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COSTLY INFORMATION ACQUISITION, STOCK PRICES AND NEOCLASSICAL GROWTH

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Abstract

An old question in economics concerns the role of the stock market: Is the stock market another Las Vegas, where a lot of people just "gamble"? Or do the activities of informed stock traders contribute in a useful way to the optimal allocation of resources by providing guidance for physical investment? A new model to adress this old question is analyzed in this thesis.

The approach taken here is to incorporate the endogeneously costly acquisition of information about future returns and the partial transmission of that information to the market via prices into an infinite horizon rational expectations neoclassical growth model. The key feature is that stock prices signal investment opportunities for physical investment and thus guide the allocation of resources. That is, the higher the stock price for a particular technology, the more likely is a high future dividend stream per capital unit, and thus, the more should be added to the physical capital using that technology.

However, this efficiency increasing effect is trading off against the redistributional effect, which arises because information acquirerers ("insiders") will earn a higher expected rate of return on their assets than the average. This tradeoff is analyzed for the model at hand with numerical

calculations. It turns out that information acquirerers perform a welfare increasing role only as long as their information is revealed sufficiently well to the market.

Among the theoretical insights, it is shown that a perfect market portfolio mutual fund rules out information acquisitional activities.

I. Introduction

The stock market is the focus of many recent policy debates. The Security Exchange Commission has stepped up their efforts in prosecuting "unfairly" informed traders. The number of highly skilled individuals attracted to Wall Street rather than to the manufacturing sector, say, is of concern to many. Various kinds of trading regulations for stocks have been proposed or even been put into effect.

The fundamental question that underlies much of this debate is actually an old one and can be phrased as follows: Is the stock market another Las Vegas, where a lot of people just "gamble"? Or do the activities of informed stock traders contribute in a useful way to the optimal allocation of resources by providing guidance for physical investment? To formulate it differently: given the arrangement of a stock market, is the presence of information acquirerers welfare improving? It is impossible to think about and judge policy regarding the stock market without taking a stand on this key question. Economic theory should enable us to guide our intuition on the answer. While some progress has been made to that end (see in particular Hayek (1945), Hirshleifer (1971) and Grossman (1977)), a computationally tractable general equilibrium type framework which can be fit to data and within which the question can be answered is still lacking.

This paper analyzes a new model to adress this old question. The approach taken here is to incorporate the costly acquisition of information about future returns and the partial transmission of that information to the market via prices into an infinite horizon rational expectations neoclassical growth model. The key feature is that stock prices signal investment opportunities for physical investment and thus guide the allocation of resources. That is, the higher the stock price for a particular technology, the more likely is a high future dividend stream per capital unit, and the more fruitful is it for outsiders without special knowledge of this technology to use real resources for the addition to the technology's capital stock.

However, the agents in the economy have to pay handsomely for that service of the stock market: information acquirerers ("insiders") will earn a higher rate of return on their assets on average than agents without special information ("outsiders"). Insiders do not participate in the production process in the model: the foregone wages are their (opportunity) costs of information. Taken together then, there is a tradeoff between the service provided by insiders (i.e. the information revealed in stock prices) and the losses they induce (e.g. the non–participation in the production process).

The paper develops a computationally tractable model in which this tradeoff can be analyzed. The model is a step in the direction of a computationally tractable general equilibrium framework which can be fit to data. While it assumes away aggregate uncertainty in order to make the analysis simpler (but at the same time remote from aggregate time series analysis), it works within a neoclassical growth model framework which has become the basis for much of modern business cycle analysis, see Kydland

and Prescott, (1982). Integrating aggregate uncertainty into the model is the next step on the agenda. It should be emphasized that the steady state growth rate is an exogeneous variable of neoclassical growth models. For that reason, the information acquisition here will only have effects on the level but not on the growth rate of the economy. Incorporating endogeneous growth could make the impact of insiders more dramatic and is left to future research.

One can view the model here as a version of Tobin's q with costly information acquisition and in an infinite–horizon neoclassical growth model. It is reasonable that there is a distinction in the resources spend on information acquisition in the stock market versus information acquisition concerning other forms of ownership of capital or claims to revenues and with other forms of financial intermediation. This paper focuses on information acquisition given the arrangement of a stock market. It might therefore help to shed light on the puzzle, that stock market prices are good predictors for investment, while data on q in general is not (see Barro (1989)).

One of the key difficulties in constructing the model is to find a inference mechanism on the stock market. It is necessary to understand why agents might want to spent resources (i.e., time) on predicting future prices of stocks instead of just inferring information from prices. Understanding why this is so is by no means trivial from a theoretical perspective. This point has been made forcefully by Radner (1979), Grossman and Stiglitz (1980), Tirole (1982), and Milgrom and Stokey (1982), culminating in versions of the No–Trade theorem. The theorem states that under certain conditions, no trade should take place just because of the injection of additional information. Although this theorem does not apply directly to our framework, since information is actually productive (it signals resource–allocation possibilities), it is nonetheless relevant here, since this productive nature of information is purely external to the information acquisition activities of the agent in this model.

Attempts have been made to generate inference mechanisms in which informed trades are both possible and profitable. There are (at least) five approaches to the problem: The first uses a one–shot market clearing in the stock market. Agents are prevented from trading contingent on their endowment shock or preference shock before the signal is realized [See, e.g., Grossman and Stiglitz (1980), Hellwig (1980), Verrecchia (1982), or Admati and Pfleiderer (1987)]. The second tells an explicit story of how stock market prices are set over time by a specialist, using a bid–ask spread [See, e.g., Glosten and Milgrom (1985) and Diamond and Verrecchia (1987)]. In the third, the informed insider explicitly recognizes the impact of his actions on the overall trades [See, e.g., Kyle (1985) and Gale and Hellwig (1987)]. For the fourth approach, markets become dynamically incomplete due to the acquistion of information, making it possible for some agent to exploit the lack of insurance possibilities of other agents for their own good (see Berk and Uhlig (1990)). A fifth approach does away with the common knowledge assumption, so that agents do not assume the same probability structure for the underlying randomness of the economy. All these models break the assumptions of the No–Trade theorem at some point. In the first three, a source of noise is introduced which prevents prices from completely revealing all information and which constitutes a source of income for the insiders. From the aggregate point of view taken here, the question of where the noise comes from moves into focus.

In this paper, the idea of the one–shot approach as e.g. in Verrechia (1982) is used. The difficulty with that framework is that it has been developed only for normally distributed random payoffs. In our model, however, the overall growth rate is essentially determined by the maximum outcome of these random variables. Furthermore, we would like to have a continuum of these random variables to avoid the difficulties that would be introduced if agents infer information from aggregate variables. It follows that independently and normally distributed random variables imply that our economy grows at an infinitely high rate. Thus, for the environment studied here, the distribution of growth rates must have compact support, and the closed form solutions as in Verrechia(1982) can no longer be used.

As in Verrecchia (1982), the No–Trade theorem does not apply because noise and stock–return information arrive simultaneously and complete insurance is ruled out. Noise enters in our model via the somewhat imperfect diversification attempts by agents without special knowledge (outsiders) when trying to save across periods. The economy does not start from Arrow – Debreu complete markets, but rather evolves over time with trading on the spot markets like in a Radner–equilibrium. Stock information is injected during the sequence of trades that otherwise might achieve the usual equilibrium.

The model displays the following features. Stock market prices partially reveal future growth rates of a particular technology. Information acquisition happens and "informed trades" are profitable, if the diversification attempts of outsiders are noisy. The number of information acquisitors is endogenous. Physical investment into the technologies is guided by the information revealed by stock market prices. Thus, the presence of information acquisitors allows a better allocation of resources to physical investment than would be possible without signaling through prices. However, given an allocation of capital, information acquisition diminuishes total output within the period, since information acquisitors do not participate in the physical production process.

Let me highlight three theoretical results of this paper. First, a Bayesian formula is derived which allows an agent to combine his own private information with information contained in stock prices to form his posterior beliefs (Theorem III.1). This formula might be of practical use. Secondly, noise trades and insider trades have to be "compatible" in equilibrium, i.e. there is a consistency condition that relates the insider trades and the noise (Theorem A.V.13, see also the discussion in section V). Essentially, insiders earn their higher returns from exploiting the noisy trades, i.e. noise traders have to be on the "wrong side" of the market on average. In order to keep the inference problem well posed, it should not be possible for outsiders to infer from prices on which side the noise traders traded: prices have to clear markets at the same time as transmitting information. Thirdly, if outsiders have access to a perfectly diversified "market portfolio" mutual fund, there can be no insiders (Theorem VI.1). The reason is in short as follows: in our model, all returns on assets are ultimately return on capital. But if outsiders can get the average return on total capital, a non–zero fraction of insiders cannot get a higher return on average!

