went on to cover further topics including conditional expectation, Markov chains, and coupling. For the end-of-semester talk, we focused on the Metropolis-Hastings algorithm for Markov Chain Monte Carlo simulation, also covering alternative methods for simulating random variables whose distributions are not fully known.

Reference(s):

- Ross, S. M.; Pekoz, E. A.; **A Second Course in Probability**. *ProbabilityBookstore.com, Boston, MA*, 2007. ISBN: 0-9795704-0-9.

Title: The Five Color Theorem and the related Four Color Theorem

Student: Junyu Yuan

Mentor: Baian Liu

Description: I mainly studied topology and graph theory this semester. Point-set topology is, simply put, a subject that studies the remaining properties of geometry without metrics via the notion of "open set" and relations between points, in something called a "topological space". The applications of this I studied on Topological Graph Theory on the Five-Color Theorem and the Four-Color Law.

Reference(s):

- Gross, J. L.; Tucker, T. W.; **Topological graph theory.** Reprint of the 1987 original with a new preface and supplementary bibliography. *Dover Publications, Inc., Mineola, NY*, 2001. xvi+361 pp. ISBN: 0-486-41741-7.

Title: Uniform Distribution Modulo 1

Student: Kabir Belgikar

Mentor: Bhawesh Mishra

Description: Uniform distribution of sequences plays a central role in Ergodic theory and ensuing number theory results. The notion of a uniformly distributed sequence is intuitive and natural. While some elementary results have suggestive proofs, others (the difference theorems, for example) require rigorous analysis. An important objective of this project was to go over proofs of theorems like this. We went over the proof of van der Corput's difference trick and the difference theorem discussed at the beginning of the semester. Later, we also covered some metric theorems, well-distribution, the basics of discrepancy, and the basics of uniform distribution in compact spaces. During our meetings, we usually discussed exercises from the book. At one point, we went over the beginning of a paper by Dr. Bergelson and discussed the results.

Reference(s):

- Kuipers, L.; Niederreiter, H.; **Uniform Distribution of Sequences.** *Dover Publications, Inc., Mineola, NY*, 2006.

Title: Methods for statistical evaluation of data

Student: Matthew Chilinski

Mentor: Benjamin Call

Description: Over the course of the spring 2022 semester, my mentor Ben and I read through *The Elements of Statistical Learning*, a book by Jerome H. Friedman, Robert Tibshirani, and Trevor Hastie. The main concepts we sought to learn more about were statistical analysis; what kind of methods exist that can allow us to easily predict future data by using sample, or test data? The section of the book that we worked through primarily focused on the least-squares method and K-nearest neighbors. These methods take test data and derive models for which we can predict future data, with the goal being to minimize the residual/error of the simulated model. We also looked at specific examples related to these different models of statistical analysis, as well as establishing some specific terminology omnipresent in statistics. Garnering a deep understanding of how these formulas were derived, the different cases in which they would be useful, and the different possible applications for these methods, I hope to use this knowledge to establish the foundations of a career in data analytics, ideally in esports analytics in the far future.

Reference(s):

- Hastie, T.; Tibshirani, R.; Friedman, J.; **The elements of statistical learning. Data mining, inference, and prediction.** Second edition. Springer Series in Statistics. *Springer, New York, 2009*. xxii+745 pp. ISBN: 978-0-387-84857-0.

Title: Essentials of Stochastic Processes

Student: Nuofan Tian

Mentor: Zihao Fang

Description: *Essentials of stochastic processes* by Rick Durrett is a good book we read for this semester. I learned many definitions, interesting examples and famous theorems from this book. For example, the definition of Markov property, transition matrix, transition probability, closed matrix, irreducible matrix, recurrent, stationary distribution and so on. These definitions are not only important for proving the theorems such as convergence theorem, but also very useful in my study in statistics. From this book, I learned how to have mental pictures of the matrix representing steps and the probabilities inside the matrix. It has detailed examples related to our daily lives, such as machine fixing and gambling rules. I enjoyed spending the time reading the book and, although some part of the problems solving or proving procedures are challenging, with the help from Zihao I fully understood the concepts, the special symbols and some useful math skills. Overall, I learned a lot from this book and how to collaborate with our graduate students, it was a valuable time and I believe the knowledge I learned is useful for my future study.

Reference(s):

- Durrett, R.; **Essentials of stochastic processes.** Third edition. Springer Texts in Statistics. *Springer, Cham, 2016*. ix+275 pp. ISBN: 978-3-319-45613-3; 978-3-319-45614-0

Title: Background and proof of the Chinese Remainder Theorem

Student: Ruoke Zhang

Mentor: Zach Davis

Description: This semester, we have studied and discussed ring theory, which we contrasted with group theory, introducing a large number of basic concepts and properties. These properties are the basis for the study of algebraic structures such as mappings and isomorphisms. In our study of the Chinese Remainder Theorem, we discussed its application to integer addition rings. We learned aboaut the problem of a system of linear congruence equations in one element applying the Chinese remainder theorem first appeared in China in the 5th century AD. A systematic solution was given one thousand years ago. The theorem stipulates that if we know the remainder of an unknown divided by some "pairwise prime" numbers (i.e., the two do not have any common prime factors), we can find the unknown itself. We think that the proof of this theorem on rings of integers is very interesting and reflects a number of features of exchange rings.

Reference(s):

- Lang, S.; **Algebra**. Revised third edition. Graduate Texts in Mathematics, 211. *Springer-Verlag, New York*, 2002. xvi+914 pp. ISBN: 0-387-95385-X

Title: Topological Data Analysis and Football

Student: Rylan Roberts

Mentor: Ling Zhou

Description: This presentation includes a brief introduction to computational topology from simplices and simplicial complexes all the way to filtrations and gives a taste of its applications in Topological Data Analysis (TDA), and in particular, how algebraic and computational topology can be used to view data in football.

Reference(s):

- Edelsbrunner, H.; Harer, J. L.; **Computational topology. An introduction.** *American Mathematical Society, Providence, RI*, 2010. xii+241 pp. ISBN: 978-0-8218-4925-5.

Title: Why Measure Theory?

Student: Sanjay Janardhan

Mentor: Adam Christopherson

Description: This semester, we studied the background and reasons for why measure theory is so prevalent today. We started with studying the Riemann integral from an introductory analysis perspective. We then looked at the Dirichlet function, and showed that it is not Riemann-integrable, thus showing that the Riemann integral falls short. Moving on to *Rudin's Real and Complex Analysis*, we learned about the concepts of measurable and topological spaces. After studying a few definitions, we delved into the properties of measurable functions, and then introduced the concept of a measure. Using a specific type of measure, the Lebesgue Measure, we were able to easily construct a definition of an integral for a simple function. Using a sequence of simple functions to approximate a more general class of functions, we were able to define an integral, the Lebesgue Integral, for a more general class of functions. Using this integral, we are now able to find a solution to the integral of the Dirichlet function.

Reference(s):

- Rudin, W.; **Real and Complex Analysis.** Third edition. *McGraw-Hill Book Co., New York, 1987*. xiv+416 pp. ISBN: 0-07-054234-1.
- Rudin, W.; **Principles of mathematical analysis.** Third edition. International Series in Pure and Applied Mathematics. *McGraw-Hill Book Co., New York-Auckland-Düsseldorf*, 1976. x+342 pp.

Title: The Contracting Mapping Principle

Student: Tyler Pondel

Mentor: James Marshall Reber

Description: For the Directed Reading Program, I worked alongside James Marshall Reber as we read *A First Course in Dynamics by Hasselblatt and Katok*. This book covered a lot of topics on the basics of dynamics and how they apply to other fields of math. Specifically, we read through the first two chapters of this textbook which set up a lot of the framework for what one may want to study within dynamics. Chapter 1 focuses primarily on where one may want to study dynamics, specifically in nature and why it is interesting. Chapter 2 focuses primarily on Contracting Maps and their applications, specifically in showing fixed points exist which can be extended to prove various other theorems. Then, I looked at where we can use dynamics, specifically in scenarios such as using logistic maps or maps on circles and tori. I also studied metric spaces where we are able to study systems with different notions of distance.

Reference(s):

- Hasselblatt, B.; Katok, A.; **A first course in dynamics. With a panorama of recent developments**. *Cambridge University Press, New York*, 2003. x+424 pp. ISBN: 0-521-58304-7; 0-521-58750-6.

