Oka's theorem: a short proof

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A *complex analytic space* is a locally ringed Hausdorff space X, which is locally isomorphic (as ringed spaces) to a closed analytic subset of $\mathbb{C}^n$ defined by the vanishing of finitely many holomorphic functions. For example this includes $\mathbb{C}^n$ itself, closed analytic subsets of $\mathbb{C}^n$, and complex manifolds. Every complex analytic space X has a canonical sheaf $\mathscr{O}_X$ of holomorphic functions. In this note we give a straightforward and self-contained proof of the following result, originally due to Oka:

Theorem 0.1. *The sheaf $\mathscr{O}_X$ is coherent as a module over itself. In other words, given any open $U \subseteq X$ and a morphism*

$$\mathscr{O}_U^k \to \mathscr{O}_U,$$

its kernel is locally finitely generated.

The proof is standard and can be found for example in [2], but is usually given with substantial setup. Here we give a much shorter proof; it is closely adapted from [1], which proves an analogous result in the context of o-minimal geometry. The exposition is self-contained except for a few basic results from complex analysis, stated here and proven in chapter 0 of [3].

Definition 0.2. Write the coordinates of $\mathbb{C}^n$ as $z_1, \ldots, z_{n-1}, w$, and let $U \subseteq \mathbb{C}^n$ be any open set. A *Weierstrass polynomial* is a *monic* polynomial in $\mathscr{O}(U)[w]$ of the form

$$f = w^d + a_{d-1}(z)w^{d-1} + \cdots + a_0(z),$$

such that $a_i(0) = 0$ for all i.

For convenience, we sometimes refer an element of $\mathscr{O}(U)[w]$ (for some U) as a *w-polynomial.*

Theorem 0.3 (Weierstrass Preparation)**.** *Suppose f is a holomorphic function defined on a neighborhood of the origin in $\mathbb{C}^n$. If the function $f(0, w)$ is not identically zero, then on some neighborhood of the origin f can be uniquely written as*

$$f = g \cdot h,$$

where g is a Weierstrass polynomial and $h(0) \neq 0$.

Remark 0.4. If f is not identically zero, then there exist choices of coordinates which meet the conditions of Weierstrass preparation — otherwise, f would vanish identically on every line through the origin. By the same token, the set of directions in which Weierstrass preparation is not possible has empty interior.

Theorem 0.5 (Weierstrass division)**.** *Let $g(z,w)$ be a Weierstrass polynomial of degree k. Then for any holomorphic function f defined near the origin in $\mathbb{C}^n$, there is a unique expression*

$$f = g \cdot h + r,$$

valid on some neighborhood of the origin, where g is holomorphic and r is a w-polynomial of degree less than k.

1 The Proof

1.1 Reduction to $\mathbb{C}^n$

Let X be a complex analytic space. First, we reduce to the case where $X = \mathbb{C}^n$. Since the statement is local, we may assume that X is a closed analytic subset of an open U in $\mathbb{C}^n$, defined by the vanishing of finitely many holomorphic functions. Thus, X is the vanishing locus of a finitely generated ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_U$. Since the ideal sheaf $\mathscr{I}$ is finitely generated and also a subsheaf of the coherent sheaf $\mathscr{O}_U$, $\mathscr{I}$ is also coherent as an $\mathscr{O}_U$-module.

Suppose we have some map

$$\varphi\colon \mathscr{O}_X^k \to \mathscr{O}_X,$$

whose kernel we want to show is locally finitely generated as an $\mathscr{O}_X$-module. It is clearly sufficient to show that the kernel is locally finitely-generated as a $\mathscr{O}_U$-module. By general properties of coherent sheaves this would follow from coherence of $\mathscr{O}_X$ as a $\mathscr{O}_U$-module. Likewise, we have the short exact sequence

$$0 \to \mathscr{I} \to \mathscr{O}_U \to \mathscr{O}_X \to 0.$$

Since $\mathscr{I}$ is coherent, so is $\mathscr{O}_X$, and we conclude.

1.2 The induction

Throughout the section, let X be a nonempty open subset of $\mathbb{C}^n$. Suppose we are given a map $\varphi\colon \mathscr{O}_X^k \to \mathscr{O}_X$, defined by a tuple of holomorphic functions $(f_1, \dots, f_k)$ on X. Let $I = \operatorname{Ker}\varphi$ be the relation sheaf; then our object is to show that I is locally finitely generated around every point. Using Weierstrass preparation and division, we reduce the question to one about w-polynomials of bounded degree. This reduces the question to one about coherent sheaves in one dimension lower, and we conclude by induction.