The question we originally posed at the beginning can be answered within this model for any specific choice of parameters. We solve for the steady state equilibrium numerically and make welfare comparisons across different steady states of the economy – one with , one without insiders and information in prices – to evaluate the trade–off. Judging from the experiments, the results support the intuition that insiders perform a welfare–increasing role only as long as their information is revealed to the market sufficiently well. Examining the welfare effects on individual agents rather than just averaging shows that distributional effects cannot be ignored: it can easily happen that while the presence of insiders induces higher output as well as higher wages, making particularly poor and particularly rich agents better off, the vast majority of agents still favors outlawing insider trades since their welfare is lowered because of the lower returns on their portfolio in the absence of private stock market information.

There are eight sections, figures and six appendices. Section II describes a quick overview and the environment. Section III describes the market arrangement we impose. Section IV defines the steady state equilibrium. Section V "dissects" the model in order to gain an understanding of the basic mechanics and to gain insights in how to compute the model. Also it discusses the consistency result. Section VI states that a market–portfolio mutual fund rules out costly information acquisition. Section VII describes the numerical results. Section VIII concludes. All figures follow section VIII. The discussions of figures 1.1.1 to 5.4.2 can be found in the section for the numerical results, part VII, whereas figures 6.1.1 to 6.8.2 are referred to in appendix VI.

Appendix I briefly describes the law of large numbers for continuum economies. Appendix II contains a table of notation for the main body of the text. Appendix III offers a table for the sequence of events within a period. Appendix IV contains the transformation of the value–function to eliminate the growth rate and gives a formula to allow meaningful average – welfare comparisons across experiments. Appendix V analyzes the model theoretically as far as possible. Appendix VI describes in greater detail the design of the computer program to calculate equilibria numerically. References follow the last appendix.

II. The Environment.

A quick overview is in order before we describe the model in detail. The assumptions made in this model are largely dictated by the key feature of incorporating costly information acquisition at the stock market and the signalling of opportunities for physical investment due to the revealed information. The model aims at simplicity, given this key feature, although it might seem complicated at first. We will try to motivate some of our assumptions below.

The basic structure is that of a neoclassical growth model: this is the most commonly used model to analyze physical investment in an infinite horizon economy. There is a continuum of technologies τ. The productivity parameter $\gamma_{\tau t}$ of a technology can grow at two different rates independently across technologies and time. By using a law of large numbers there are no aggregate uncertainty and no aggregate fluctuations in the model. That way, the inference problem agents face does not have to include aggregate variables since they do not contain any information. This simplifies the model.

The same output good can be produced with any technology using technology–specific capital and labor as input. Aggregate output is split between consumption and investment at time t. Non–negative investment can be undertaken in the (old) capital committed to a specific technology τ, resulting in new capital, which becomes productive in the next period.

Stocks are ownerships of capital, and dividends are capital shares. This is just the usual neoclassical growth model structure.

Agents are endowed with one unit of time per period each. Each period, they can choose to devote this unit of time to participate in the physical production process for a wage ("outsider") or to acquire information ("insider"). The opportunity costs of foregone wages are the costs of acquiring information. Independently from each other, insiders will then learn a private message about the new growth rate of the productivity of a particular technology. Exploiting their message, insiders will engage in stock market trades beyond the "liquidity trades" which accomodate the usual savings decisions of agents from period to period. Since agents individually face risky streams of income (with risky portfolios), a representative agent analysis seemed to be too complicated (though potentially still possible, see e.g. Prescott and Rios–Rull (1988)). Instead, we use many agents. Again for simplicity, there is a continuum of agents with different asset–holdings and a constant relative risk aversion utitity function for per–period consumption. In steady–state equilibrium then, we only need to consider stationary asset distributions to describe our population of agents. There will also be a continuum of insiders per stock. That way, we avoid complications due to information monopoly power (for a discussion of these issues, see e.g. Gale and Hellwig, 1988).

Noise enters in the way agents pick a stock for their portfolio at random. Their access to diversification is imperfect: they can buy a mutual fund which makes the same "errors" in picking a stock as individual agents do. The mutual fund here only plays the role of modelling the attempts at diversification of uninformed agents and allows the analysis to proceed with the consideration of only very low–dimensional portfolio choices. That is, the mutual fund gets rid of the risk due to the idiosynchratic shock in each technology, but it does not get rid of the noise inherent in the stock choices of agents. We need this noise to make costly information acquisition and incomplete revelation possible: this is discussed in section VI. The noise here is generated via randomness in the demand for stocks rather than randomness in the supply of stocks: this assumption is more reasonable since the number of shares is actually a number that is easy to find out for any publicly traded company.

Due to the noise, the information available to the insiders will be revealed only incompletely to the market: on average, higher stock prices signal higher productivity next period. That information can be used productively, since physical investment (undertaken by a competitive investment sector) takes place at the same time. Thus, by equating margins, more investment will flow to the capital with higher – priced capital and thus to the capital which will on average be more productive next period. It is here, where the insiders provide a positive informational externality to the market.

The realized growth rate for a particular technology can then be read off the actual production in the next period, i.e. insiders have an informational advantage for only one period and the externality lasts only one period. Thus we may as well assume, that all past uncertainty is resolved at the beginning of the next period. Extending the uncertainty across multiple periods would make the model harder to analyze, but could at the same time enhance the impact of insiders.

A period then has roughly the following structure (see alse Appendix III). Observe that the key ingredient is the stock market in the fifth part:

1. All past information becomes public and agents can price their portfolios without assymetric information.
2. Insider–/Outsider–decision
3. Physical production. Wage– and dividend payments.
4. New random variables are drawn. Insiders learn their message.
5. The stock market. Agents make their consumption–/savings–decision. Insiders exploit their superior information. Stock prices aggregate that information only incompletely due to noisy mutual fund demands. Physical investment happens and is directed more towards the higher – priced capital, thus exploiting the informational externalities of the insiders.
6. Agents consume.

In steady state, the distribution of capital across the various types of technologies, the foregone wage as the opportunity cost of information as well as the asset distribution and the split into outsiders and insiders will be generated as part of the solution.

For the analysis of this model as well as for numerical solutions, observe that part 5 of a period requires to solve for the information contained in prices, given the noisy demands by the mutual fund. It turns out, that this is almost impossible to do. The theoretical reason for that is the consistency condition, which is discussed in section V. Instead, we will impose the information revealed by prices as a parameter and solve for the noisy demands of the mutual fund consistent with the amount of revealed information in equilibrium. This is an application of the backsolving idea, see Sims (1984, 1989, 1990).

Certain restrictions are imposed on the behavior of agents. These restrictions will not be derived from first principles and some initial environment. This allows us to keep the model simple and to get interesting results. These restrictions are:

1) The value of all assets an agent holds has to be nonnegative at all times, in particular in part 1 of the period. This is the form of a borrowing constraint as introduced in Foley and Hellwig (1975) or Scheinkmann and Weiss (1986) and guarantees bounds to risky speculations.
2) Agents are prohibited from contracting on the side to use their observations for other traders. A careful discussion of side trades or contractual arrangements other than the markets postulated in this

model would be well beyond the scope of this paper.

3) Agents can buy only limited portfolios (e.g. due to some unmodelled broker–fees for trading in arbitrarily diversified portfolios). We make the extreme assumption that agents can hold at most two types of assets: a particular stock and a possibly imperfect mutual fund. This allows the introduction of noise into the model, and noise is necessary in order to have insiders at all (see section VI).

We will now describe the economy in detail. For convenience a table of notation and a table for the sequence of events within a period have been assembled in Appendix II and III.

The stochastic nature of the environment is completely spelled out in this and the following section. However for the definition of a stationary equilibrium in section IV, we appeal to a law of large numbers (see Appendix I) and only deal with aggregate descriptions of our economy. The description of the environment serves to motivate the definition of a steady state equilibrium in section IV.

a. Preferences and Endowments.

There are countably many time periods, $t = 0, 1, 2, \ldots$ There is one consumption good c per period. There is a continuum of infinitely lived agents $j \in [0,1]^2$. They care about a consumption stream $c_j = (c_{jt})_{t=0}^{\infty}$ via expected utility

$$U(c_j) = E_0[\sum_{t=0}^{\infty} \beta^t u(c_{jt})],$$

where

$$u(c) = \frac{c^{1-\eta} - 1}{1 - \eta}$$

is the utitity function with constant relative risk aversion parameter $\eta > 0$, $\eta \neq 1$, and $0 < \beta < 1$ is a discount factor .