Title: Matrix Lie groups and Lie algebras: An Introduction

Student: Wo Wu

Mentor: Pan Yan

Description: This semester we went through the first three chapters of B. Hall's book *Lie Groups, Lie Algebras, and Representations*. In the first chapter, we studied matrix Lie groups and their homomorphisms, followed by a series of examples of matrix Lie groups. The example part includes general and special linear groups, unitary and orthogonal groups, symplectic groups, etc. It also includes a few topological properties. In the second chapter, we studied the matrix exponential and matrix logarithm, which provide a connection between Lie groups and Lie algebras. In the third chapter, we studied the abstract notion of a Lie algebra and various examples of Lie algebras associated to the matrix Lie groups from the first chapter. Finally we studied the theorem which says that a Lie group homomorphism between two Lie groups gives rise in a natural way to a map between the corresponding Lie algebras.

Reference(s):

- Hall, B.; **Lie groups, Lie algebras, and representations. An elementary introduction.** Second edition. Graduate Texts in Mathematics, 222. *Springer, Cham, 2015*. xiv+449 pp. ISBN: 978-3-319-13466-6; 978-3-319-13467-3.

FALL 2022

Title: Factorization and the Rho Method

Student: Aaron Yoder

Mentor: Bhawesh Mishra

Description: During the autumn 2022 semester, we chose to read the book *A Course in Number Theory and Cryptography* by Neal Koblitz with my mentor Bhawesh Mishra. The first two chapters of the book covered some basic number theory and set the stage for its application in the later chapters. In the next two chapters, we studied some classical ciphers such as *Caesar Cipher* and *Vigenère Cipher*, as well as public-key cryptography and zero-knowledge proofs.

Chapter five of the book was on primality and factoring, and the most intense part of our study. After studying some classical primality-tests and factorization methods, we got to the *rho-method* of factorization, which ultimately was the focus of my final presentation. The rho-method of factorization utilizes the number-theoretic techniques from the first two chapters to cleverly factorize large integers n with impressive accuracy and speed. We learned several novel achievements as well as some limitations of this method of factorization. For instance, we learned that a large integer n can be factored at worst in $\log^3(n)n^{1/4}$ operations and the probability of succeeding in factorization with a random choice of starting parameter is extremely close to 1.

Reference(s):

- Koblitz, N.; **A course in number theory and cryptography.** Second edition. Graduate Texts in Mathematics, 114. *Springer-Verlag, New York*, 1994. x+235 pp. ISBN: 0-387-94293-9.

Title: Fermat's Dream: Fermat's Last Theorem for regular primes

Student: Alina Li & Wo Wu

Mentor: Stefan Nikoloski

Description: This semester, we studied *Number Fields* by Daniel Marcus and *A Classical Introduction to Modern Number Theory* by Ireland and Rosen. During the first half of the semester, we studied the first four chapters of Marcus's *Number Fields*, especially number rings, unique factorization of ideals in Dedekind domains, ideal class groups, and splitting of primes in extensions. During the second half of the semester, we studied the proof of Fermat's Last Theorem in the case of regular primes. The proof is adapted from §11 on Chapter 17 of *A Classical Introduction to Modern Number Theory*.

Reference(s):

- Ireland, K.; Rosen, M.; **A classical introduction to modern number theory.** Second edition. Graduate Texts in Mathematics, 84. *Springer-Verlag, New York*, 1990. xiv+389 pp. ISBN: 0-387-97329-X.
- Marcus, D. A.; **Number fields.** Second edition. With a foreword by Barry Mazur. *Universitext. Springer, Cham*, 2018. xviii+203 pp. ISBN: 978-3-319-90232-6.

Title: The Birth of Analytic Number Theory – How *L*-functions arose from counting primes

Student: Christine Lyu & Randall Rickel

Mentor: Shifan Zhao

Description: For our project, we chose to read *Introduction to Analytic Number Theory* by T.M. Apostol. In this project, we sought to understand a proof of Dirichlet's theorem in arithmetical progressions and connect to broader concepts in analytic number theory. Dirichlet's theorem states there are infinitely many prime numbers in an arithmetic progression where the common difference and the initial term are coprime. To understand this, we began by reviewing properties of integers and introducing arithmetic functions, such as Euler's totient function and the Möbius function. Then, we covered topics in analysis such as summation techniques, notably Euler's summation formula, and applied them to arithmetic functions. We introduced the topic of characters from group theory, more specifically Dirichlet characters. Finally, we incorporated all of these topics to understand a proof of Dirichlet's theorem in arithmetical progressions, focusing on Dirichlet L-functions. Dirichlet L-functions initiated further study into leading problems and conjectures in analytic number theory. Some of these topics include Siegel zeros and the Generalized Riemann Hypothesis Siegel zeros are real zeros of Dirichlet L-function very close to 1. Generalized Riemann Hypothesis conjectures that all non-trivial zeros of Dirichlet's L-function lie on the critical line.

Reference(s):

- Apostol, T. M.; **Introduction to analytic number theory.** Undergraduate Texts in Mathematics. *Springer-Verlag, New York-Heidelberg, 1976.* xii+338 pp.

Title: Exploring Complex Systems: Analytic Methodologies and Cross-Disciplinary Applications of Network Theory

Student: Daniel Tcheurekdjian

Mentor: Amogh Parab

Description: Being networked is a fundamental characteristic of complex systems, and network theory is the study of that central unity.

Lying at the intersection of pure mathematics and empirical discovery, network theory develops analytic methodologies to analyze complex systems, using graphs as representations of symmetric and asymmetric relationships between discrete objects, while representing that graph through matrices.

As we read through our chosen text, we developed an understanding of network theory from the basis of graph theory and linear algebra: how to apply adjacency matrices, spectral analyses, graph invariants, Gershgorin's theorem, Perron-Frobenius theorem, graph Laplacians, the Fiedler vector, Degree Distributions, Clustering Coefficients and Random Models of Networks, and Spectral Node Centrality to derive semantic meaning from networked systems.

We finally used those tools to derive network-based, rigorous models of natural phenomena, from the motion of a harmonic oscillator to the construction of the fundamental equations of Quantum Field Theory.

Reference(s):

- Estrada, E.; Knight, P. A.; **A first course in network theory**. *Oxford University Press*, 2015. 254 pp.

Title: On the Poincaré Group

Student: Harrison Blake

Mentor: Makana Silva

Description: My DRP program was about the Poincaré group, which essentially derives why the angular momentum of particles is discrete from fundamental symmetries in nature. Using group theory, Lie group theory, and representation theory, I derived that since the Poincaré group expresses several fundamental symmetries of nature, the labels of its irreducible representations give us many of the properties of fundamental particles, such as mass, charge, and angular momentum. Not only do these labels express that angular momentum for particles are discrete, but it also gives us their magnitudes. This fact affects every part of physics from the inception of quantum mechanics up to the modern day.

I learned many aspects of mathematics that will aid me in my journey to becoming a theoretical physicist. For solving problems in high energy physics and quantum field theory, the group theoretical approach is crucial due to its ability to make general statements about symmetry untethered to physical problems. I will be using this widely in my future, and I am thankful I had a space to practice early on in my career.

Reference(s):

- Schwichtenberg, J.; **Physics from symmetry.** Second edition. Undergraduate Lecture Notes in Physics. *Springer, Cham, 2018*. xxi+287 pp. ISBN: 978-3-319-66631-0.

- Jeevanjee, N.; **An introduction to tensors and group theory for physicists.** Second edition. *Springer, Cham, 2015*. xvi+305 pp. ISBN: 978-3-319-14794-9.

Title: An Introduction to Algebraic Curves

Student: Jacob Monzel

Mentor: Deniz Genlik

Description: We worked on the first three chapters of Fulton's famous *Algebraic Curves* book. The first chapter is mainly the preparational chapter for the algebraic tools that we will need to use later on and fundamental definitions. Hilbert's Basis Theorem and Hilbert's *Nullstellensatz* are the two important theorems that we covered in the first chapter. The second chapter is the place where interaction between algebra and geometry comes into play. These include coordinate changes, maps of algebraic varieties, functions over algebraic varieties, and forms. The last chapter we read is the third chapter which is an introduction to the realm of algebraic geometry; namely, algebraic plane curves.

Reference(s):

- Fulton, W.; **Algebraic curves.** An introduction to algebraic geometry. Notes written with the collaboration of Richard Weiss. Reprint of 1969 original. Advanced Book Classics. *Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA*, 1989. xxii+226 pp. ISBN: 0-201-51010-3.

Title: A Glimpse of p-adic Analysis

Student: Kabir Belgikar

Mentor: Jake Huryn

Description: The p-adics numbers are an exotic object that is a meeting ground for analysis and algebra (to be fair, many things are). Additionally, they are also quite ubiquitous in number theory. That made them an attractive topic of study for both of us, especially since we'd already done some algebraic number theory beforehand.