Fix a point of $\mathbb{C}^n$. By changing coordinates, we can assume that this point is the origin, and that f_1 admits a Weierstrass preparation relative to the coordinates $z_1, \dots, z_{n-1}, w$ on $\mathbb{C}^n$. Suppose that $V \subseteq \mathbb{C}^n$ is a sufficiently small neighborhood of the origin, and let $U := \pi(V) \subseteq \mathbb{C}^{n-1}$ denote the corresponding neighborhood of the origin in $\mathbb{C}^{n-1}$. Then we have an expression

$$f_1 = P \cdot u,$$

where $u(0) \neq 0$ and $P \in \mathscr{O}(U)[w]$ is a Weierstrass polynomial. Since f_1 and P differ by a unit near the origin, when finding local generators for I we may assume without loss of generality that $f_1 = P$ is a Weierstrass polynomial of degree d.

Our next object is to use the Weierstrass division theorem to replace the other functions f_j by polynomials in $\mathscr{O}(U)[w]$ with bounded degree. Indeed, since f_1 is a Weierstrass polynomial, Weierstrass division implies that after possibly shrinking V further, the following holds for all $j \neq 1$:

$$f_j = q_j f_1 + r_j,$$

where q_j is a holomorphic function, and $r_j \in \mathscr{O}(U)[w]$ is a polynomial of degree less than d. It follows that the tuples $(f_1, \ldots, f_k)$ and $(f_1, r_2, \ldots, r_k)$ differ by an invertible affine transformation, and therefore have isomorphic sheaves of relations. Thus, we can assume that for all i, $f_i \in \mathscr{O}(U)[w]$ is a polynomial of degree at most d.

Observe that the sheaf I has the following sections defined on V:

$$\sigma_j = -f_j e_1 + f_1 e_j.$$

Proposition 1.1. *Any section of I is generated around 0 by the σ_j and by w-polynomials in I of degree at most d.*

Proof. Suppose we have a relation

$$g = \sum_i g_i e_i \in I$$

defined on some neighborhood of 0. By Weierstrass division, possibly shrinking the neighborhood, we have the following:

$$g_j = q_j f_1 + r_j,$$

where r_j is a w-polynomial of degree $< d$. Thus, by subtracting appropriate multiples of the σ_j, we can assume that the g_j are w-polynomials of degree at most d, for $j \neq 1$. Now, since

$$\sum_j g_j f_j = 0,$$

it follows that $Q := g_1 f_1$ is likewise a w-polynomial of degree at most $2d$. We claim that g_1 is a w-polynomial as well. Indeed, for a fixed value z_0, both $Q(z_0, w)$ and $f_1(z_0, w)$ are ordinary polynomials, of degree $\leq 2d$ and d respectively. Since their ratio is a holomorphic function g_1, it follows that the set of roots (with multiplicity) of f_1 is a contained in that of Q, which implies that f_1 divides Q as polynomials. It follows that for any fixed z_0, $g_1(z_0, w)$ is a polynomial of degree at most d. Since g_1 is holomorphic, its partial derivatives are as well. The function

$$g_{1,d} = \frac{1}{d!} \frac{\partial^d}{\partial w^d} g_1,$$

when restricted to fixed z_0, yields the top coefficient of the polynomial $g_1(z_0)$. By subtracting off $g_{1,d} w^d$ and iterating, we arrive at an expression for g_1 as a w- polynomial, which must have degree at most d. $\square$

Now, consider the map

$$\psi\colon (\mathscr{O}(U)[w]^{\leq d})^k \to \mathscr{O}(U)[w]^{\leq 2d},$$

given by

$$\psi(r_1, \ldots, r_m) = \sum_i f_i r_i.$$

Since degrees are bounded, this equates to a map

$$\psi\colon \mathscr{O}_U^{kd} \to \mathscr{O}_U^{d}$$

of finite free $\mathscr{O}_U$-modules. By induction, the kernel of ψ is locally finitely-generated as a $\mathscr{O}_U$-module. Picking generators R_i for $\mathrm{Ker}(\psi)$ on some small neighborhood of 0, they generate I along with the σ_j, and we conclude.

References

[1] Benjamin Bakker, Yohan Brunebarbe, and Jacob Tsimerman. o-minimal GAGA and a conjecture of Griffiths. *Invent. Math.*, 232(1):163–228, 2023.

[2] Hans Grauert and Reinhold Remmert. *Coherent analytic sheaves.* Springer-Verlag, Berlin, 1984.

[3] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.