Agents are endowed with one unit of time each period, which they supply inelastically. If used in physical production, this unit of time has productivity N_{jt}, i.e. the worker can provide N_{jt} standard units of labor. N_{jt} is drawn iid across time and agents from some distribution F_N with $E[N] = 1$. The random variable N is introduced to ensure a mixing behaviour for the stochastic process of the asset variable a_{it}, to ensure a resupply of insiders in every period and to ensure that the equilibrium interest rate is not big enough as to make insiders become ever richer.

Agents are also endowed initially with asset holdings a_{j0} drawn from some distribution F_a on $\mathbb{R}_{++}$. For now, we regard a_{j0} as claims to the consumption good at time 0. Finally, there is a random variable $\tau_{jt} \in [0,1]$, which assigns each agent to a technology τ (see section III).

<u>b. Technologies.</u>

There is a continuum of technologies, indexed by $\tau \in [0,1]$. For each technology and period in time, there is a technology specific capital good, the total physical units of which are denoted by $k_{\tau t}$. $k_{\tau t}$ is produced in period t ("new" capital) and is productive in period t+1 ("old" capital).

In each period, each technology is used by a competitive market of production firms for the production of the same output good. Denote with $n_{\tau t}$ the total input of labor units for technology τ at time t. Total output from technology τ is then given by

$$y_{\tau t} = \gamma_{\tau t}\, k^{\rho}_{\tau t-1}\, n^{1-\rho}_{\tau t}$$

where $\gamma_{\tau t}$ is the productivity specific to technology τ and $0 < \rho < 1$ is the capital share. Aggregate output $\bar{y}_t$ is then given by

$$\bar{y}_t = \int y_{\tau t}\, d\lambda(\tau)$$

with λ the Lebesgue measure and the integral here as in the sequel is to be read as a Pettis–integral or a nonstandard sum over an appropriate hyperfinite grid (see Appendix I or Uhlig (1987)). Let

$$\bar{c}_t = \int c_{tj}\, d\lambda^2(j)$$

denote aggregate consumption and

$$\bar{x}_t = \int x_{\tau t} \, d\lambda(\tau)$$

aggregate investment, where $x_{\tau t}$ is total non–negative investment into τ–specific capital $k_{\tau t-1}$. Aggregate output can be used for either consumption or investment, i.e feasibility requires that

$$\bar{x}_t + \bar{c}_t \leq \bar{y}_t.$$

The capital for the next period t+1 ("new" capital) is produced by a competitive market of investment firms according to

$$k_{\tau t} = f(k_{\tau t-1}, x_{\tau t}),$$

where f is linear homogeneous of degree 1. Some further assumptions about f are made in appendix V of this paper. For the numerical examples, we use in particular

$$f(k,x) = (\kappa_1 k^\alpha + \kappa_2 x^\alpha)^{1/\alpha}$$

with $0 < \alpha < 1$, $\kappa_1 > 0$ and $\kappa_2 > 0$. Observe that with $\alpha = 1$, the function

$$f(k,x) = \kappa_1 k + \kappa_2 x$$

gives rise to the usual linear investment technology and $\kappa_2 = 0$, i.e.

$$f(k,x) = \kappa_1 k$$

corresponds to "Lucas–trees" (see Lucas (1978)).

Total aggregate ("old") capital in physical units is denoted by $\bar{k}_t$,

$$\bar{k}_t = \int k_{\tau t}\, d\lambda(\tau).$$

Observe that we add up "apples and oranges" in this equation, since capital specific to a technology τ cannot be used for another technology. Also, $\bar{k}_t$ does not correspond to aggregate capital in an accounting sense, since we simply add up physical units and do not weigh by their market value. $\bar{k}_t$ is simply a number helpful for our further analysis.

Initial "old" capital $k_{\tau,-1}$ in each technology is assumed to be the same amount k_{-1} across all technologies. Hence, $\bar{k}_{-1} = k_{-1}$.

c. Information.

We need to draw the random growth rates of the productivity parameter as well as the messages, insiders will receive about these random growth rates. Since we proceed in a "backsolving mode", we will also draw an index which will correspond to the information revealed by prices in equilibrium.

Thus, in each time period t and for each technology τ, a random variable

$$Z_{\tau t} = (g_{\tau t}, i_{\tau t}, (m_{j\tau t})_{j \in [0,1]^2})$$

is drawn iid from some distribution F_Z on the domain dom_Z. The parts of Z are

$g_{\tau t}$ $\in \{ 0, 1 \}$ is the index for the random **growth** rate $\Gamma_{g_{\tau t}}$ of the productivity parameter of technology τ. We assume

$$\Gamma_0 > \Gamma_1 > 0$$

$i_{\tau t}$ $\in \{ 0,..,I \}$ together with g <u>indexes</u> the noise in the demand for a particular stock. This is modelled as the noisy part of the mutual funds asset holdings decision, see section III. The index is correlated with g, i.e. provides an **instrument** for g. In equilibrium, it also indexes the publicly available **information** about the growth rate $\Gamma_{g_{\tau t}}$ of a stock as revealed by prices,

$m_{j\tau t}$ $\in \{ 0,..,M \}$ describes the **message** agent j receives if he decides to be an "insider" in period t and if technology τ is assigned to him.

Thus,

$$dom_Z = \{ 0, 1 \} \times \{ 0,..,I \} \times \{ 0,..,M \}^{[0,1]^2}$$

with the usual product–σ–algebra.

We let $\overline{\gamma}_t$ denote the maximal productivity parameter at time t and across all technologies τ. I.e.

$$\overline{\gamma}_t = \max\{ \gamma_{\tau t} \mid \tau \in [0,1] \}$$

and we restrict ourselves to equilibria in which $\overline{\gamma}_t$ exists. We will restrict ourselves to economies, in which the fraction of technologies with $\gamma_{\tau t} = \overline{\gamma}_t$ is not zero (and this restriction has to be compatible with the stationarity of the equilibrium). It follows by the law of large numbers, that

$$\overline{\gamma}_t = \Gamma_0^t \, \overline{\gamma}_0.$$

Observe that it is here where we need an upper bound on the distribution of growth rates.

Initial productivity levels $\gamma_{\tau 0}$ are assumed to be

$$\gamma_{\tau 0} = \xi_{1_{\tau 0}} \, \overline{\gamma}_0$$

where the nonnegative integer $1_{\tau 0}$ is drawn at random and iid across technologies according to some distribution F_1 on the nonnegative integers and where

$$\xi_l = (\Gamma_1 / \Gamma_0)^l.$$

As a result, the only values $\gamma_{\tau t}$ can take are products of $\overline{\gamma}_t$ with an integer power of (Γ_1 / Γ_0):

$$\gamma_{\tau t} = \xi_{l_{\tau t}} \overline{\gamma}_t.$$

We call $l_{\tau t} = l$ the __level__ of technology τ at time t. Since the production sector of the economy is competitive, "names" of technologies do not matter – the only relevant characteristic at the beginning of a period is the level of a technology. Thus let

$$F_{kt}(l) = \int 1_{\{l_{\tau t}=l\}} k_{\tau t} \, d\lambda(\tau) / \overline{k}_t$$

be the fraction of total capital which is specific to technologies on level l at time t. We can and will proceed as if all technologies are renamed so that $k_{\tau t} \equiv \overline{k}_t$ and $F_k(l)$ is the fraction of all technologies $\tau \in [0,1]$ on level l (e.g. imagine that firms reorganize each period so that all firms on the same level redistribute their capital and then start on their own technology path for next period independently from the other level–l firms). We note that $F_{k0} = F_k$.

We assume that the indices $g_{\tau t}$ and $i_{\tau t}$ are drawn independently across technologies according to some probability distribution

$P(\, i, g \mid l\,)$

Furthermore, we assume that different insiders in the same technology receive messages independently from each other according to some probability distribution

$$P(\, m \mid l, i, g\,) = P(\, m \mid g\,).$$

I.e. we also assume that the probabilities for a message only depend on the random growth rate of a technology.

We will always assume that there is a continuum of agents per technology whose asset distribution and whose split in insiders and outsiders mirrors the corresponding aggregate distribution. Strictly speaking, this requires a Fubini–type of law of large numbers, which is not proven in Appendix I. We do not rely on the mathematics of the law of large numbers to formulate our equilibrium, however, but rather define an equilibrium directly as if the law holds. In particular, the combined knowledge of the insiders in a technology would completely reveal $\Gamma_{g_{\tau t}}$ (unless of course there are no insiders). That information will be only partly revealed via prices (see section III). We will be interested in equilibria in which the random information index $i_{\tau t}$ but not the growth rate $\Gamma_{g_{\tau t}}$ is revealed via prices. I.e. in equilibrium the level $l_{\tau t}$ and the parameter $i_{\tau t}$ of a technology will be public information. We call (l,i) the **type** of a technology and (l,i,g) the category of a technology. The type of a technology will be known at the end of period t, but the category will not be revealed until the beginning of the next period. We restrict the analysis to equilibria that are symmetric in the sense that technologies of the same category are treated in the same way, technologies of the same type are priced the same etc.