The construction of $\mathbb{Q}_p$, the p-adic numbers, is straightforward enough. Indeed, it is natural once one sees the construction of $\mathbb{R}$. After an exploration of its (frankly nonsensical) topology and algebraic structure, we learned the requisite linear algebra to prepare for deeper waters.

Apparently, some mathematicians thought that $\mathbb{Q}_p$ was inadequate. In the same way that $\mathbb{C}$ extends $\mathbb{R}$, they sought a complete, algebraically closed field Ω_p that extends $\mathbb{Q}_p$. Most of the semester was devoted to studying its construction. That required dealing with finite extensions of $\mathbb{Q}_p$, which connected nicely with some of the algebraic number theory we'd done.

In the last few weeks, we ended up doing some analysis in Ω_p in a desperate attempt to justify the title of our project. To finish the semester, we read about zeta functions and their connections to p-adic analysis.

Reference(s):

- Koblitz, N.; **p-adic numbers, p-adic analysis, and zeta-functions.** Second edition. Graduate Texts in Mathematics, 58. *Springer-Verlag, New York*, 1984. xii+150 pp. ISBN: 0-387-96017-1.

Title: Symbolic dynamics and Shannon's theory of communication

Student: Lena Zhang & Tong Du

Mentor: Caleb Dilsavor

Description: We studied Lind and Marcus' *An Introduction to Symbolic Dynamics and Coding*, with the goal of applying concepts there to understand Shannon's theory of communication. Shannon's paper studies a setup in which an information source communicates via a *channel*, which is some device which can relay symbols to a receiver, possibly with restrictions on which sequences of symbols can be used by the channel. On the other hand, Lind and Marcus' object of study is something called a *shift space*, which is a certain kind of collection of bi-infinite sequences of symbols. The connection between these two is that a channel gives rise to a shift space by defining the shift space as the collection of all possible bi-infinite sequences which can be output by the channel over a bi-infinite timeline. Under this identification, the term 'capacity' from Shannon's paper, which describes the maximum information rate at which a channel can transmit, is exactly the 'entropy' (or 'topological entropy') of the shift space as defined in Lind and Marcus.

Reference(s):

- Lind, D.; Marcus, B.; **An introduction to symbolic dynamics and coding.** Second edition. Cambridge Mathematical Library. *Cambridge University Press, Cambridge, 2021*. xix+550 pp. ISBN: 978-1-108-82028-8.

Title: Smooth manifolds and the Frobenius theorem

Student: Pallav Pant

Mentor: Ivo Terek

Description: The goal of this reading project was to start learning about differential geometry. As an introduction to manifolds, we covered the first two chapters from the book. In particular, we also covered tangent spaces, vector fields and flows, Lie derivatives, vector bundles, and distributions, all of which built towards the Frobenius Theorem, which was the topic of the presentation. Over the course of the project we also covered some basic topology and advanced linear algebra to complement the material from the textbook. Overall I was exposed to a lot of new mathematics and I think I learned a lot during this semester.

Reference(s):

- Robbin, J. W.; Salamon, D. A.; **Introduction to differential geometry.** Springer Studium Mathematik—Master. *Springer Spektrum, Wiesbaden*, 2022. xiii+418 pp. ISBN: 978-3-662-64339-6.

Title: Mathematical models in nature: the SIR model and Lotka-Volterra's competition model

Student: Ryan Hardig & Yuchen Zhao

Mentor: James Marshall Reber

Description: Our group covered the book Dynamic Data Analysis, which is data modeling with differential equations. We mainly focused on linear differential equations and systems, numerical solutions such as Runge–Kutta method, and qualitative behaviors. With those tools, we practiced on SIR model and competitive Lotka–Volterra equations, which consist the real-world applications. The study of the SIR model is short for susceptible, infected, and recovered modeling. It is an excellent model in the field of epidemiology. The competitive Lotka–Volterra equations modeling allows us to visualize the potential outcome of the competition from two or more sides over limited resources. It is a great model for ecologists to track animals' dynamic populations.

Reference(s):

- Ramsay, J.; Hooker, G.; **Dynamic data analysis.** Modeling data with differential equations. Springer Series in Statistics. *Springer, New York, 2017*. xvii+230 pp. ISBN: 978-1-4939-7188-6.
- Strogatz, S. H.; **Nonlinear dynamics and chaos**: With Applications To Physics, Biology, Chemistry, and Engineering. Second edition. *CRC Press*, 2015. 528 pp. ISBN: 978-0-81334-910-7.

Title: Statistical Learning

Student: Vyom Dave

Mentor: Mario Gómez

Description: Throughout the semester, we covered the first 6 chapters of *An Introduction to Statistical Learning*. We started by learning some basic definitions/concepts such as the Bias and Variance trade-off and the two types of machine learning. Supervised learning contains data that is already labeled, while Unsupervised learning requires patterns to be recognized from unlabeled data. Simple Linear Regression (SLR) is an example of a method used on supervised datasets. It attempts to quantify the linear relationship between two continuous variables by estimating the following coefficients: β_1 and β_0. Where β_1 is the "slope" and β_0 is the "intercept" of the line. Values such as the Residual Sum Error (RSE) and the R^2 statistic can be used to evaluate this estimation. RSE estimates the standard deviation of the error term in the linear approximation and the R^2 value estimates of how well the data fits the line. Moreover, Multiple Linear Regression can be used when there are multiple predictors of the dependent value. This is a direct extension of SLR.

Additionally, Logistic Regression is used when the data is Qualitative rather than Quantitative. We do not use a linear model to avoid obtaining probabilities greater than 1 and below 0. Instead, we use a Logistic Function that outputs values in the interval $[0, 1]$. Logistic regression classifies the observation into one of 2 classes. The predictor belongs to the class for which the Logistic Function outputs the highest probability. Through, Multinomial Logistic Regression, it can be extended to classify a predictor X into one of K classes.

Lastly, we also explored Linear Discriminant Analysis (LDA) with 1 predictor, as well as multiple predictors. Like Logistic Regression, LDA is also a model for classification problems. However, it is different because instead of directly calculating the probabilities, it uses Bayes Theorem to obtain its estimates. Note: For LDA it is assumed that the distribution of the predictors X is approximately normal.

With all these approaches, one question persists. How can we go about validating our estimates? Multiple Cross-Validation methods can be used to validate our estimates. One such technique is called the "Validation Set Approach". This approach is performed by randomly separating the datasets into 2 parts: a training set and a validation set. The model is fit on the training set and the fitted model is used to predict the responses for the observations in the validation set. Then, the set error rate (calculated through MSE) can be used as an approximation for the test error rate.

Reference(s):

- James, G.; Witten, D.; Hastie, T.; Tibshirani, R.; **An introduction to statistical learning—with applications in R**. Second edition. Springer Texts in Statistics. *Springer, New York, 2021*. xv+607 pp. ISBN: 978-1-0716-1417-4.

Title: Measure, Integration and Dominated Convergence Theorem

Student: Yumin Shen

Mentor: Thomas O'Hare

Description: This semester, we have thoroughly studied chapters 0 to 2 of the selected book. We studied the abstract measure theory, from the most fundamental notion of σ-algebra to measure, outer measure, Caratheodory's Theorem, to Lebesgue measure, then we studied measurable functions, constructed product measures, and understood the proof of the Fubini-Tonelli theorem. We also didn't just follow the textbook, but also discussed some details about the usage of these theorems, and some topics in the functional analysis like Riesz representation and introductory L^p spaces.

Reference(s):

- Folland, G. B. **Real analysis. Modern techniques and their applications.** Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. *John Wiley & Sons, Inc., New York*, 1999. xvi+386 pp. ISBN: 0-471-31716-0.

SPRING 2023

Title: What is the Role of Intuition in Mathematics?

Student: Anton Fediukov

Mentor: Amogh Parab

Description: This semester, I've read about set theory, and how we can define all familiar notions of mathematics (such as functions, ordered pairs, and even numbers themselves) as sets. Essentially, by assuming only a few basic "axioms", which are logical statements we take for granted, we can rigorously define all concepts we wish to use in a meaningful way, leaving out any potential ambiguity. I was particularly interested in the construction of the natural numbers as sets, and how "mathematical induction" is the foundation for the natural numbers, rather than a consequence of them. This also led me to researching two alternative definitions to a finite/infinite set.

Reference(s):

- Stoll, R. R.; **Set theory and logic.** Corrected reprint of the 1963 edition. *Dover Publications, Inc., New York, 1979.* xiv+474 pp. ISBN: 0-486-63829-4.

