Notes on conformal geometry, uniformization, and Ricci flow

Spencer Dembner

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1 Introduction

In this paper, we explain some of the connections between Riemann surfaces and 2-dimensional Riemannian manifolds. As long as we assume orientability, the two concepts are very nearly the same: a Riemann surface is just an oriented Riemannian surface whose metric is defined only up to rescaling (a "conformal manifold"). This allows us to translate between complex analytic statements about Riemann surfaces and geometric statements about Riemannian manifolds.

The most important result about Riemann surfaces is the uniformization theorem, which says that the only simply connected Riemann surfaces are the disk, the complex plane, and the Riemann sphere. In the rest of the paper, we focus on interpreting this result in the language of Riemannian geometry. First, we show that uniformization is equivalent to a more explicitly geometric condition: Any 2-dimensional Riemannian manifold admits a unique constant curvature metric conformal to its original metric. Finally, we discuss a strategy, due to Hamilton and others, to prove certain cases of the uniformization theorem using Ricci flow. Our discussion here falls very short of a proof, but highlights some of the key technical points, and offers further references for the interested reader.

Throughout the paper, all manifolds are taken to be smooth. Additionally, we will require that Riemann surfaces be smooth manifolds in the usual sense. In particular, the fact that all Riemann surfaces are second-countable is part of our definition.

2 Complex structure and isothermal coordinates

The first thing we need to understand is the relationship between Riemannian structure and complex structure. There clearly should be some connection, because both give rise to a definition of angle on each tangent space. However, the details are not quite as straightforward as they could be. Not every 2-dimensional Riemannian manifold can be a complex manifold. Indeed, *every* 2-dimensional manifold admits some Riemannian metric. But all Riemann surfaces are orientable, so a non-orientable surface, such as $\mathbb{R}P^2$, cannot admit a complex structure. Conversely, a complex structure clearly does not determine a Riemannian structure because, for instance, complex analysis cannot distinguish between the disk of radius 1 and the disk of radius 2. The issue, essentially, is that rescaling the metric at a single point does not give a meaningfully different structure on the tangent space from the point of view of complex analysis

Taking account of these issues, the correspondence ends up being the following: Riemann surfaces are the same as oriented 2-dimensional manifolds with a *conformal* structure. In this section, we show how to go from conformal structure to complex structure, relying on a result

(existence of isothermal coordinates) which we will not prove. In the next section, we go the other way.

Definition 2.1. Let M be a manifold. A conformal structure on M is an equivalence class of Riemannian metrics g on M, where we identify a metric h with all metrics of the form

$$h' = \rho h,$$

for $\rho\colon M \to \mathbb{R}_{>0}$ a smooth function. An isomorphism $f\colon M \to N$ of conformal manifolds (which we will sometimes call a conformal equivalence) is a diffeomorphism which preserves the metric up to conformal equivalence: In other words, the pullback metric $f^*(g_N)$ on M is conformal to the metric g_M.

Remark 2.2. Given a vector space V, two inner products on V define the same notion of angle on V if and only if they are rescalings of each other. See [Spi79a] for the (reasonably straightforward) proof.

Many notions from Riemannian geometry will not be independent of the choice of metric within a conformal equivalence class; for instance, the usual definition of curvature is not conformally invariant. The freedom to choose any metric within a conformal equivalence class will be important later on, when we study the uniformization theorem.

An important observation which we will use later is that bijective holomorphic maps between open subsets of $\mathbb{C}$ are conformal equivalences. Indeed, if $f\colon U \to V$ is bijective, a basic result in complex analysis is that f' never vanishes on U. However, multiplication by a complex number, as a linear transformation, is the composition of a rotation and a dilation. Therefore, up to rescaling, it is an isometry. Conversely, it is also true that any conformal equivalence $f\colon U \to V$ between open subsets of $\mathbb{C}$ is a holomorphic map, as long as f preserves orientation. Indeed, the derivative of f is of the form

$$v \mapsto rA(v),$$

where $A\colon \mathbb{C} \to \mathbb{C}$ is an isometry, and $r > 0$. But A must be orientation-preserving, so A is a rotation, and we see that the derivative is given by multiplication by a complex number.