III. Markets, Portfolios, Structure of a Period and Agents' Maximization Problem.

Instead of deriving the market structure endogeneously, we will impose a certain structure on our economy. We will rule out certain types of insurance arrangements or information sharing arrangements. Both can be motivated by alluding to the fact that information is costly in this model. Thus these trading restrictions might be thought of as resulting from further specifications of the costly informational requirements for engaging in contracts other than the ones described below.

We assume that there are only two types of assets trades: stocks, representing ownership of a unit of physical capital in a particular technology, and shares of a mutual fund, which we will describe in some more detail below.

a. Structure of a Period.

A period t consists of six consecutive parts, enumerated by roman letters I,II through VI. For a quick overview, see Appendix II.

In part I, the growth rates $\Gamma_{g_{\tau t-1}}$ (and hence the category $(l_{\tau t-1}, i_{\tau t-1}, g_{\tau t-1})$ become public information. A market in ownerships of capital opens (stock market 1) and prices $q_1(l)$, $l = 0,1,..$ for one unit of ("old") capital on level l in terms of the period–t–consumption good are set.

Agents thus know the total value of their beginning–of–the–period asset holdings a_{jt}. We prohibit agents at this point from ever holding portfolios with negative values, i.e we restrict agents to

$$a_{jt} \geq 0.$$

This is a form of borrowing constraint as e.g. used in Foley and Hellwig (1975) or Scheinkmann and Weiss (1986) and introduces risk aversion with respect to holding assets for one period. In equilibrium, agents will sell all their individual stocks to the mutual fund and only hold shares of the mutual fund, thus insuring themselves against technology–shocks within the period.

In part II. agents can enter a betting market in which they can trade : asset holdings for any actuarialy fair lottery. Agents that buy a lottery ive (non–negative) random amount of assets whose mean is the purchase :e of the lottery. This part of the period exists for purely technical sons to ensure the concavity of the value function of agents (lotteries are en used to ensure convexities, see e.g. Prescott and Rios–Rull (1988)). ;ents then decide after the outcome of possible lotteries, whether they want be an "insider" or an "outsider".

In part III, the agent–specific labor–productivity shock N_{jt} is :ealized. There is a market in labor. Production of output takes place. Wages w and capital shares d(l) are paid. "Insiders" are busy acquiring information.

In part IV, the technology–specific random variables $Z_{\pi t}$ are realized. Agents pick their stock that they might include in their portfolio (see below). Insiders learn their message m about their stock.

In part V, a market in ownerships of old and new capital opens with prices $q_2(l,i)$ for a unit of old capital of type (l,i) and $q_3(l,i)$ for a unit of new capital of type (l,i) (stock market 2). A consumption good/ investment good – market opens. A competitive sector of investment firms buys all old capital and buys the investment good to produce the new capital which they sell. Agents and the mutual fund make their portfolio– and saving/consumption–decisions.

In part VI, agents consume and the period ends.

b. Available Portfolios.

We now spell out the portfolio choices available to agents: restrictions are imposed to introduce noise on the stock market.

In "reality" we can observe that it is costly to trade in many assets at the same time: broker fees have to be paid, time is involved. There are indivisibilities: it is not possible to buy arbitrary fractions of one IBM–stock say. We also observe that agents typically are not extremely diversified, but hold only few assets, some of which might be shares of a diversified basket of assets.

Instead of modelling explicitely the costs of diversification, we restrict agents directly to not holding more than an a priori fixed number of different assets in their portfolio: this allows us to proceed with a simpler although not quite "micro–founded" model. Proceeding this way, we use the extreme assumption that agents can hold at most two assets ("two" instead of "one" still allows for some non–trivial portfolio decisions): a stock and a mutual fund.

A stock, i.e. one unit share, of technology τ at time t is a certificate of ownership of one unit of capital in technology τ at time t. At the second stock market, i.e. in part V of the period, old stocks are in effect exchanged to new stocks at the rate of the marginal contribution of old capital to the production of new capital in that technology. This "exchange" of stock should not be taken too literally: choosing a unit for a stock is just a matter of accounting.

We now describe the way an agent picks a stock he might include in his portfolio in part IV of the period before the second stock market opens in part V. It is at this point, where noise possibly enters the model. We assume that agents pick a stock of category (l,i,g) randomly with probability π(l,i,g), where the random wheel π is part of the nature of the economy. At this time in the period, the agent can only observe the level of a technology. The agent can reject the technology and pick again at random according to the probabilities π and so on until he finds a level l he likes. Hence, to keep things simple, we restrict our analysis to probability distributions so that agents are indifferent between stocks of different levels, i.e. that they are indifferent between stocks based on public information up to that point. It follows from this restriction that the conditional probability $\pi(i \mid l)$ for drawing a parameter i and the conditional probability $\pi(g \mid l,i)$ for the growth rate Γ_g does not depend of the level l. I.e. we have

$$\pi(l,i,g) = \pi(l)\pi(i)\pi(g \mid i).$$

Further implications of this assumption are analyzed in section V of this paper.

However we do not assume that the probabilities $\pi(l,i,g)$ coincide with the distribution of capital units across categories (which have to be scaled appropriately for that comparison to account for the price–differences, see appendix V). Instead agents pick their technologies with an "unbalanced" wheel, throwing them off a "fair" choice. This unbalanced wheel π is part of the nature of the economy and cannot be affected by actions of the agents. Another way to motivate this wheel is to consider a "fair" wheel the benchmark model of completely rational agents and to consider an unbalanced wheel as a deviation from rational stock – picking, which is not further explained in this model (but where agents then proceed to price assets well knowing that they picked stocks in an irrational fashion). Reasons why agents might pick stocks in irrational, unbalanced ways are given e.g. in Shiller(1984), who argues that agents follow trends and fashions in their portfolio decisions.

The mutual fund, which we are going to describe now, is the only other asset in this economy. This asset has been included, since it pulls an interesting aspect of the model out into the open and since it simplifies the model. The mutual fund holds a very diversified portfolio, thus yielding a rate of return with certainty by the law of the large numbers. The mutual maximizes this rate of return. However, the technology for diversification on the second stock market for this mutual fund is as imperfect as the choice of stocks by the agents. More precisely, we assume that the mutual fund chooses numbers $\varphi_t(l,i)$ to buy

$$\varphi_t(l,i,g) = \varphi_t(l,i)\ \pi(\ l,i,g\ )$$

units of all the (new) capital of technologies of category (l,i,g). Observe that the mutual fund can decide not to buy a certain type (l,i) at all (via $\varphi(l,i) = 0$) or to sell it short (via $\varphi(l,i) < 0$). As a story, one might envision that the mutual fund only operates by asking each agent to buy on account of the fund an amount $\varphi(l,i)$ of shares of the technology picked by him and to put these shares into a pool available to the fund. Thus the noise is a part of the nature of this procedure. We assume that neither agents nor the mutual fund itself can see the fraction $\varphi(l,i,g)$ the mutual fund buys. Just $\varphi(l,i)$ is public information. Agents can buy shares of this mutual fund.

By the law of large numbers, the mutual fund earns a sure return on all its stock holdings of technologies of type (l,i). This return is given by

$$R_t(l,i) = \frac{\sum_{g=0}^{1} \pi(\, g \mid i \,)\, q_{1,t+1}(l+g)}{q_{3,t}(l,i)}.$$

Since the mutual fund maximizes its return, we must have

$$R_t(l,i) = R_t$$

for any technology–type (l,i) included in the mutual fund, where R_t is the overall return on the mutual fund.

We assume that there is no noisiness in the portfolio decisions of the mutual fund between part I and part V of the period and that the return on any included level has to be equal to 1 to keep the analysis simple.

In this model, agents would like to diversify at least part of their portfolio, since they are risk–averse: this is what the mutual fund allows them to do although "imperfectly". It will be shown below, that outsiders would like to hold the "market" if they could. In absence of a mutual fund but with other types of restrictions on complete, perfect diversification (like e.g. the above–mentioned transaction costs etc), we might substitute the phrase "mutual fund" by "net result of the possibly noisy diversification attempts by agents". This is the function and role of the mutual fund in this model rather than an explanation for how mutual funds actually behave. The noisiness in the diversification attempt will allow the existence of insiders and the incomplete revelation of the aggregate insider information to the market.