Title: Set Theory and Logic

Student: Cherish Brumley

Mentor: Amogh Parab

Description: The book used this semester was Robert R. Stoll's Set Theory and Logic. Chapters that were studied included 1, 2, and 3, which covered introductions to sets and relations, the natural number sequence, and the extension of the naturals to the reals, in that order. We began with chapter 1 before quickly moving to chapters 2 and 3. The focus from this point on was learning to derive the reals from the naturals, which requires one to first define the naturals, then derive integers, then derive rationals, before finally reaching the reals. Overall, I learned that the reals are an equivalence class of rationals and became more familiar with properties and proofs of the reals through studying Stoll's book.

Reference(s):

- Ornes, S.; **Math Art: Truth, Beauty, and Equations.** *Union Square & Co., NY, 2019.* 208 pp. ISBN: 1-4549-3044-6.
- Stoll, R. R.; **Set theory and logic.** Corrected reprint of the 1963 edition. *Dover Publications, Inc., New York, 1979.* xiv+474 pp. ISBN: 0-486-63829-4.

Title: Riemann Surfaces and Complex Analysis

Student: Christine Lyu

Mentor: Stefan Nikoloski

Description: This semester, Stefan and I read the first three chapters of *Algebraic Curves and Riemann Surfaces* by Rick Miranda. In the first chapter, the book introduced the definition of a Riemann Surface (it's a one-dimensional complex manifold: if you zoom in enough, it will look like the complex plane) via complex charts and compatibility between them. Then the book gives examples of Riemann surfaces and charts defining it, such as the complex plane, complex tori, the Riemann sphere, and smooth affine plane curves. We learnt about how to do standard complex analysis on Riemann surfaces via charts, and we have the definition of holomorphic and meromorphic functions on Riemann surfaces. We have also covered Hurwitz's formula to compute genuses, which links complex analysis and topology together.

Reference(s):

- Miranda, R.; **Algebraic curves and Riemann surfaces.** Graduate Studies in Mathematics, 5. *American Mathematical Society, Providence, RI, 1995.* xxii+390 pp. ISBN: 0-8218-0268-2.

Title: Sturm's Theorem

Student: Gabriel Black

Mentor: Min Shi

Description: We began the semester with no specific theorem in mind to focus on proving. Rather, we were looking at various concepts from number theory and abstract algebra. This Included topics such as, the uniqueness and existence of a prime factorization of Principal Ideal Domains and a proof of the division algorithm over a ring of polynomials. Just over a month before the semester was going to end, we shifted our focus to looking at Sturm's Theorem and its proof. After we had proved it, we had various questions about the different applications of the theorem. We each began to think of several practical uses of the theorem. Some of these were its use in proving polynomial non-negativity for one variable polynomials and a root finding algorithm that is based on the theorem.

Reference(s):

- Jacobson, N.; **Basic algebra. I.** Second edition. *W. H. Freeman and Company, New York, 1985.* xviii+499 pp. ISBN: 0-7167-1480-9.

Title: Matroids and the Greedy Algorithm

Student: Gowrav Mannem

Mentor: Kyle Binder

Description: Over the SP23 term, my mentor and I explored Chapter 13 of B. Korte's and J. Vygen's book *Combinatorial Optimization*. We started off by understanding what an independent system is and the various applications of independent systems. We then refined our focus to Matroids, a special type of independent systems, and the common representations of matroids on finite sets, matrices, undirected graphs, and digraphs. The exploration of the interesting properties and theorems associated with matroids allowed us to better understand them. This foundational knowledge paid off when we studied the greedy algorithm, matroid intersections, matroid partitioning, and weighted matroid intersections. We frequently discussed how these concepts applied to real-world optimization problems.

During the later part of the semester, we explored linear programming and the simplex algorithm as a tangent from our main focus of Matroids. Overall, I expanded my knowledge of matroids and specifically the use of them within the Greedy Algorithm. I was amazed to see the role Combinatorics played in commonly used optimization problems within computer science.

Reference(s):

- Korte, B.; Vygen, J.; **Combinatorial optimization. Theory and algorithms.** Sixth edition. Algorithms and Combinatorics, 21. *Springer, Berlin, 2018.* xxi+698 pp. ISBN: 978-3-662-56038-9.

Title: Four Polynomials: Motivations for Combinatorial Reciprocity Theorems

Student: Haochen Wang

Mentor: Yang Yang

Description: In this past semester, our focus was on the book *Combinatorial Reciprocity Theorems* by Matthias Beck and Raman Sanyal. We conducted a thorough analysis of four distinct polynomials, each with combinatorial significance when evaluated using reciprocity. Specifically, we examined chromatic polynomials, which are used to count graph colorings, flow polynomials, which count flows on graphs, as well as order polynomials and Ehrhart polynomials. In addition to these polynomials, we also acquired the necessary background knowledge required to comprehend their intricacies. These polynomials serve as examples and motivations for further studies in this field.

Reference(s):

- Beck, M.; Sanyal, R.; **Combinatorial reciprocity theorems. An invitation to enumerative geometric combinatorics.** Graduate Studies in Mathematics, 195. *American Mathematical Society, Providence, RI, 2018.* xiv+308 pp. ISBN: 978-1-4704-2200-4.

Title: Wiener Process for Stock Price

Student: Haoran Ma & Shuo Chen

Mentor: Michael Tychonievich

Description: In this program, the book we read is *Options, Futures, and Other Derivatives* by John C. Hull. We firstly learned several basic financial concepts like options, hedges, arbitrage, forwards and futures contracts. After have a briefly understanding about finance and market, we did the project on simulating a stock price by using Wiener process.

Wiener process is a type of Markov stochastic process with a mean change of zero and a variance rate of 1.0 per year. We can plot the graph of Wiener process by formula $\Delta z = \varepsilon\sqrt{\Delta t}$, where Δz is the change of a variable z following a Wiener process, and Δt is a small period of time. Generalized Wiener process for a variable x can be defined in terms of $\mathrm{d}z$ as $\mathrm{d}x = a\,\mathrm{d}t + b\,\mathrm{d}z$, where a, b are constants and $\mathrm{d}z$ is basic Wiener. In this equation, $a\,\mathrm{d}t$ represents the amount of variable x increases in a period time; $b\,\mathrm{d}z$

can be considered as adding variability to the path followed by x. To have a better understanding on the formula, we used MATLAB programming to simulate the Wiener process.

Then we began to work on using Wiener process to simulate the process for a stock process. Since a stock price follows a generalized Wiener process, we are able to get the formula about change of stock price, S: $\Delta S = \mu S \, dt + \sigma S \, dt$. μ is the stock's expected rate of return, and we use the risk-free rate for the rise-neutral measure. σ is the volatility of the stock price. By plugging numbers in the MATLAB programming that we built before, we got the simulation of the stock price. Furthermore, we also computed call option price in our model. Comparing with result from Black-Scholes-Merton model, it shown that our estimated price is close to the BSM price.

Reference(s):

- Hull, J.; Sankarshan, B.; **Options, Futures, and Other Derivatives.** 11th edition. Pearson, 2022. 900 p.p. ISBN: 0-1369-3997-X.

Title: Recovering a function in $\mathbb{R}^3$ from its Radon Transform

Student: Isaac Shellabarger

Mentor: Nicholas Castillo

Description: This semester we began by studying the basics of Fourier Analysis and the Fourier Transform. Once we had covered the ideas central to this topic we decided to investigate one application of this topic, the Radon Transform. The Radon Transform is an application of the Fourier Transform that allows one to recover a function in $\mathbb{R}^3$ from a set of averages. This is used in everyday applications of X-ray technology, as it allows one to reconstruct the image from collected X-ray data.

Reference(s):

- Stein, E. M.; Shakarchi, R.; **Fourier analysis. An introduction.** Princeton Lectures in Analysis, 1. *Princeton University Press, Princeton, NJ, 2003.* xvi+311 pp. ISBN: 0-691-11384-X.

Title: Counting Rational Plane Curves

Student: Jacob Monzel

Mentor: Deniz Genlik

Description: We studied the first 5 chapters of *Enumerative Geometry and String Theory* book by Sheldon Katz. The first three chapters are a soft introduction to enumerative geometry. They give an idea of what modern enumerative geometry is by introducing certain algebraic varieties and counting certain algebraic objects in these varieties. Chapter 4 is an introduction to topological spaces and manifolds. Chapter 5 is a crash course on cohomology of manifolds. After these five chapters we started reading certain sections of *An Invitation to Quantum Cohomology* book by Israel Vainsencher and Joachim Kock. Here we focused on counting rational curves on projective plane.