Now, we show how to endow an oriented 2-dimensional Riemannian manifold M with a compatible complex structure. The question we are interested in really has two parts:

1. Local structure: What (smoothly-varying) structure do we have on a single tangent space?
2. Global structure: How do these structures "integrate" over the whole manifold?

The local structures are as follows: Given a (necessarily even-dimensional) vector space V, a complex structure on V is just an automorphism $J\colon V \to V$ such that $J^2 = -1$. If the surface M has the structure of a Riemann surface, then every tangent space T_pM has a complex structure. If M is a Riemannian manifold, by definition T_pM has an inner product structure, so it makes sense to discuss orthogonal transformations of T_pM. If M is also 2-dimensional, then there are very few such orthogonal transformations. Along with the choice of an orientation, this constrains the possible complex structure we could impose. The orientation condition is clearly necessary, because of examples like the real projective plane, which was discussed above.

Definition 2.3. Let (M, g) be a 2-dimensional oriented Riemannian manifold. A smoothly-varying choice J_p of complex structure on T_pM is compatible with the metric and orientation if we have:

1. For all p, J_p is an isometry.

2. For all $v \neq 0$ in T_pM, $(v, J_p(v))$ is a positively-oriented basis.

The second condition is a natural one, because it is equivalent to requiring that the canonical orientation coming from the complex structure is the same as the pre-existing orientation on T_pM. We know that SO(2) is the rotation group, and the only rotations which square to -1 are rotation by $\pm\pi/2$. Of these, only rotation by $\pi/2$ satisfies the condition that v, Jv should be a positively-oriented basis. Therefore, we see that the choice of a Riemannian metric and an orientation gives a *unique* compatible complex structure on each tangent space.

Observe that this choice of a complex structure on the tangent space is invariant under rescaling the metric. This is because rescaling the inner product on a given vector space does not affect which transformations are orthogonal with respect to the inner product. Thus, the complex structure on T_pM coming from a metric is the same for conformally equivalent metrics.

There still remains the global problem. A Riemannian metric need not satisfy any global conditions, as long as it varies smoothly from point to point. This could be seen as one reason why all manifolds admit Riemannian metrics: there is no global problem to solve, so local patching via partitions of unity does not run into difficulties. However, complex structure on a surface M requires more than a smoothly-varying choice of complex structure on each tangent space. In fact, such a choice defines something strictly weaker, an *almost-complex structure.*

Definition 2.4. Let M be a manifold. An almost-complex structure on M is a smoothly-varying choice of automorphism $J_p \colon T_pM \to T_pM$ such that $J_p^2 = -1$.

We have just proven the following:

Proposition 2.5. *Let M be a 2-dimensional oriented Riemannian manifold. Then M admits a unique almost-complex structure compatible with its orientation and metric.*

It turns out that there are almost-complex manifolds which do not admit a complex structure, at least in real dimension 4. However, there are no such examples in dimension 2 (our case of interest), and an open conjecture of Yau states that no examples exist in real dimension at least 6 (see a discussion in [Var]). This shows that almost-complex structures which do not come from complex structures are something fairly unusual, and indeed we will see below that the almost-complex structure defined above gives a complex structure.

We have not yet said what it means for a complex structure to be compatible with a Riemannian structure. Requiring that the charts be isometries is clearly too strong: It rules out the standard charts on the sphere, since the transition between the two standard charts is not an isometry. Instead, motivated by the idea that complex structure is similar to conformal structure, we make the following definition:

Definition 2.6. Let M be a 2-dimensional oriented Riemannian manifold. A complex structure on M is compatible with the metric if the charts for the complex structure are orientation-preserving conformal equivalences.

Proposition 2.7. *If M has a compatible complex structure, the corresponding almost-complex structure is compatible as well.*

Proof. The complex structure on M gives rise to an almost-complex structure, in the following way: Pick some $p \in M$, and some corresponding chart $\varphi_p \colon \mathbb{D} \xrightarrow{\cong} U_p$. Suppose $\varphi_p(z) = p$. This induces an identification of $T_z\mathbb{C} = \mathbb{C}$ with T_pM. We let $J_p \colon T_pM \to T_pM$ be the automorphism corresponding, under this identification, to the usual multiplication by i map on $\mathbb{C}$. This is independent of the chart (U_p, φ_p) chosen, since the transition maps have $\mathbb{C}$-linear derivatives.