In equilibrium, the aggregate insider demand for a stock, the demand of the mutual fund and the private demand of outsiders (which will turn out to be zero) has to sum up to the total supply of a category of a stock to clear the market: this will put restrictions on the the probability structures in the model and generate the consistency condition. This condition is discussed in section V as well as a more direct way to think about the asset holdings of the mutual fund and the random wheel π.

In steady state, the shares hold by the mutual fund move with the growth rate ζ of the economy. I.e. we have

$$\varphi_t(1,i) = \zeta^t \varphi(1,i).$$

c. The Decision Problem of the Agent.

Let us now analyze the maximization problem, an agent faces in period t, who holds assets valued at a_{jt} (in terms of the period t consumption good) at the beginning of part II of the period. He takes as given prices, wages, interest rates and the distribution of the various variables. First, the agent can buy a fair lottery on assets and receives the outcome. The agent then chooses to become either an outsider or an insider, makes his consumption/investment decision in part V and ends up with asset holdings a_{jt+1} at the beginning of the next period. Since the mutual fund delivers a sure return R_t, it acts like a bond as far as the portfolio decisions of an agent are concerned. We thus denote the value (in terms of period–t consumption) of the mutual fund shares an agent hold by b_{jt}. We reserve the word stock holdings, denoted s_{jt}, for the units of new capital agent i buys in the assigned technology τ_{jt}. We restrict ourselves to solutions to the agent problem, in which only his asset holdings and aggregate state variable matter. I.e. suppose we have a list $\aleph_t$ of state variables which describes the aggregate state of the economy at the beginning of period t. We then look for value functions v, v^{ins}, v^{outs}, which satisfy

$$v^{ins}(a,\aleph_t) = E_{(1,i,m)}[\max_{c\ ,b,s} \{ \frac{c^{1-\eta}-1}{1-\eta} + \beta E_g[v(a',\aleph_{t+1}) \mid i,m] \mid$$
$$c + q_{3,t}(l,i)s + b \leq a,$$
$$0 \leq a' = R_t b + q_{1,t+1}(1+g)s \ \}], \tag{3.1}$$

$$v^{outs}(a,\aleph_t) = E_{(N,l,i)}[\max_{c\ ,b,s} \{ \frac{c^{1-\eta}-1}{1-\eta} + \beta E_g[v(a',\aleph_{t+1}) \mid i] \mid$$
$$c + q_{3,t}(l,i)s + b \leq a + w_t N,$$
$$0 \leq a' = R_t b + q_{1,t+1}(1+g)s \ \}], \tag{3.2}$$

and

$$v(\hat{a},\aleph_t) = \max_{\mu}\{ \int \max\{ v^{ins}(a,\aleph_t) , v^{outs}(a,\aleph_t) \} d\mu(a) \mid$$
$$\mu \text{ is a probability distribution on } \mathbb{R}$$
$$\text{with } \int a \, d\mu(a) = \hat{a} \ \} \tag{3.3}$$

For the analysis in appendix V, we assume as a tie–breaking rule, that agents hold the portfolio with the minimal amount of variance, if they are indifferent between several portfolios.

Note, that we used our restriction to symmetric treatment of technology categories in the formulation of the insiders problem, i.e. his maximization problem only depends on the type (l,i) of the chosen technology and his message m.

The expectations in the expressions above are taken with respect to the probabilities $\pi(g \mid l,i)$ for an outsider resp. $\pi(g \mid l,i,m)$ for an insider. The following formula gives the correct probabilities the insider infers from his knowledge. Note, that this formula only requires the insider to know the probabilities of the various growth rates based only on his private message as well as based only on public information i. This formula might therefore be of practical use: one just has to take a guess at these distributions as "input", the formula takes care of the rest.

<u>THEOREM III.1:</u>

Assume that $\pi(g) \neq 0$ for all g, where $\pi(g)$ is defined below. Assume that $P(m|g) \neq 0$ for at least one growth rate g such that $\pi(g|i) \neq 0$. The probability $\pi(\bar{g}|i,m)$ of growth rate $\bar{g}$, given the information i contained in stock prices and the private message m is calculated from the distributions for the growth rates $\pi(g \mid m)$, $\pi(g \mid i)$ and $\pi(g)$ (that is, given only the message m, given only public information i, and given no information at all) according to

$$\pi(\bar{g}|i,m) = \frac{\pi(\bar{g}|i)\ \pi(\bar{g}|m)\ /\ \pi(\bar{g})}{\sum_{g=0}^{1} \pi(g|i)\ \pi(g|m)\ /\ \pi(g)},$$

where

$$\pi(g) = \sum_{i=0}^{I} \pi(g|i)\ \pi(i).$$

PROOF:

Observe, that this probability is independent of the level l of the technology by our assumptions about π. By Bayes' rule, we have

$$\pi(\bar{g}|i,m) = \frac{P(m|\bar{g},i)\ \pi(\bar{g}|i)}{\pi(m \mid i)}, \tag{3.4}$$

where

$$\pi(m \mid i) = \sum_{g=0}^{1} P(m|g,i)\ \pi(g|i).$$

Furthermore by Bayes' rule,

$$\pi(g \mid m) = \frac{P(m|g)\quad \pi(g)}{\pi(m)}$$

or

$$P(m|g,i) = P(m|g) = \frac{\pi(g|m)\ \pi(m)}{\pi(g)}, \tag{3.5}$$

where

$$\pi(m) = \sum_{g=0}^{1} P(m|g)\ \pi(g).$$

Substituting (3.5) into (3.4) yields the result.•

Below, we will concentrate entirely on steady state equilibria of our economy. Then $\aleph_t \equiv \aleph$ and we can delete $\aleph_t$ and time subscripts in (3.1) to (3.3). Furthermore, there will be a constant growth rate ζ. In Appendix IV, we show that any solution v^{ins}, v^{outs}, v to the following "transformed" problem yields a solution (with the same decision rules) to the original problem described in equations (3.1) to (3.3) in steady state:

$$v^{ins}(a) = E_{(l,i,m)}[\max_{c\,,\,b,s} \{ \frac{c^{1-\eta}-1}{1-\eta} + \zeta^{1-\eta}\beta E_g[v(\tilde{a}') \mid i,m] + \frac{\beta}{1-\beta}\frac{\zeta^{1-\eta}-1}{1-\eta} \mid c + q_3(l,i)s + b \le a, \; 0 \le \tilde{a}' = R\, b/\zeta + q_1(l+g)s/\zeta \; \}] \tag{3.6}$$

$$v^{outs}(a) = E_{(N,l,i)}[\max_{c\,,\,b,s} \{ \frac{c^{1-\eta}-1}{1-\eta} + \zeta^{1-\eta}\beta E_g[v(\tilde{a}') \mid i] + \frac{\beta}{1-\beta}\frac{\zeta^{1-\eta}-1}{1-\eta} \mid c + q_3(l,i)s + b \le a + w\,N, \; 0 \le \tilde{a}' = R\, b/\zeta + q_1(l+g)s/\zeta \; \}] \tag{3.7}$$

and

$$v(\hat{a}) = \max_{\mu}\{ \int \max\{ v^{ins}(a)\,,\, v^{outs}(a) \}\, d\mu(a) \mid \mu \text{ is a probability distribution on } \mathbb{R}_+ \text{ with } \int a\, d\mu(a) = \hat{a} \}.$$

We denote the decision rules by $c^{ins}(a,l,i,m)$, $c^{outs}(a,N,l,i)$, $b^{ins}(a,l,i,m)$, $b^{outs}(a,N,l,i)$, $s^{ins}(a,l,i,m)$, $s^{outs}(a,N,l,i)$ and $\mu_{\hat{a}}$, where $\mu_{\hat{a}}$ denotes the choice of the lottery given initial assets $\hat{a}$. These decision rules result in nonnegative random asset holdings next period denoted by $\tilde{a}'(\hat{a})$ in the transformed problem.

Since v is the concavification of the maximum of two functions, which are concave themselves, the choice of the lotteries can be made simpler: we show in appendix V, that it is enough to consider only lotteries which randomize over two points at most. I.e. we can write the decision problem for the lottery as

$$v(a) = \max_{a_i, a_o} \{ P_o\, v^{outs}(a_o) + P_i\, v^{ins}(a_i) \mid \text{either } 0 \le a_o \le a_i \text{ or } 0 \le a_i \le a \le a_o, \text{ and } P_o, P_i \in [0,1], \text{ so that } P_o + P_i = 1 \text{ and } P_o\, a_o + P_i\, a_i = a \}. \tag{3.8}$$

We denote the resulting decision rules by $a_o(a)$, $a_i(a)$, $P_o(a)$ and $P_i(a)$.