Reference(s):

- Katz, S.; **Enumerative geometry and string theory.** Student Mathematical Library, 32. IAS/Park City Mathematical Subseries. *American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 2006.* xiv+206 pp. ISBN: 0-8218-3687-0.

- Kock, J.; Vainsencher, I.; **An invitation to quantum cohomology.** Kontsevich's formula for rational plane curves. *Progress in Mathematics, 249. Birkhäuser Boston, Inc., Boston, MA, 2007.* xiv+159 pp. ISBN: 978-0-8176-4456-7

Title: Exploring Machine Learning

Student: Mark Kikta & Umar Jara

Mentor: Aziz Burak Gülen

Description: During the semester, we studied the theory and use of contemporary machine learning (ML) algorithms. *Hands-On Machine Learning with Scikit-Learn, Keras & TensorFlow* introduced techniques relevant to industry application such as data preparation, feature engineering, and validation. The book then covers some theoretical background of several classical but powerful ML models including Logistic Regression, Support Vector Machines (SVM), and Decision Trees. Along with theory, we learned how and when to use each model under different circumstances. We applied what we learned by fitting SVM and Random Forest models on white wine datasets to predict their quality (Cortez et al., 2009). The input features of the dataset were 11 different physicochemical tests such as pH, total sulfur oxide, density, etc. The goal of the task was to accurately predict the quality of wine samples on a 1-10 scale. Using SVM and Random Forest models, we achieved 69% prediction accuracy – beating the benchmark of 64.6% from Cortez et al.

Reference(s):

- Géron, A.; **Hands-On Machine Learning with Scikit-Learn, Keras, and TensorFlow: Concepts, Tools, and Techniques to Build Intelligent Systems**, 3rd Edition. *O'Reilly Media, CA, 2022.* xv+850 pp. ISBN: 1-098-12597-5.

- Cortez, P., Cerdeira, A., Almeida, F., Matos, T., & Reis, J. (2009). **Modeling wine preferences by data mining from physicochemical properties.** *Decision Support Systems*, *47(4)*, 547-553. https://doi.org/10.1016/j.dss.2009.05.016

Title: Poker Over the Phone: An Example of Mathematical Cryptography.

Student: Nathan Howe

Mentor: Jonathan Hales

Description: This semester, we started by building a foundation in the number theory that is used in encryption techniques. Then, we examined various encryption cryptosystems such as affine ciphers, the Vigenère cipher, one-time pads, and RSA. We examined how each system secures information, the amount of security that each provides, and the attacks that are used against them. Finally, we explored additional information sharing problems such as how to divide a key among multiple people where not all are needed to recover the information in secret sharing schemes and how to reveal that you have a key to a cryptosystem without revealing what the key is in zero-knowledge techniques.

Reference(s):

- Trappe, W.; Washington, L. C.; **Introduction to cryptography with coding theory.** Second edition. *Pearson Prentice Hall, Upper Saddle River, NJ, 2006.* xiv+577 pp. ISBN: 0-13-186239-1.

Title: The Weierstrass Approximation Theorem via Bernstein Polynomials

Student: Nigel Krekeler

Mentor: Michael Lane

Description: As a summary of what I worked on this semester, I learned some foundations for functional analysis using Dzung Ming Ha's Functional Analysis: A Gentle Introduction. In particularly, I learned about continuity, convergence, boundedness, compactness, etc. There was also much discussion on metric spaces and topological spaces. I also learned about the Stone-Weierstrass approximation theorem. I used Bersntein polynomials to prove the Weierstrass approximation theorem. Overall, this semester has taught me about some important ideas in analysis and encouraged further interest into analysis.

Reference(s):

- Ha, D. M.; **Functional analysis. Vol. 1: a gentle introduction.** *Matrix Editions, Ithaca, NY, 2006.* xvi+640 pp. ISBN: 978-0-9715766-1-2.
- Rudin, W.; **Principles of mathematical analysis.** Third edition. International Series in Pure and Applied Mathematics. *McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976.* x+342 pp.

Title: The Schwarzschild metric and Birkhoff's theorem

Student: Pallav Pant

Mentor: Ivo Terek

Description: For this project I studied differential geometry and general relativity. For the background we covered the first three chapters of Carroll's book, including manifolds, tensors, differential forms, covariant derivatives, parallel transport, and geodesics. We used Kühnel's textbook as a reference for some insight into curves and surfaces in Euclidean space and a more geometry-focused approach to the topics.

As the main objective, we studied the Schwarzschild metric and Birkhoff's Theorem. The Schwarzschild metric describes the geometry of spacetime in the vicinity of a charge-free and non-rotating black hole. Birkhoff's Theorem gives us a local isometry between a four-dimensional, spherically symmetric, and Ricci-flat spacetime and a Schwarzschild black hole.

Reference(s):

- Carroll, S.; **Spacetime and geometry. An introduction to general relativity.** Addison Wesley, San Francisco, CA, 2004. xiv+513 pp. ISBN: 0-8053-8732-3.
- Kühnel, W.; **Differential geometry. Curves—surfaces—manifolds.** Third edition. Student Mathematical Library, 77. *American Mathematical Society, Providence, RI, 2015.* xii+402 pp. ISBN: 978-1-4704-2320-9.

Title: The Invariant Subspace Problem

Student: Yumin Shen

Mentor: Thomas O'Hare

Description: We first studied Chapter 5 of Folland's *Real Analysis* as an introduction to functional analysis. Then, we used Rudin's *Functional Analysis* and studied some big theorems on Banach algebras, including some spectral theory. We proved Lomonosov's Invariant Subspace Theorem as an application of spectral theory for compact operators, and also covered some applications of functional analysis in dynamical systems.

Reference(s):

- Lax, P.; **Functional analysis.** Pure and Applied Mathematics (New York). *Wiley-Interscience (John Wiley & Sons), New York, 2002.* xx+580 pp. ISBN: 0-471-55604-1.
- Folland, G. B. **Real analysis. Modern techniques and their applications.** Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. *John Wiley & Sons, Inc., New York*, 1999. xvi+386 pp. ISBN: 0-471-31716-0.
- Rudin, W.; **Functional analysis.** Second edition. International Series in Pure and Applied Mathematics. *McGraw-Hill, Inc., New York, 1991.* xviii+424 pp. ISBN: 0-07-054236-8.

Title: Real Analysis and Baire's Category Theorem for the Real Line

Student: Zhengyang Liu & Qinuo Shi

Mentor: Isaac Brown

Description: This semester we studied *Understanding Analysis* by Stephen Abbott. We began with introductions to sequences and series in $\mathbb{R}$ along with some standard theorems such as the monotone convergence theorem, Bolzano-Weierstrass theorem, and Cauchy condensation test. Following this, we began discussing the basic topology of $\mathbb{R}$ including open and closed sets, compact sets, the density of $\mathbb{Q}$ in $\mathbb{R}$, and the Heine-Borel theorem. The culmination of this semester's DRP project was a study of F_σ sets and G_δ sets, leading to Baire's category theorem for the real line. Informally, this theorem states that if the real numbers are broken up into different sets, then we must have either many different sets (uncountably many), or at least one of our sets must have many elements in a "short" range of real values (dense in some subset of $\mathbb{R}$). To contrast this, in the case of $\mathbb{Q}$ we can break up the rationals into countably many singleton sets, so Baire's category theorem for the real line tells us that (informally speaking) the set of real numbers is "infinitely bigger" than the set of rationals.

Reference(s):

- Abbott, S.; **Understanding analysis.** Second edition. Undergraduate Texts in Mathematics. *Springer, New York, 2015.* xii+312 pp. ISBN: 978-1-4939-2711-1.

FALL 2023

Title: The Theory of Special Relativity

Student: Andrew Clark

Mentor: Zach Davis

Description: This semester we studied the Theory of Special Relativity using three references. In the beginning we worked with the familiar Galilean transformation to understand its properties and, more importantly, when it failed. We were then introduced to the Lorentz transformation as a more general description of shifting between inertial frames. This involved studying the fundamental effects of the loss of simultaneity, time dilation, and length contraction with exercises from texts by Resnick and Morin. Many of our meetings were focused on tackling some of the more challenging exercises. We also studied spacetime as defined by the Minkowski metric and visually represented one dimensional Lorentz transformations with Minkowski diagrams. Towards the end of the semester we studied rapidity and its simplification of successive Lorentz boosts. We ended by discussing 4-vectors, their invariance under Lorentz transformations, and some familiar results using their framework.

Reference(s):

- Gourgoulhon, É.; **Special Relativity in General Frames**, *Springer-Verlag Berlin Heidelberg*, 2013. 784 pp. ISBN: 978-3-642-37275-9.