The first property to check is that $J_p \colon T_pM \to T_pM$ is an isometry. Since being an isometry is invariant under rescaling of the metric, and since the charts are conformal equivalences, it's enough to check this property for the corresponding map $J \colon \mathbb{C} \to \mathbb{C}$. But the property in this case is trivial.

The second property is that $(v, J_p(v))$ forms a positively-oriented basis for $v \neq 0$. Again, since the charts are orientation-preserving, we only need to check the corresponding property for $\mathbb{C}$. Again, the property for $\mathbb{C}$ is trivial. □

2.a Isothermal coordinates

In order to endow the 2-dimensional Riemannian manifold M with a complex structure, we need to be able to relate its metric, on coordinate patches, to the standard metric on $\mathbb{R}^2$. A useful property about $\mathbb{C}$, phrased rather tautologically, is that $\mathbb{C}$ is globally isometric to $\mathbb{R}^2$ with the standard flat metric. In general, there is no reason why M should be globally identifiable with $\mathbb{R}^2$, and even locally, the complex charts on M need not be isometries. However, they *do* need to be conformal equivalences, which motivates the following definition:

Definition 2.8. Let M be a Riemannian manifold, and let $U \subset M$ be open. An isothermal coordinate chart on U is a chart $U \to \mathbb{R}^n$ such that, written in the corresponding coordinates, the metric on M has the form

$$g = \rho(dx_1^2 + \cdots + dx_n^2)$$

for some $\rho > 0$.

Observe that isothermal coordinate charts are exactly coordinate charts such that the chart is a conformal equivalence.

If M admits a compatible complex structure, then the metric on a coordinate patch $U \subset M$ is conformal to the metric pulled back from $\mathbb{C}$. This shows that M admits isothermal coordinates around each point. Conversely, we have:

Lemma 2.9. *Assume a 2-dimensional oriented manifold M admits isothermal coordinates. Then M admits a compatible complex structure consisting of orientation-preserving isothermal coordinate charts.*

Proof. First, observe that if M admits isothermal coordinates, M admits orientation-preserving isothermal coordinates, since we can just compose with an orientation-reversing isometry of $\mathbb{C}$. To show these charts give a complex structure, we only need to show that the transition maps between different positively-oriented isothermal coordinate charts are holomorphic. As we observed above, the isothermal coordinate charts are orientation-preserving conformal equivalences. It follows by composition that the transition maps are orientation-preserving conformal equivalences, between open subsets of $\mathbb{C}$. But we have already seen that all such maps are holomorphic. The fact that the corresponding complex structure is compatible is immediate from the definition. □

It follows that in order to see that M admits a compatible complex structure, we only need to know that M locally admits isothermal coordinates. In fact, it turns out that all 2-dimensional Riemannian manifolds admit isothermal coordinates, but the proof is quite hard, and we will not give it.

Theorem 2.10. *Let M be a 2-dimensional Riemannian manifold. Then M admits isothermal coordinates around every point.*

Proof. For a discussion in the context of surfaces in $\mathbb{R}^3$, see page 135 of [Sch14]. For a full proof, see Addendum 1 to Chapter 9 of [Spi79b]. □

This shows how to give an oriented Riemannian manifold a complex structure. In higher dimensions, isothermal coordinates do not generally exist, but there are explicit conditions, such as the vanishing of a tensor, to decide when they do exist.

3 Reconstructing a conformal structure

Now, we handle the other direction. Let M be a Riemann surface. Our goal is to exhibit a single metric g that the complex structure is compatible with, keeping in mind that any metric conformally equivalent to g will have the same property.

If $\varphi\colon \mathbb{D} \to \mathbb{C}$ is a chart, then the metric on $\mathbb{D}$ as an open subset of $\mathbb{C}$ pushes forward to a metric on $\varphi(\mathbb{D}) \subset M$. As we have repeatedly observed, this metric depends heavily on the choice of chart. However, its conformal equivalence class does not. Indeed, the transition maps have open subsets of $\mathbb{C}$ as domain and codomain. We saw above that the transition maps, which are bijective holomorphic maps, are conformal equivalences with respect to the inherited metrics. This does not quite finish the argument, because it only shows how to *locally* define a conformal structure. However, a standard patching argument readily gives us what we want.