IV. Equilibrium

Definition:

A **steady state equilibrium** is a vector (ζ, $\underline{a}$, F_a, F_a^{ins}, F_a^{outs}, F_k, v^{ins}, v^{outs}, v, a_o, a_i, P_o, P_i, c^{ins}, c^{outs}, b^{ins}, b^{outs}, s^{ins}, s^{outs}, μ, $\tilde{a}'$, R, w, q_1, q_2, q_3, $\bar{k}$, $\bar{y}$, $\bar{n}$, $\bar{x}$, $\bar{c}$, x, n) consisting of

- a growth rate ζ,
- a cut–off level $\underline{a}$,
- asset distributions F_a (pre–lottery) and F_a^{ins} and F_a^{outs} (post–lottery),
- distribution of old capitals over levels F_k,
- value functions v^{ins},v^{outs},v,
- decision rules c^{ins}, c^{outs}, b^{ins}, b^{outs}, s^{ins}, s^{outs}, a_i , a_o, P_i, P_o
- "next–periods" random asset holdings $\tilde{a}'$, corrected for the growth rate,
- interest rate R,
- wage w,
- pricing functions q_1, q_2 and q_3,
- aggregate capital $\bar{k}$ (equal to initial capital $\bar{k}_{-1}$), output $\bar{y}$, labor $\bar{n}$, investment $\bar{x}$ and consumption $\bar{c}$,
- investment rules x and labor hiring rules n per unit of capital,

such that the following conditions (i) through (vi) are met:

(i) the problem of the agent (3.6), (3.7), and (3.8) is solved,

(ii) firms maximize profits. I.e. we have

– for the **production firms**:

$$d(l) = \max_n \left(\xi_l \gamma_0 n^{1-\rho} - wn \right), \tag{4.1}$$

$$n(l) = \operatorname{argmax}_n \left(\xi_l \gamma_0 n^{1-\rho} - wn \right) \tag{4.2}$$

$$y(l) = \xi_l \gamma_0 n(l)^{1-\rho}, \tag{4.3}$$

– for the **investment firms**:

$$q_2(l,i) = \max_x \left(q_3(l,i)\, f(l,x) - x \right),$$

$$x(l,i) = \operatorname{argmax}\left(q_3(l,i)\, f(l,x) - x \right),$$

– and for the **mutual fund**:

$$R(l,i) = R$$

for all technology–types (l,i) included in the portfolio across periods and

$$q_1(l) = d(l) + \sum_{i=0}^{I} P(i\,|\,l)\, q_2(l,i)$$

for all technology–levels l included in the portfolio within the period.

(iii) markets clear:

consumption goods market:

$$\bar{c} = \sum_{l=0,i=0,g=0}^{\infty \quad I \quad 1} \pi(l\,,\,i,g) \int c^{outs}(a,N,l,i)\,d(F_a^{outs} \times F_N) + \sum_{l=0,i=0,g=0}^{\infty \quad I \quad 1} \pi(l,i,g) \sum_{m=0}^{M} P(m\,|\,g) \int c^{ins}(a,l,i,m)\,dF_a^{ins},$$

$$\bar{x} = \bar{k} \sum_{l=0}^{\infty} F_k(l) \sum_{i=0}^{I} P(i\,|\,l)\; x(l,i)$$

$$\bar{y} = \bar{k} \sum_{l=0}^{\infty} F_k(l)\; y(l), \tag{4.4}$$

and

$$\bar{c} + \bar{x} = \bar{y}.$$

stock market:

for all categories (l,i,g):

$$F_k(l)\; P(i,g\,|\,l)\; f(1,x(l,i))\; \bar{k}$$

$$=$$

$$\pi(l,i,g)\; \varphi(l,i) \quad +$$

$$\pi(l,i,g) \int s^{outs}(a,N,l,i)\; d(F_a^{outs} \times F_N) \quad +$$

$$\pi(l,i,g)\sum_{m=0}^{M} P(m|g)\int s^{ins}(a,l,i,m)\ dF_a^{ins},$$

<u>mutual fund market</u>:

$$\bar{k}\sum_{l=0,i=0,g=0}^{\infty\quad I\quad 1} \pi(l,i,g)\ \varphi(l,i)\ q_3(l,i)$$

$$=$$

$$\sum_{l=0,i=0,g=0}^{\infty\quad I\quad 1} \pi(l,i,g)\int b^{outs}(a,N,l,i)d(F_a^{outs}\times F_N) +$$
$$\sum_{l=0,i=0,g=0}^{\infty\quad I\quad 1} \pi(l,i,g)\sum_{m=0}^{M} P(m|g)\int b^{ins}(a,l,i,m)\ dF_a^{ins},$$

<u>labor market:</u>

$$F_a(\{a < \underline{a}\}) = \bar{n} = \bar{k}\sum_{l=0}^{\infty} F_k(l)\ n(l)$$

(iv) <u>distributions are stationary[1]:</u>

for all $0 \leq a^* \leq \infty$:

[1]The term "distribution" always refers to "probability distribution", i.e. a distribution yields nonnegative weights for measurable sets and integrates out to 1.0 over the whole set.

$$F_a(a^*) = \int P(\tilde{a}'(a) \leq a^*)\, F_a(da),$$

for all $0 \leq a^* \leq \infty$:

$$F_a^{ins}(a^*) = \int P_i(a)\, 1_{\{a_i(a) \leq a^*\}}\, F_a(da)$$

and

$$F_a^{outs}(a^*) = \int P_o(a)\, 1_{\{a_o(a) \leq a^*\}}\, F_a(da),$$

for all $l = 0,1,2, \ldots$:

$$F_k(l) = 1/\zeta \sum_{\substack{g=0 \\ l-g \geq 0}}^{l} F(l-g) \sum_{i=0}^{I} P(i,g \mid l-g) f(l, x(l-g,i))$$

and

$$F_k(0) > 0$$

(v) <u>identifiability:</u> for all l,

- either $q_3(l,i) \neq q_3(l,i')$ whenever $i \neq i'$
- or $q_3(l,i)$ and all other functions that depend on (l,i) are independent of i.

(vii) <u>ζ is the steady growth rate</u>,

i.e. let $y(1,\tilde{\gamma},\tilde{w})$ be the solution to (4.1), (4.2) and (4.3) with $\tilde{\gamma}$ substituted for γ_0 and $\tilde{w}$ substituted for w. Similarly let $\bar{y}(\tilde{k},\tilde{\gamma},\tilde{w})$ be aggregate output defined analogously to (4.4) with $\tilde{k}$ replacing $\bar{k}$. Then

$$\zeta\bar{y} = \bar{y}(\ \zeta k,\ \Gamma_0\gamma_0,\ \zeta w\ ).$$

V. Analysis of the Model

We now want to "dissect" the model, i.e. prove the existence of an equilibrium and understand its properties. This is done as far as possible in Appendix V. Here we only give the intuition behind the analysis. It turns out, that equilibria can only exist, if a certain consistency condition is met, which links the "subjective" probabilities π with the "objective" production uncertainties P. We will discuss this in more detail below.

Fix the probability structure (P , π). The key to understanding the model is to recognize that it consists of two parts, each of which can be analyzed in a fairly standard way and which are linked by one parameter and one market clearing condition.

The first part is the production side of the economy. These are all those equations and conditions of the equilibrium which pertain to production:

- profit maximization, part (ii),
- labor market clearing,
- stationarity of the capital distribution,
- determination of the steady state growth rate.

There are two items here that require some work. First, a stationary solution is to be obtained for the prices q_i and the investment rule x. Profit maximization for the mutual fund maps prices q_2 into prices q_1 and prices (via a no–arbitrage condition with respect to expected values) and maps q_1 into q_3 via

$$Rq_3(l,i) = \pi(g=0|i)q_1(l) + \pi(g=1|i)q_1(l+1) \qquad (5.1)$$

given an interest rate R, for the equilibria in which the mutual fund holds shares of all types. q_3 is mapped into q_2 and linked to x and the unit price for the consumption good for a given wage w via the maximization condition of the investment firms. A contraction mapping argument then yields the fixed point. The formal argument is in Theorem A.V.2. The second item concerns the stationarity of the capital distribution across levels: this is not a fixed point argument for the whole distribution but instead a fixed point argument for level 1: the equilibrium wage has to be chosen just right so that the investment rule for the fraction $P(g=0|l=0)\ F_k(0)$ of the capital which is on the highest level and will again be on the highest level in the next period exactly replicates the fraction $F_k(0)$: such a wage can be found. The rest of the distribution can now be calculated in a straightforward manner. The steady state level of capital can then be found from the aggregate labor supply (which is determined when solving the agents problem), and the labor demand, arising from the distribution F_k (see appendix V).