Title: Probability, Concentrations, and Programming

Student: Avery Stevens

Mentor: Wenyan Luo

Description: For this project we studied the fundamentals of probability theory, some concentration inequalities, and a small amount of distribution classes. This began by studying George Casella, and Roger Berger's Statistical Inference, building the foundation for probability theory, specifically dealing with distributions and random variables. After building this foundation, we studied Roman Vershynin's High-Dimensional Probability, making it through the first two chapters of the book. These chapters covered many important inequalities, with a focus on concentration inequalities, as they are some of the main tools used when bridging between traditional probability theory and high dimensional probability. The last of our time was spent mainly on the applications of the concentration inequalities studied earlier. These inequalities were primarily focused on random algorithms, and bounding these algorithms.

Reference(s):

- Casella, G.; Berger, R.; **Statistical Inference**. Second edition. *Duxbury, California*, 2002. ix+649 pp. ISBN: 0-534-24312-6. 12, 47, 48. ISBN: 978-1-138-63409-1.

- Roman, V.; **High-Dimensional Probability: An Introduction with Applications in Data Science**. *Cambridge University Press, Cambridge*, 2018. xiii+281 pp. ISBN: 978-1-108-41519-4.

Title: Visual Beauty and Chip-Firing

Student: Cherish Brumley

Mentor: Kyle Binder

Description: The focus this semester was learning how to create artistic and visually pleasing mathematical models. We first went over a few options including chip-firing and Mandelbrot sets. This project was exciting because it allowed us to use creative freedom and was an opportunity to show the visual beauty of mathematics that is sometimes forgotten. We ultimately decided to work on chip-firing's cousin, the abelian sandpile model, because it was a doable project along with learning to use Python to create interesting looking models. We used Caroline J. Klivans' book, The Mathematics of Chip-Firing, throughout the semester to help learn the basics of chip-firing and sandpiles. Additionally, I learned the basics of Python and we found examples of code online that could be used to make sandpile images. We then modified the code to create GIFs and changes of color in the images to show avalanches and reconfigurations on finite graphs. What we ended up with was several variations of an abelian sandpile model, and I was happy to use some artistry to choose colors and create visual contrast in the movements between configurations.

Reference(s):

- Klivans, J. C.; **The Mathematics of Chip-Firing**, *CRC Press*, 2019, pp. 3, 11, 12, 47, 48. ISBN: 978-1-138-63409-1.

Title: The Baire Category Theorem and Elementary Analysis

Student: Eric Chen

Mentor: Adam Christopherson

Description: During this semester we learned about the following topics: Metric spaces, Arzela-Ascoli's Theorem, Baire Category Theorem, Differential calculus in $\mathbb{R}^n$, Banach Spaces, Mean Value Theorem, Inverse Functions Theorem in $\mathbb{R}^n$, Implicit Function Theorem in $\mathbb{R}^n$, Integral calculus in $\mathbb{R}^n$, Lebesgue's Criterion, Fubini's Theorem, and differentiable manifolds.

Reference(s):

- Loomis, L. H.; Sternberg, S.; **Advanced Calculus**, *WSPC*, 2014. 594 pp. ISBN: 978-9814583930.

Title: Tychonoff's Theorem

Student: Jeremy Case

Mentor: Linus Ge

Description: This semester we read the book Topology James Munkres. Most of the semester we were focused on studying some of the basic concepts of point set topology such open/closed sets, continuous functions, compact spaces, connected spaces, normal spaces, and metric spaces. Toward the end of the semester, we started discussing the proofs of some of the "deeper" theorems of topology like the Urysohn Metrization Theorem and the Tychonoff theorem.

Reference(s):

- Munkres J.; **Topology**, *Pearson College Div*, 2010. 537 pp. ISBN: 978-0131816299.

Title: The 4-Color Theorem

Student: Jiaming Cheng

Mentor: Chloe Ireland

Description: In this project, we explored Graph Theory in depth, starting with a review of mathematical induction as our foundational tool. We then revisited basic graph properties to establish common ground before progressing to more complex topics. Our journey led us to planar graphs, where we examined their characteristics and understood what qualifies a graph as planar. Building on this knowledge, Euler's formula for planar graphs was introduced—an elegant equation linking the number of vertices, edges, and faces. Our studies further delved into the Five Color Theorem, where we learned about its proof and implications within graph coloring. This theorem paved the way for our next topic—the Four Color Theorem. Although brief, examining the Four Color Theorem's proof made us appreciate its complexity and intricate verification methods. We also explored its historical context, the controversy surrounding it, and eventually, how computer-aided methods proved its validity. Overall, this project offered an exhaustive overview of several key concepts in graph theory while deepening our appreciation for relationships between graph properties and coloring theorems. It solidified my understanding of core principles in Graph Theory and enhanced my problem-solving skills and mathematical reasoning.

Reference(s):

- Diestel, R.; **Graph Theory**, *Springer*, 2010. 410 pp. ISBN: 978-3642142789.

Title: Combinatorics and the Catalan Numbers

Student: Luc Azen

Mentor: Caleb Dilsavor

Description: This semester, Caleb and I read *Concrete Mathematics* by Graham, Knuth, and Patashnik. We studied the first four chapters, which first discussed how to approach and solve a combinatorics problem. We were first introduced to recurrences, the repertoire method, and generating functions. After that, we explored finite calculus, which gave us a way to 'integrate' summations using 'rising' and 'falling' powers, which are closely related to the factorial function. Finally, just before beginning our final project, we discussed the prime numbers, and their importance in number theory. There are not necessarily any unifying theorems of combinatorics, so I chose to discuss the Catalan numbers for my presentation. Next, we read part of Catalan Numbers by Richard P Stanley, and studied the various constructions of the Catalan numbers. His book contained over 200 different ways to construct the sequence. I chose two, and presented how we use generating functions and recurrence relations to derive the sequence.

Reference(s):

- Graham, R. M.; Knuth, D. E.; Patashnik, O.; **Concrete Mathematics**, *Addison-Wesley Professional*, 1994. 672 pp. ISBN: 978-0201558029.
- Stanley, R. P.; **Catalan Numbers**, *Cambridge University Press.*, 2015. 224 pp. ISBN: 978-1107427747.

Title: Mathematics from a Pedagogical Standpoint

Student: Neila Sarkis

Mentor: Brett Hungar

Description: This reading project delves into the pedagogical aspects of teaching mathematics at the undergraduate level. First, we focused on where specifically students were lacking understanding in math. We identified key areas of struggle amongst students which included concepts, definitions, and the ways in which we view mathematical processes. After gaining more insight on where students struggle the most in math, we then assessed how professors and lectures can best help students really grasp the mathematical content. This was shown through interaction patterns in the classroom, class and assessment structure, emotional involvement in problem-solving, and teacher engagement. Through this project, we hoped to bring to light new and different strategies to incorporate in the classroom to ensure undergraduate students' success in mathematics.

Reference(s):

- Maher, R. J.; **Innovative Approaches to Undergraduate Mathematics Courses Beyond Calculus**, *The Mathematical Association of America*, 2005. 200 pp. ISBN: 978-0883851777.

Title: The Hydrogen Atom and Representation Theory

Student: Steven Speck

Mentor: David Green

Description: I studied representation theory with David Green with a focus on better understanding quantum mechanics. We learned about irreducible representations and Lie Algebras. We learned about how representation theory relates to functions on the sphere and the hydrogen atom.

Reference(s):

- Sternberg, S.; **Group Theory and Physics**, *Cambridge University Press*, 1995. 444 pp. ISBN: 978-0521558853.

Title: An Introduction to Stochastic Processes

Student: Thomas Thackston

Mentor: Jake Huryn

Description: This semester we studied the basics of stochastic processes and mathematical finance through *Elementary Probability Theory With Stochastic Processes and an*

Introduction to Mathematical Finance, by Kai Lai Chung and Farid AitSahlia. We first started with Chapter 8, which began with random walks and limiting schemes, eventually generalizing to Markov chains, stochastic matrices, and steady state probabilities. Some of these concepts were then used in Chapter 9 to investigate the Mean-Variance pricing model. Throughout the course of this semester, I gained perspective on the big picture of stochastic processes, as well as some basic financial math models.

Reference(s):

- Chung, K. L.; AitSahlia, F.; **Elementary Probability Theory: With Stochastic Processes and an Introduction to Mathematical Finance**, *Springer*, 2003. 418 pp. ISBN: 978-0387955780.