Theorem 3.1. *Let M be a Riemann surface. Then, up to conformal equivalence, there is a unique metric on M such that the complex structure on M is compatible with the metric and the canonical orientation.*

Proof. For all $p \in M$, our discussion above gives some open U_p containing M, and a metric g_p on U_p such that the restriction of any chart to U_p is an orientation-preserving conformal equivalences. Since transition maps are conformal equivalences, if $U_p \cap U_q \neq \emptyset$, then the two metrics g_p, g_q are conformally equivalent on the overlap.

Since M is second-countable and hence paracompact, the cover U_p admits a locally finite refinement, which we index as $\{U_\alpha\}$. Likewise, we index the corresponding metrics as $\{g_\alpha\}$. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$. Define a metric g on M by

$$g = \sum_\alpha \rho_\alpha g_\alpha.$$

This sum is locally finite, and hence well-defined. Since the complex structure is compatible with each g_α, it is immediate that the complex structure is compatible with g.

It remains to show uniqueness. However, by compatibility, we know that the metric on M is locally conformal to the metric induced by any complex chart. This determines the metric near any point up to conformal equivalence. □

4 The statement of uniformization

We have now worked out a fairly robust correspondence between Riemann surfaces and oriented Riemannian 2-manifolds up to conformal equivalence. In the remaining sections, we focus on the most important basic result about Riemann surfaces, the uniformization theorem. First, we use the correspondence to state the uniformization theorem in a more geometric language, at least for closed surfaces. In the final sections, we explore one way to try to prove this version of uniformization geometrically, by using Ricci flow. There are many details to check, some of them difficult, and we will omit almost all of them. In addition, this strategy does not obviously make sense for noncompact surfaces, and until somewhat recently it was not even known to work for the 2-sphere. However, the proof via Ricci flow gives a beautiful geometric intuition for why uniformization "should" hold, at least for closed surfaces.

The usual statement of uniformization is the following:

Theorem 4.1 (Uniformization)**.** *Let M be a connected, simply connected Riemann surface. Then M is biholomorphic to one of the following:*

- *The complex plane, $\mathbb{C}$.*
- *The disk, $\mathbb{D}$, itself biholomorphic to the upper half plane $\mathbb{H}$.*
- *The Riemann sphere, S_2 (or $\mathbb{C}_\infty$).*

Proof. The usual analytic proof is presented in many textbooks, such as [Sch14]. □

Of course, we now know that Riemann surfaces are just oriented conformal 2-manifolds. Furthermore, the statement of uniformization looks suspiciously similar to the following basic result from Riemannian geometry:

Theorem 4.2 (Space forms)**.** *Let M be a complete simply connected Riemannian manifold with constant sectional curvature K. Then M is isometric to:*

1. *The upper half plane $\mathbb{H}^n$, if $K = -1$.*
2. *$\mathbb{R}^n$, if $K = 0$.*
3. *S^n, if $K = 1$.*

Proof. See theorem 4.1 from Chapter 8 of [Car92]. □

We have seen that biholomorphic maps are conformal equivalences. Furthermore, the metrics on $\mathbb{H}, \mathbb{R}^2$ and S^2 are compatible with their complex structures. Indeed, this follows immediately from the fact that all three Riemann surfaces admit (local or global) charts which are conformal equivalences with respect to their usual metrics. This suggests the following candidate statement:

Theorem 4.3 (Uniformization, version two)**.** *Let M be an orientable 2-dimensional Riemannian manifold. Then M admits a conformally equivalent metric of constant curvature.*

Note that if the metric g has constant curvature K, then the metric ag, where $a > 0$ is a constant, has constant curvature $a^2 g$. It follows that if $K > 0$, then M is isometric to S^n with a uniformly rescaled metric, and likewise in the case $K < 0$. Therefore, we really only need to consider the cases $K = 0, 1, -1$.

Remark 4.4. We could require that the metric M is complete. However, any metric is conformal to a complete metric, so in fact the weaker and stronger statements are equivalent.