The second part of the model concerns the decision problem of the agents or the consumption side of the model. The arguments here are for most parts rather standard. Firstly, the decision problem of the agent can be written simpler: the wage can be normalized to 1, since it is only a scaling factor and prices q_1 and q_3 can be completely eliminated and be substituted by probabilities π, once relationship (5.1) is established. One can then analyze the decision problem of the outsider and the insider separately. Each of these decision problems map a future value function v' into a present value function v^{outs} and v^{ins}. Via the simplified two – point lotteries, we obtain a present value function v. A standard contraction mapping argument under rather mild assumptions about the given interest rate R yields a fixed point.

Let

$$R_{max} = R \max \left\{ \frac{\pi(g \mid i,m)}{\pi(g \mid i)} \mid i,g \right\}$$

be the maximal return, an insider can possibly earn on his portfolio. Furthermore, let

$$\bar{R} = \zeta^{\eta} / \beta$$

be the equilibrium interest rate in the benchmark neoclassical growth model with a representative agent. If $R < \bar{R}$ and messages are not too informative, we will have $R_{max} < \bar{R}$. By comparison with a standard consumption – savings – problem (see Stokey – Lucas with Prescott (1989), p. 126 ff), we conjecture that extremely rich insiders should eventually eat most of their "cake" and thus return to the region of more average agents.

This will be the condition on the stochastic transition kernel for asset holdings which ensures, that agents will not become too rich. We will thus be able to prove the existence of an invariant asset distribution and the continuity of this distribution in the parameter vector, for which we eventually solve in the final fixed point theorem, by an extension of the Theorems 11.12 and 12.13 in Stokey – Lucas, with Prescott (1989).

The common link between the production side and the consumption side of the economy will be market clearing on the mutual funds market or (equivalently, by Walras' law) market clearing on the consumptions goods market: the interest rate R is the parameter that has to adjust until market clearing is accomplished. What will force the equilibrium interest rate R to satisfy $R < \overline{R}$ (which is important to prove the existence of a stationary asset distribution) is the random labor income N for outsiders: with low initial asset holdings, they have to save more than in the standard consumptions – savings –problem with a fixed income of 1 unit of labor, since they might get a bad draw and have little labor income in the next period.

In order to calculate market clearing on the mutual funds market, we first have to determine the asset holdings $\varphi(1,i)$ of the mutual fund, however. This is done via the stock market clearing condition (4.12), which is used as a defining equation for $\varphi(1,i)$. Observe however, that with two possible values for g, we get two equations (4.12) which have to be linearly dependent in order to define $\varphi(1,i)$ consistently: this is the consistency condition. The condition is roughly of the form

$$P(g \mid l,i) = (1 + \hat{\chi}(l,i,g))\, \pi(g \mid i), \tag{5.2}$$

where $\hat{\chi}$ depends on the aggregate insider demands, among other things. Without this consistency condition being satisfied, an equilibrium cannot exist.

This condition should not be too surprising. After all, prices have to do two things here at the same time: they have to clear markets as well as aggregate information. Suppose, a price q_3 is a convex combination

$$q_3 = \alpha\, q_3^{max} + (1 - \alpha)\, q_3^{min}$$

of the maximal possible price

$$q_3^{max} = q_1(l) / R,$$

which would prevail, if the good growth rate $g = 0$ was certain and the minimal price

$$q_3^{min} = q_1(l+1) / R,$$

which would prevail, if the bad growth rate $g = 1$ was certain. This price q_3 conveys the public information, that based purely on the price, there is a probability of α, that the good growth rate will prevail for the stock bought by some agent on account of the mutual fund or his own account, since there would be arbitrage possibilities for the mutual fund otherwise (when using a large number of stocks and applying the law of large numbers). Under both possibilities – the good growth rate g=0 and the bad growth rate g=1 – the price q_3 must be a market clearing price: otherwise, the growth rate would be immediately known and q_3 not an equilibrium price. From the properties of the model, agents can calculate the aggregate demand functions and thus the probabilities α' and $1-\alpha'$, that market clearing came about from $g = 0$ and $g = 1$, given that the market cleared at q_3 and given that this technology has been picked for that agent. It then has to be the case that this probability α' coincides with the probability α conveyed by the price being a convex combination of the maximal and the minimal price. This is another way of stating the consistency condition.

Let us examine this argument in some more detail. How does the information α get into the price q_3 in the first place? The best way to think about this is to start by fixing the size of the noisy demand and solving for the information contained in prices "forward" rather than fixing the information in prices π and working "backwards" to the noise: we will show below, how the one approach maps into the other one. Let us fix some technology τ on some level l. Let ω be some state of nature, drawn from some probability space (Ω,Σ,P), which we assume to be finite for the sake of the argument: $\Omega = \{ \omega_1,...,\omega_n \}$. Since we only need two random variables below, we might as well identify Ω with the space of the outcomes of these variables: $\Omega = \{ (\sigma_1, g_1), ..., (\sigma_n, g_n) \}$. Nature decides on the size of the population $\sigma(\omega) > 0$, trading in the stock of our technology τ. Following the framework of Grossman (1981), the mutual fund decides on some function $\varphi(q_3)$, taking into account the information contained in q_3, if q_3 was actually the market clearing price. Nature then decides on how many assets $D^{mut}(q_3,\omega) = \varphi(q_3)\sigma(\omega)$ the mutual fund actually buys, since every agent present on the market buys $\varphi(q_3)$ assets on account of the mutual fund. $\varphi(q_3)$ is known to all agents, but nobody can observe the size of $\sigma(\omega)$: this is the source of noise. The mutual fund is risk – neutral, thus equilibrium prices have to make the mutual fund indifferent between buying more or less of the stock. Now suppose the random growth rate g for our stock is the good one: $g(\omega) = 0$. The Walrasian auctioneer calls out various prices q_3. Since the insider combine their own private message with the information α contained in the price q_3, if it were the market clearing price, their aggregate demand $D^{ins}(g{=}0,q_3)$ will vary with q_3. Aggregate supply $S(q_3)$ will also vary but in a known fashion via the investment decisions of the firms in the investment sector who only equate margins. The "noisy" part of the demand of the mutual fund $\sigma(\omega)$ does not change with q_3, but the overall demand $\varphi(q_3)\sigma(\omega)$ might. For the sake of this argument, let us assume that there is only one market clearing price $\hat{q}_3 = q_3(\omega)$, $q_3^{min} < \hat{q}_3 < q_3^{max}$. Let $\hat{\alpha}$ be the parameter that expresses $\hat{q}_3 = \hat{\alpha}\, q_3^{max} + (1 - \hat{\alpha})\, q_3^{min}$ as a convex combination of the maximal and the minimal price.

What would happen, if this was the only way, the price $\hat{q}_3$ could come about? In that case, everybody would know at that price, that $g(\omega) = 0$, i.e.

the price would be completely revealing. But by the argument given above, the revealed information should just be $\hat{\alpha}$ and not 1.0, i.e. $\hat{q}_3$ cannot be an equilibrium price. Another way to put that is that at a price $q_3 < q_3^{max}$, which completely reveals the good growth rate in equilibrium, agents and the mutual fund will have taken that into account when formulating their demand functions and thus there can only be excess demand for the stock, contradicting market clearing. For $\hat{q}_3$ to be an equilibrium price at ω, it thus has to be the case, that there is another possibility: there is some ω' with the same (for the argument here, unique) market clearing price $q_3(\omega') = q_3(\omega) = \hat{q}_3$ and $g(\omega') = 1$. This is the same as the observation above that the two equation (4.12) have to be linearly dependent. Since g takes only two values in our model and since we identified ω with distinct (σ,g) – pairs, this exhausts all possibilities of arriving at the market clearing price $q_3(\omega)$.

The expected return on the portfolio of the mutual fund, given $\hat{q}_3$, thus satisfies

$$\mathrm{Ret}(\hat{q}_3) = \frac{P(\omega)\sigma(\omega)}{P(\omega)\sigma(\omega)+P(\omega')\sigma(\omega')} \frac{q_1(l)}{\hat{q}_3} + \frac{P(\omega')\sigma(\omega')}{P(\omega)\sigma(\omega)+P(\omega')\sigma(\omega')} \frac{q_1(l+1)}{\hat{q}_3},$$

taking into account that the size of the portfolio of the mutual fund depends on the state of nature ω. To make the mutual fund indifferent between buying more or less of the stock (which is the same as the absence of an arbitrage opportunity when averaging over many stocks), it is necessary, that $\mathrm{Ret}(\hat{q}_3) = R$, or,

$$\hat{q}_3 = \alpha' \, q_3^{\max} + (1-\alpha') \, q_3^{\min},$$

where

$$\alpha' = \frac{P(\omega)\sigma(\omega)}{P(\omega)\sigma(\omega)+P(\omega')\sigma(\omega')}$$

is the relative probability $\pi(g\,|\,\omega)$ for a member of the population, that he was chosen for a stock with the good growth rate $g = 0$, given that the market clears at $\hat{q}_3$. This "subjective" probability is the wheel π, which we chose for parameterizing our model. $\hat{q}_3$ can be a market clearing price only if $\alpha = \alpha'$.