Title: Functorial Algebraic Geometry

Student: Wo Wu

Mentor: Luke Wiljanen

Description: This semester we read some algebraic geometry with some algebraic number theory and commutative algebra intertwined. Our main reference for algebraic geometry is Qing Liu's book. We investigated the geometry of some local rings and localizations, in what sense p-adic fields are local, and how an affine group could be viewed as a functor. For the final presentation, we decided to try to convey how one could view a scheme/variety as a functor(so-called the functor of points) and why one could want to do that, following the Introduction of Part B of Bosch's book and the last chapter of Eisenbud and Harris' book.

Reference(s):

- Eisenbud D.; Harris, J.; **The Geometry of Schemes**, *Springer*, 2000. 304 pp. ISBN: 978-0387986371.
- Bosch, S.; **Algebraic Geometry and Commutative Algebra**, *Springer-Verlag London* , 2013. x+504 pp. ISBN: 978-1-4471-4829-6.
- Liu, Q.; **Algebraic Geometry and Arithmetic Curves**, *Oxford University Press, USA*, 2006. 577 pp. ISBN: 978-0199202492.

Title: A Method of Proofs based on Markov's Inequality and the Moment Generating Function

Student: Yicheng Lin

Mentor: Hao Xing

Description: Throughout this semester, we read the book High-Dimensional Probability by Roman Vershynin. We covered the first few chapters of the book which include a review of basic probability theory, some concentration inequalities, and sub- Gaussian distributions. We had a weekly meeting to discuss any questions or problems we have encountered while reading the book. We completed and understood many of the exercise problems like checking some variance properties and whether a sub-Gaussian norm is actually a norm. In the first few chapters of the book some tricks and techniques were introduced and demonstrated. For example, simplifying the problem at first or trying to find some equivalent problems to figure out a potential idea might be helpful in solving a problem.

Reference(s):

- Vershynin, R.; **High Dimensional Probability**, *Cambridge University Press*(1st edition), 2018. 296 pp. ISBN: 978-1108415194.

SPRING 2024

Title: Chaos & Croissants

Student: Aaliyah Julius

Mentor: Tianyu Zhao

Description: In Spring Semester 2024 I studied *Chaos: An Introduction to Dynamical Systems* by Alligood, Sauer and Yorke with Tianyu Zhao. We read a chapter a week for most weeks but stopped progressing after the fourth or fifth chapter to dive further into topics. The first few chapters involved discussions of fixed points, sinks, and Lyapunov exponents which were crucial to understanding the complex maps of later chapters. Some topics that also interested me were Cantor sets and how they could be used to prove that the set of irrational numbers are uncountably infinite. One major point of discussion was that of the horseshoe map as it was the first map that truly piqued my interest in chaos theory. Other topics were discussed outside of the book such as Liouville Theorem and tensors.

Reference(s):

- Alligood K.T; Sauer T.D.; Yorke J.A.; **Chaos: An Introduction to Dynamical Systems.** Corrected edition. *Springer*, 1996. 620 pp. ISBN: 978-0387946771.

Title: Continuum Hypothesis and Forcing

Student: Adarsh Praturi

Mentor: Caleb Dilsavor

Description: The continuum hypothesis states that if A is an infinite set of real numbers, then A is countable or has cardinality of the continuum. Our goal this semester was to get a proof of the independence of the continuum hypothesis from ZFC. For this, we read *Forcing for mathematicians* by Nik Weaver. We started by developing the ZFC axioms and developing ordinals and cardinals in ZFC to the extent that we needed. We then introduced the theory ZFC+ which satisfies all the ZFC axioms and has a countable transitive model M of ZFC. We showed that assuming ZFC is consistent, we can prove theories stronger than ZFC are consistent by defining models of these theories in ZFC+. We then started developing the theory of forcing - one of the techniques used to define models in ZFC+. Working in ZFC+, we introduced many notions important to forcing - Forcing notions, Generic Ideals, P-names and values for these P-names and defined the generic extension M[G]. At this point we had run out of time in the semester and could not finish what we wanted to. We were also supposed to cover the fundamental theorem of forcing and use all this to prove that the continuum hypothesis is independent from ZFC+.

Reference(s):

- Weaver N.; **Forcing for Mathematicians.** *World Scientific Publishing Company*, 2014. 152 pp. ISBN: 978-9814566001.

Title: Nonlinear Dynamics

Student: Andrew Clark

Mentor: David Green

Description: This semester we predominantly studied *Nonlinear Dynamics and Chaos* by Steven Strogatz, but also looked at examples from Marsden and Ratiu's *Introduction to mechanics and symmetry* towards the end of our project. We began by studying flows on the line and how to examine the behavior of systems without finding an analytic solution. This included studying fixed points, their stability, and their creation/destruction through bifurcations. We then looked at a special case of one dimensional flows with flows on the circle. Afterwards we studied two dimensional linear systems and how to analyze stable points. We then looked at nonlinear two dimensional systems where we used the Jacobian matrix to make linear approximations about fixed points and learned how to qualitatively analyze the phase plane. We ended the semester with a brief introduction to Index Theory and looked at arguably the most important result from nonlinear dynamics in the Poincare-Bendixson Theorem.

Reference(s):

- Strogatz S.H.; **Nonlinear Dynamics And Chaos.** 1st edition. *CRC Press*, 2000. 512 pp. ISBN: 978-0738204536.
- Marsden J.E; Ratiu T.S; **Introduction to Mechanics and Symmetry.** 2nd edition. *Springer*, 1999. 604 pp. ISBN: 978-0387986432.

Title: Topology for Computing

Student: Andy Liu & Kira Watase

Mentor: Dennis Sweeney

Description: Simplicial complexes are a framework for building a topological space out of triangles of various dimensions. We studied simplicial homology, which is a method of understanding and counting the holes of a simplicial complex in each dimension. Taking coefficients modulo 2 allows using the tools of linear algebra for computations, so we mainly discussed mod-2 Betti numbers: the nth Betti number is found by subtracting the ranks of the incoming and outgoing boundary maps from the number of n-simplices. Finally, we discussed homeomorphism-invariance of Betti numbers and interpretations of Betti numbers as "gaps", "holes", and "caverns" in dimension 0, 1, and 2.

Reference(s):

- Zomorodian A.J.; **Topology for Computing.** *Cambridge University Press*, 2009. 260 pp. ISBN: 978-0521136099.

Title: Practical Statistics for Data Scientists

Student: Bonian Jin

Mentor: Jasmine Hunt

Description: For data scientists, analysts, and everyone else interested in using statistical techniques in data science, *Practical Statistics for Data Scientists: 50+ Essential Concepts* is an extensive reference. The book, written by Peter and Andrew Bruce, offers a succinct and understandable summary of the major statistical ideas and methods applied in data science.

The book is organized to address the fundamental concepts that are necessary for every data scientist to grasp, such as probability distributions, regression analysis, statistical inference, machine learning strategies, and resampling approaches. It places a strong emphasis on useful applications, showing how statistical ideas may be used to solve actual data science issues.

Readers may expect a hands-on learning experience with examples, R and Python code snippets, and explanations of statistical ideas throughout the book. Its goal is to enable readers to become proficient in both statistics and data science, enabling them to deal with data, develop predictive models, and make defensible decisions through statistical analysis. In addition, the book delves into more complex subjects like clustering and Bayesian statistics, giving readers a comprehensive understanding of the statistical methods and instruments that are essential to contemporary data science.

Reference(s):

- Bruce P.; Bruce A.; Gedeck P. **Practical Statistics for Data Scientists: 50+ Essential Concepts.** 2nd edition. *O'Reilly Media*, 2020. 360 pp. ISBN: 978-1492072942.

Title: Options, Futures and Other Derivatives

Student: Connor Engel & Ryan Elrich

Mentor: Michael Tychonievich

Description: For our project we read *Options, Futures, and Other Derivatives* by John C. Hull. We learned about the Wiener Process and Itô's Lemma in Financial Mathematics and how it can be used to model a stock price. We also learned about Pitchfork Bifurcation as a unique stock price model to branch out. We were given Python and MATLAB code to work with these models ourselves and experiment to see how they work and move with different parameters.

Reference(s):

- Hull J.; **Options Futures and Other Derivatives.** 11th Edition. *Pearson*, 2021. 880 pp. ISBN: 978-1292410654.

Title: Lattice Diagrams of Music Rhythm

Student: Haochen Wang

Mentor: Hannah Sheats

Description: In this past semester, we read the book *Exploring Musical Spaces: A Synthesis of Mathematical Approaches* by Julian Hook. We had a comprehensive coverage of mathematical approaches to music theory, combining notes, chords, scales, and rhythmic values to mathematical set theory, graph theory, group theory, geometry, and topology.