Proposition 4.5. *The first version of uniformization implies the second one.*

Proof. Let M be an oriented 2-dimensional Riemannian manifold. Since M is oriented, our work in the previous sections shows that M admits a compatible complex structure. Let $\widetilde{M}$ be the holomorphic universal cover of M. Then by uniformization, $\widetilde{M}$ is biholomorphic to one of the three standard Riemann surfaces. Thus, we have a biholomorphic identification of M with

$$N/\Gamma,$$

where $N = \mathbb{H}, \mathbb{C}, \mathbb{D}$, and $\Gamma < \mathrm{Aut}(N)$, the group of complex automorphisms of N. Since biholomorphic maps are conformal automorphisms in the Riemannian geometry sense, in order to show that M has a conformal metric of constant curvature, we only need to show the corresponding result for N/Γ. Therefore, assume that $M = N/\Gamma$.

The usual metric on $\widetilde{M}$, which is one of the three standard surfaces, has constant curvature. Therefore, it would be enough to show that this metric descends to M. In turn, this would follow if we knew that the elements of Γ were isometries. This is immediate for $\mathbb{D}$, since all (complex) automorphisms of $\mathbb{D}$ are hyperbolic isometries. Likewise, all automorphisms of $\mathbb{C}$ without fixed points are translations, and hence isometries, and all automorphisms of S^2 have a fixed point. □

The reverse implication is not quite so clear. In general, knowing that a simply connected Riemann surface admits some constant curvature metric does not tell us much, because the constant curvature metric might not be complete, even if some other metric conformal to it is. This is true, for example, if we take the Euclidean metric on $\mathbb{D}$ coming from $\mathbb{C}$, rather than the conformally equivalent hyperbolic metric. However, if M is a cover of a closed surface, then the compactness of the base allows us to dispense with any issues related to completeness.

Proposition 4.6. *The second version of uniformization implies the first one, in the case where the surface is the universal cover of a closed surface.*

Proof. Let M be a closed Riemann surface, and let $\widetilde{M}$ be its universal cover. We will show that $\widetilde{M}$ is biholomorphic to one of the model Riemann surfaces. Let h be any metric compatible with the complex structure on M. Then $\widetilde{h}$, the lift of h to $\widetilde{M}$, is compatible with the complex structure on $\widetilde{M}$. By the second statement of uniformization, the surface M admits a metric g of constant curvature, conformal to h. This lifts to a metric $\widetilde{g}$ on $\widetilde{M}$, which likewise has constant curvature. By working locally, we see that $\widetilde{g}$ is conformal to $\widetilde{h}$, since g is conformal to h. Therefore, $\widetilde{g}$ is compatible with the complex structure on $\widetilde{M}$.

Since M is compact, the Hopf-Rinow theorem shows that all metrics on M are complete. Therefore, g is complete. But this means that $\widetilde{g}$ is complete as well: Indeed, since the covering map $(\widetilde{M}, \widetilde{g}) \to (M, g)$ is a local isometry, the lift of a geodesic from M to $\widetilde{M}$ is a geodesic. Since geodesics on M are defined for all time, it follows that the same is true on $\widetilde{M}$, so $\widetilde{M}$ is complete.

Therefore, by Theorem 4.2, it follows that $(M, \widetilde{g})$ is isometric to one of the model Riemann surfaces (note again that the standard metrics on the model Riemann surfaces are compatible with their complex structures). However, we have already seen that $\widetilde{g}$ is compatible with the complex structure. It follows that the isometry from $(M, \widetilde{g})$ to a model Riemann surface is biholomorphic, completing the proof. □

5 Ricci flow and uniformization

Now that we have a statement of uniformization which is geometric in flavor, we turn our attention to the question of how to prove uniformization in a "geometric" way. For the rest of the paper, we assume that M is closed unless stated otherwise. Many of the results we state are from [CK04], and we give only an impressionistic view of most of the key ideas.

A motivating heuristic is the idea that where curvature is negative, geodesics spread out and volume grows quickly, while where curvature is positive, geodesics converge and volume is small. This suggests that we should try to negate this change in volume by rescaling the metric around each point, making it larger near points where the curvature is overly negative, and making it smaller near points where the curvature is overly positive. In fact, this is essentially what we will do.

Definition 5.1. Let (M, g) be any Riemannian manifold. An (unnormalized) Ricci flow on M is a smooth family g_t of metrics on M satisfying

$$\frac{\partial}{\partial t} g_t = -2 \operatorname{Ric}_{g_t},$$

where Ric is the bilinear Ricci curvature tensor.

