Set # 2
Ross Program, Number Theory
June 20, 2006
Answer the questions, then question the answers. – Glenn Stevens

Terminology

Q1. What is the "greatest common divisor" of two integers?

Exploration

P1. Factor $a^n - b^n$ and $a^n + b^n$, for $n = 2, 3, 5, 7, \ldots$. Any conjectures?

P2. Perform Euclid's Algorithm on the numbers 163 and 519. Use it to do the following:
(a) Find the greatest common divisor of 163 and 519.
(b) Use "forward substitution" to find integers x and y satisfying $163x + 519y = 1$.
(c) Use "backward substitution" to express 1 as a combination of 163 and 519.

P3. Continue your work from P2:
(a) Show that

$$\frac{519}{163} = 3 + \cfrac{1}{5 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{4}}}}.$$

This is usually denoted as $\frac{519}{163} = [3, 5, 2, 3, 4]$, and is called the (simple) continued fraction for $\frac{519}{163}$.
(b) Using this notation, simplify the continued fractions: $[4], [3, 4], [2, 3, 4], [5, 2, 3, 4], [3, 5, 2, 3, 4]$. Where have you seen these numerators and denominators?
(c) Now simplify the continued fractions: $[3], [3, 5], [3, 5, 2], [3, 5, 2, 3], [3, 5, 2, 3, 4]$. What do you notice? Note: These are called the convergents to the continued fraction $[3, 5, 2, 3, 4]$.
(d) Finally, construct the "Magic Table" for the fraction $\frac{519}{163}$ as mentioned in class. Where have you seen these numbers before?

Prove or Disprove and Salvage if Possible

P4. $-1 \cdot -1 = 1$

P5. For $n \geq 0$, $(-1)^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$.

P6. $a|b \Rightarrow b|a$ for all a and b in $\mathbf{Z}$.

P7. $n \in \mathbf{Z}$ and $2 \nmid n \Rightarrow 8|(n^2 - 1)$.

P8. $d|a$ and $d|b \Rightarrow d|(ar + bs)$ for every r and s. True in $\mathbf{Z}$.

P9. $a = bq + r \;\Rightarrow\; (a, b) = (b, r)$. True in $\mathbf{Z}$.

Numerical Problems (Some food for thought)

P10. List all the perfect squares in $\mathbf{U}_3$. How many are there? Do the same for $\mathbf{U}_5$. For $\mathbf{U}_7$. For $\mathbf{U}_{11}$. For $\mathbf{U}_{13}$. Conjectures?

P11. Find an integral solution to the diophantine equation $7469x + 2463y = 1$. Find the multiplicative inverse of 2463 (mod 7469).

Extra problems for advanced students.

A1. (i) Given $b > 0$, does every n have a unique base b expression $n = \sum_{k \geq 0} r_k b^k$ where $0 \leq r_k < b$?

(ii) Does n have a unique factorial-base expression $n = \sum_{k \geq 1} s_k k!$ where $0 \leq s_k \leq k$?

A2. A polynomial f is called *integer-valued* if $f(n) \in \mathbf{Z}$ for every $n \in \mathbf{Z}$. Certainly every $f \in \mathbf{Z}[x]$ in integer-valued. What about $\frac{1}{5}x^5 + \frac{2}{3}x^3 + \frac{2}{15}x$? Find all the integer-valued polynomials.

A3. Recall how Pascal's triangle is built. Label the rows so that row 0 is: 1, and row 1 is: $1, 1$. Write out the first ten rows of the "mod 2" Pascal triangle (all entries in $\mathbf{Z}_2$) and contemplate the patterns. Here are some questions that come to mind:
For which n is the n^{th} row all odd?
Can you find a formula for the number of odd entries appearing in row n?
Cut off the triangle after n rows. For which n does that "equilateral triangle" have rotational symmetry?

A4. Which two digit numbers a, b have the longest euclidean algorithm? Which three digit numbers?

A5. On the set of ordered pairs of integers (x, y) define operations L and R by: $L(x, y) = (x - y, y)$ and $R(x, y) = (x, y - x)$. If (a, b) is a pair of relatively prime positive integers, must there be a sequence of L's and R's reducing it to $(1, 0)$? For example, for $(5, 7)$ we find $RRLLR(5, 7) = (1, 0)$.
Is that sequence of Rs and Ls unique?