Reference(s):

- Hook J.; **Exploring Musical Spaces: A Synthesis of Mathematical Approaches.** *Oxford University Press*, 2022. 682 pp. ISBN: 978-0190246013.

Title: Primes of the Form $4q + 3$

Student: Jason Wang

Mentor: Mehmet Basaran

Description: During this past semester, we studied the basics of number theory, with material ranging from divisibility to congruences. The topic we spent the most time on was prime numbers, and we explored different types of primes, such as those of the form 4q+3 and Fermat and Mersenne primes, arriving at some incomprehensibly large numbers. In addition, we also discussed the interesting properties unique to prime numbers and the relevancy of primes to cryptography. Another fascinating topic was congruences, particularly the Chinese Remainder Theorem, which is believed to have been used to predict planetary movements and celestial phenomena, and which we discussed right around the time of the solar eclipse.

Reference(s):

- Jones G.A.; Jones J.M; **Elementary Number Theory.** Corrected edition. *Springer*, 1998. 200 pp. ISBN: 978-3540761976.

Title: Sub-Exponential/Gaussian Distribution Properties and Examples

Student: Jinglong Wang and Yicheng Lin

Mentor: Hao Xing

Description: This is the second semester during which we have been reading the book *High-Dimensional Probability* by Roman Vershynin. We have covered the second half of Chapter 2 and the first half of Chapter 3, which include sub-Gaussian distributions, sub- exponential distributions, Bernstein's inequality, concentration of the norm, covariance matrices, and examples of high-dimensional distributions. We have had weekly meetings to discuss any questions or problems we encountered while reading the book. For our final presentation, our topic is sub-exponential/Gaussian random variables. We discussed the equivalent definitions of sub-exponential/Gaussian random variables, sub- exponential/Gaussian norm, examples, and their applications in Bernstein's inequality and concentration of the norm.

Reference(s):

- Vershynin, R.; **High Dimensional Probability**, 1st edition. *Cambridge University Press*, 2018. 296 pp. ISBN: 978-1108415194.

Title: Random Graphs: Flows, Recurrence, and Beyond

Student: Jonathan Mosley

Mentor: Xingyi Li

Description: Over the course of this semester, we studied the fundamental elements of probabilistic graph theory, beginning with its motivation from the analysis of electrical networks. The methods learned here were then applied to alternate forms of several famous proofs, including Polya's recurrence theorem and Wilson's algorithm with loop erased random walks. The theory of uniform spanning trees was outlined, and this was then connected to the continuous behavior of Schramm-Löwner evolutions. We then moved to the topic of percolation and studied how higher dimensional lattices display discontinuous behaviors regarding the critical probabilities of open clusters. Variations on percolation were also covered, including oriented percolation (which introduced the notion of a dual graph) and phase criticality. This bled directly into the topic of influence theorems and Russo's formula with sharp thresholds. Finally, we covered a selection of notable results in graph theory, including the BKG inequality and Erdós-Rényi graphs.

Reference(s):

- Grimmett G.; **Probability on Graphs: Random Processes on Graphs and Lattices.** *Cambridge University Press*, 2010. 247 pp. ISBN: 978-0-521-14735-4.

Title: Some Ideas about Category Theory

Student: Mingyi Xu

Mentor: Daniel Wallick

Description: During this semester, we followed the pace of the book we chose, started by building some background examples for category theory, and went over some universal properties in category theory. We began by seeing the connection between general mathematical descriptions and categorical descriptions. Then, we started to build up some mathematical examples that I am not familiar with in undergraduate courses, such as monoids, partially ordered sets, groups, and topological spaces. After that, we stepped into really doing category theory by exploring the property of isomorphisms. Other universal properties of categories, such as monics and epics, initial and terminal objects, and products and coproducts, are also introduced with the examples what I am familiar with, such as sets, matrices, and vector spaces. After this whole project, I was able to explore more mathematical structures and gained some basic understandings of using category theory.

Reference(s):

- Cheng E.; **The Joy of Abstraction: An Exploration of Math, Category Theory, and Life.** *Cambridge University Press*, 2022. 438 pp. ISBN: 978-1108477222.

Title: A Probabilistic Proof of the Euler Proof Formula

Student: Nayeon Kim

Mentor: Bart Rosenzweig

Description: This semester we covered basic notions in measure theory and probability theory using Achim Klenke's *Probability Theory: A Comprehensive Course*. This included: classes of sets and the pi-lambda theorem, set functions and their properties, the Caratheodory extension theorem for measures, Lebesgue and Lebesgue-Stieltjes measures, measurable maps, basics of random variables, probability distributions and independence, and 0-1 laws. For the final presentation we used many of these notions to read through Klenke's proof of the Euler product formula for the Riemann zeta function.

Reference(s):

- Klenke A.; **Probability Theory: A Comprehensive Course.** 3rd edition. *Springer,* 2020. 730 pp. ISBN: 978-3030564018.

Title: Property of Group

Student: Qingze Han

Mentor: Jake Huryn

Description: This semester I studied group theory, which is a branch of abstract algebra. Our textbook is called *Visual Group Theory* by Nathan Carter. Jake and I went through the Cayley Table to learn about the five basic types of groups in group theory: cyclic group, abelian group, dihedral group, symmetric group, and alternating group. We used these groups to learn about later theories such as subgroups and product and quotient groups. We delved into the Lagrange theorem, a fascinating theory, and I learned about its applications! I learned about the application and the proof of this theory by showing that each coset is non-overlapping and equal in size. Besides, I also learned about the power of homomorphism, which allowed me to understand better the relationship between elements in a group and the whole. Later in the semester, we worked together on the Fundamental theorem of abelian groups, a very practical theory!

Reference(s):

- Carter N.; **Visual Group Theory.** *American Mathematical Society,* 2021. 295 pp. ISBN: 978-1470464332.

Title: A Glance of Local Class Field Theory

Student: Wo Wu

Mentor: Min Shi

Description: This semester we read through the first 6 chapters of *An Introduction to Automorphic Representations* by Getz and Hahn (which was available as an online note before March and as a book after). Topics include affine algebraic groups, adeles, Haar measures, automorphic forms, etc. In the end-semester presentation, we explained how local class field theory gives the local Landlangs correspondence in dimension one.

Reference(s):

- Bushnell C.J; Henniart G.; **The Local Langlands Conjecture for GL(2).** *Springer*, 2008. 364 pp. ISBN: 978-3540819936.

- Getz J.R; Hahn H.; **An Introduction to Automorphic Representations.** *Springer*, 2024. xviii+609 pp. ISBN: 978-3-031-41151-9.

Title: Universal Cover of Non-Negatively Curved Surface

Student: Yumin Shen

Mentor: Linus Ge

Description: This semester we learned algebraic topology, following Hatcher's chapters 0 and 1, and did some problems together to apply the theorems we studied. Basic topics that we've gone through include but are not limited to homotopy, fundamental groups, Van Kampen(we've also talked about the groupoid version of it), covering spaces, deck transformations, and K(G,1) spaces. We also studied some applications in areas related to but outside of algebraic topology, such as analysis, differential geometry, knot theory, and graph theory.

Reference(s):

- Hatcher A.; **Algebraic Topology.** *Cambridge University Press, Cambridge*, 2001. xii+544 pp. ISBN: 0-521-79540-1.

Title: Lovász Local Lemma and Colorings of the Real Line

Student: Zhengyang Liu

Mentor: Jingheng Wang

Description: This semester we read some additive combinatorics. The main reference to additive combinatorics is the book Additive Combinatorics written by Terence Tao and Van Vu. We mainly discussed probabilistic methods in additive combinatorics, including some inequalities such as Chebyshev's inequality and its applications in additive number theory. We also learned how to use the probabilistic method to demonstrate the existence of a mathematical object or property by showing that the probability of randomly selecting such an object or property is greater than zero. In the final presentation, we presented Lovász local lemma and how it contributes to our real-life problem. As one of the applications, we talked about Erdos problem of the Coloring of the real line. We briefly introduce how they are related and try to give the audience a sense of using the probabilistic method of additive combinatorics to solve problems.

Reference(s):

- Tao T.; Van H.V.; **Additive Combinatorics.** 1st Edition. *Cambridge University Press*, 2009. 532 pp. ISBN: 978-0521136563.
- Erdos P.; Lovász L. **Problems and results on 3-chromatic hypergraphs and some related questions.** In Infinite and Finite Sets (Colloq., Keszthely, 1973; dedicated to P. Erdos on his 60th birthday), Vol. II, Colloq. Math. Soc. Janos Bolyai, 10, *North-Holland*, 1975, 609–627 pp.