This is a partial differential equation depending in local coordinates on the components g_{ij} of the metric. Our plan is to study the solutions g_t as $t \to \infty$ of this equation, or some variant of it. If we get lucky, then they will converge to some metric g_∞ with positive curvature.

In our case, there is only one sectional curvature, so the Ricci curvature is easy to intepret. Indeed, from the discussion on page 199 of [BG05], we see that if $|x| = 1$ and $x = x_1, x_2$ is an orthonormal basis at T_pM, then

$$\operatorname{Ric}(x, x) = K(\Pi_{(x_1, x_2)}) = K(p).$$

Therefore, for all x, we have $\operatorname{Ric}(x, x) = |x|^2 K(p)$. A symmetric bilinear form is determined by the corresponding quadratic form, and one symmetric bilinear form giving the quadratic form $\operatorname{Ric}(x, x)$ is the form $A(x, y) = g(x, y)K(p)$. Thus, we have

$$\operatorname{Ric} = Kg,$$

and the equation becomes

$$\frac{\partial}{\partial t} g_t = -2K_t g_t,$$

where K_t is the sectional curvature corresponding to g_t. We can read off from the partial differential equation that if $g_t \to g_\infty$, then g_∞ *must* have constant curvature. In fact, it must have constant curvature zero. Of course, this just means that we haven't yet defined the right flow. If we start with the 2-sphere, the above flow will shrink the metric steadily to 0. If we start with the upper half plane $\mathbb{H}$, then the metric will simply grow toward infinity. Thus, we see that if we want any convergence at infinity, we need to normalize so that the flow preserves volume.

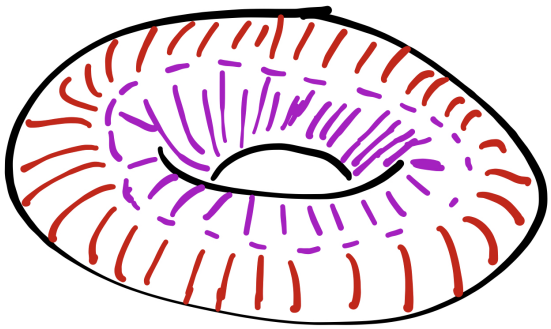


Figure 1: We consider Ricci flow on the embedded torus, which has areas of both positive (red) and negative (purple) curvature. Since the torus has $\chi(T) = 0$, the Gauss-Bonnet theorem tells us that a constant curvature metric on T must be flat. Therefore, we expect Ricci flow to uniformize the metric by evolving it toward the usual flat metric on the torus. The fact that T has average curvature 0 shows that the un-normalized Ricci flow already preserves the volume of T. However, the Ricci flow on T cannot be easily visualized in $\mathbb{R}^3$: It seems to immediately "break" the torus by stretching the outer regions and contracting the inner regions. This is unsurprising, since the flat torus is not embeddable in $\mathbb{R}^3$.

Definition 5.2. The *normalized* Ricci flow on the surface M is defined by the equation

$$\frac{\partial}{\partial t} g_t = (k_t - K_t) g_t,$$

where

$$k_t = \frac{\int_M K_t d\mu_t}{\mu_{g_t}(M)}$$

is the average sectional curvature of M.

Note that we dropped the constant 2 in front of the curvature term. This is essentially unimportant, and corresponds to a rescaling of time.

We now have an equation that looks more promising in several ways. First, the equation for Ricci flow, only in dimension 2, is of the form

$$\frac{\partial}{\partial t} g_t = A g_t,$$

where A is a function. It follows that for all times t, s, the metrics g_t and g_s are conformal to each other. Therefore, if g_t converges as $t \to \infty$ to some g_∞, then g_∞ will be conformal to g_0.

Second, now that we have adopted the appropriate normalization, the Ricci flow preserves volume on M.

Theorem 5.3. *We have*

$$\frac{\partial}{\partial t} d\mu = (k_t - K_t) d\mu.$$

In particular, the Ricci flow preserves the volume of M.

Proof. See corollary 5.6 of [CK04]. To see that the flow preserves volume, write

$$\frac{\partial}{\partial t}\int_M d\mu = \int_M \frac{\partial}{\partial t} d\mu = \int_M (k_t - K_t) d\mu$$

$$= k_t \int_M d\mu - \int_M K_t d\mu = 0,$$

and integrate over t. □

Now that we know the flow preserves volume, we can observe that k_t is independent of t. Indeed, by the Gauss-Bonnet theorem along with the definition of k_t, we have

$$k_t = \frac{1}{\int_M d\mu_t}(2\pi\chi(M)),$$

and the righthand side is independent of t. Therefore, we write $k_t \equiv k$, a constant.

Existence and uniqueness of solutions is far from clear. The key step involves obtaining upper and lower bounds on curvature:

Theorem 5.4. *Let g_t be a normalized Ricci flow on M. Then there exists a constant $C > 0$ depending only on g_0 such that:*

- *If $k < 0$, then*
$$k - Ce^{kt} \leq K_t \leq k + Ce^{kt}.$$
- *If $k = 0$, then*
$$\frac{-C}{1+Ct} \leq K_t \leq C.$$
- *If $k > 0$, then*
$$-Ce^{-kt} \leq K_t \leq k + Ce^{kt}.$$

Proof. See proposition 5.18 of [CK04] □

Remark 5.5. The hardest case when giving all details of this proof is $\chi(M^2) > 0$, corresponding to the 2-sphere. This is already evident in the asymptotics of the theorem we just stated: in the case $k < 0$, the upper and lower bounds converge to the mean curvature k. In the case $k = 0$, the lower bound converges to 0, while the upper bound is a positive constant. But in the case $k > 0$, the lower bound is merely 0, while the upper bound actually grows exponentially. Thus, we have the worst control over the curvature in this case.

In fact, the proof in the case $\chi(M^2) > 0$ is so much harder that at first, every proof that the Ricci flow uniformizes the metric on a 2-sphere was reliant on the uniformization theorem itself. This is no longer true, as discussed in [CLT06].

Having a good bound on the curvature of M with respect to g_t that is valid for all t actually turns out to be enough to guarantee existence of the flow g_t for all time. In more detail, we have the following:

Corollary 5.6. *The flow g_t extends to a maximal interval $[0, T)$ with $0 < T \leq \infty$. If $T < \infty$, then we have*

$$\lim_{t \to T^-} \left(\sup_{p \in M} |K_t(p)| \right) = \infty.$$

Proof. See Corollary 7.2 of [CK04], and the discussion on pages 115-116. □

Theorem 5.7. *The flow g_t extends to all of $[0, \infty)$, on any closed oriented Riemannian surface M.*

Proof. By theorem 5.4, we can uniformly bound the value of $K_t(p)$ for all $p \in M$ and all t in a compact interval $[a, b]$. By the previous theorem, it follows that g_t exists for all times $t \geq 0$. □

It remains to show that g_t converges in some suitably strong sense to a smooth metric g_∞, and that the metric g_∞ has constant curvature. A "moral proof" of this in the case $k < 0$ could go as follows: Since $k < 0$ the curvature estimates given above actually converge to k. If the convergence to g_∞ is good enough, it should follow that g_∞ itself has constant curvature k, completing the proof.

The case $k = 0$ admits a similar heuristic argument. Indeed, the estimates above show that g_∞ would have curvature in the interval $[0, C]$. However, since the mean curvature $k = 0$ is independent of t, g_∞ must have mean curvature 0 as well. It follows that its curvature, which is nonnegative, must vanish identically. However, the case $k > 0$ does not admit this sort of argument: the bounds on curvature guarantee that g_∞ would have nonnegative curvature, but since $k > 0$, this does not guarantee constant curvature.

Turning these heuristics into proofs requires significantly more work. Doing so is outside the scope of our paper, but the interested reader can find the details in Chapter 5 of [CK04].

As a final remark, note that the behavior of the Ricci flow in 2 dimensions is significantly different from its behavior in higher dimensions. In 2 dimensions, we have seen that an appropriately normalized Ricci flow will exist for all t, and one hopes it will converge as $t \to \infty$ to a "nice" metric. In higher dimensions, the Ricci flow will usually not even exist for all time, which means that proofs using Ricci flow will employ rather different techniques than those which appear in proving uniformization.

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