Observe that we get two instead of one market clearing conditions for $\hat{q}_3$, because the price $\hat{q}_3$ aggregates rather than reveals information: this is what the consistency condition is all about. In order to have both market clearing conditions hold, it has to be the case, that the noises $\sigma(\omega)$ and $\sigma(\omega')$ come in the right (relative) sizes (to get market clearing twice) and with the right probabilities (to be consistent with the information revealed by prices by avoiding arbitrage for the mutual fund). I.e., given $\pi(g{=}0\,|\,q_3) = \alpha$ or q_3 itself, we can find the required $\sigma(\omega)$ and $\sigma(\omega')$ from the market clearing conditions and the required $P(\omega)$ / $P(\omega')$ from the equation on α'.

If we started describing the model by choosing some arbitrary distribution for 2I (σ,g) – pairs, equilibrium with incomplete revelation could only exist, if we can find a list of distinct prices $q_3^0,..,q_3^I$, and for each price q_3^i two pairs $(\sigma^{i,0},0)$ and $(\sigma^{i,1},1)$ with their associated probabilities $P(\sigma^{i,0},0)$ and $P(\sigma^{i,1},1)$ such that the conditions implied by q_3 described above hold. By starting from the probabilities $\pi(g|q_3^i) = \pi(g|i)$ instead (which correspond to α^i) we automatically ensure $\alpha = \alpha'$ and fix $\frac{P(\omega|i)\sigma(\omega)}{P(\omega'|i)\sigma(\omega')}$. Market clearing (where we solve for the right mutual fund demand factor $\varphi(i)$) implicitely delivers the right relative population size $\frac{\sigma(\omega)}{\sigma(\omega')}$. Thus, the relative fundamental probability $\frac{P(\omega|i)}{P(\omega'|i)}$ has to be found to be consistent with $\pi(g|i)$: this is the way, the consistency condition has been formulated in equation (5.2).

The consistency condition is not a nuisance of this particular model, but rather a typical feature which we should expect in general equilibrium rational expectations models, in which prices both aggregate information and clear markets and where noise is generated via noisy demand. The relevance of the consistency condition for models in which the market mutual fund shows risk aversion as well, in which there are more than two (and possibly a continuum of) possible future states of the world, and in which there is a continuum of possible prices should be examined. In that context, it will also be important to understand, in how far our backsolving approach imposes these difficulties arising from the consistency condition. If the consistency condition and the difficulties associated with it are a persistent phenomenon, it should be clear that the interpretation of rational expectation models with partial price revelations is not a straightforward matter. That is, if the distribution on (σ,g) – pairs is viewed as part of the fundamental description of the economy, the equilibria analyzed here or in models of this type should generically not exist and instead we will generically have prices that are always revealing. A different view is that there are market forces not spelled out in these models, which force the population sizes and probabilities to be of the right kind. A third approach is to see these models as an approximate description of a real phenomenon which will not be quite as knife – edged as the theoretical results, thereby relaxing the non – genericity.

In order to finally show that there are equilibria at all (where we allow ourselves to choose the right fundamental probabilities $P(g|i,l))$, we fix the "subjective" probabilities π and, apply a fixed point argument to find the equilibrium interest rate and the equilibrium probabilities P which achieve market clearing for the mutual funds market and satisfy equation (5.2). Observe that the decision problem of the agents does not change as long as we do not change R, since it only depends on the "subjective" probabilities π: this simplifies the theoretical as well as the numerical analysis.

VI. Access to a market portfolio mutual fund rules out information acquisition.

Suppose that besides the assets introduced, there is another mutual fund, which can hold "the market" in the sense that it can choose a fraction ψ and then includes that fraction of all capital in its fund.

Theorem VI.1:

Suppose that agents can hold shares of the second mutual fund in their portfolio besides the stock and the mutual fund described above. Then the fraction of agents that acquire information is zero.

Proof:

Let K be the total value of the capital stock at the end of the period and K' be the total value of the capital stock at the beginning of the next period. I.e. we have

$$K = \bar{k} \sum_{l=0,i=0,g=0}^{\infty \quad I \quad 1} P(i,g|l) F_k(l) f(l,x(l,i))\, q_3(l,i)$$

and

$$K' = \bar{k} \sum_{l=0,i=0,g=0}^{\infty \quad I \quad 1} P(i,g|l) F_k(l) f(l,x(l,i))\, q_1(l+g).$$

Let $K^{outs} \geq 0$ resp $K^{,outs}$ be the value of all assets the outsiders possess at the end of the period resp. at the beginning of the next period. Define likewise K_{+}^{ins} resp. $K_{+}^{,ins}$ be the value of all assets the insider hold long and $K_{-}^{ins} \geq 0$ resp. $K_{-}^{,ins} \geq 0$ the value of all assets the insiders hold short. Since all assets are ultimately ownerships of capital, we must have

$$K = K^{outs} + K_{+}^{ins} - K_{-}^{ins} \tag{6.1}$$

and

$$K' = K^{,outs} + K_{+}^{,ins} - K_{-}^{,ins} \tag{6.2}$$

Let $\overline{R} = K' / K$ be the gross return on the aggregate capital stock. Since outsiders have now access to a mutual fund with the same rate of return $\overline{R}$, we must have $K^{,outs} / \overline{R} \geq K^{outs}$. Now suppose that there is a non–zero fraction of insiders. Then, since information is costly, we have $K_{+}^{ins} \neq 0$ and $K_{+}^{,ins} / \overline{R} > K_{+}^{ins}$ or we have $K_{-}^{ins} \neq 0$ and $K_{-}^{,ins} / \overline{R} < K_{-}^{ins}$ or we have both. Substituting these inequalities into (6.1) and multiplying by $\overline{R}$ yields

$$K' = \overline{R}\, K > K^{,outs} + K_{+}^{,ins} - K_{-}^{,ins}$$

in contradiction to (6.5). This proves the Theorem. •

This Theorem is a version of the logic convincingly stated in Grossman and Stiglitz (1980), that if markets were efficient, they couldn't be if information acquisition is costly: nobody would have an incentive to acquire the information. The Theorem above requires even weaker conditions. A market portfolio mutual fund avoids the possibility that outsiders ever get "exploited" by the "insiders". It shows, how crucial our assumption is that the mutual fund introduced in section III has only access to a noisy diversification technology. It also shows that the other market restrictions we impose on agents are not such big restrictions after all.

VII. Numerical experiments

We have performed 66 numerical experiments, in which we varied two parameters:

- six values for the spread of the prices (and thereby the public information contained in prices) $\pi(g=0 \mid i=0) - \pi (g=0 \mid i=1)$:
  spread = .1, .2, .3, .4, .5, .6,
- eleven values for the informativeness of the private messages $P(m=0|g=0) = P(m=1|g=1)$:
  signal quality = .64, .66, .68, .685, .69, .695 ,.70 ,.705, .710, .715, .72.
  The finer step size between .68 and .72 was chosen since previous experiments pointed to this region as the most interesting one.

The other parameters were common:

The unconditional growth probability $\pi(g=0) = .7$,

The unconditional information index probability $\pi(i=0) = .7$, (thus computing $\pi(g|i)$ to

$\pi(g=0|i=0) = .7 + \text{spread}*.3$ and

$\pi(g=0|i=1) = .7 - \text{spread}*.7$)

$\Gamma_0 = 1.05$, $\Gamma_1 = 1.02$,

$\beta = 0.95$, $\eta = 1.5$,

$\alpha = .5$, $\kappa_1 = .94$, $\kappa_2 = 1.0$,

$\rho = .3$,

an exponential distribution with $\lambda = 1$ was used for N.

The way the equilibria are computed numerically is described in some more detail in appendix VI. Essentially, the program follows the same logic as in the analysis of the "disection" of the model. To solve for the consumption–/savings–rules, we used a value function iteration approach by iterating on linearly interpolated functions described by its values on a grid of asset values a, which are equally spaced in logs. This method is similar to that of Coleman(1990), except that we used the value function rather than the Euler equations to update the policy rules. Colemans method has performed well in a comparison of different solution methods as performed in Taylor and Uhlig(1990), but a comparison of the results of different solution methods have not been performed for our model here.

In order to judge the effect of insiders, we compared each equilibrium of an economy with insiders with the "same" economy without insiders, where prices do no longer convey information (i.e. $I = 0$) and where we took care, that the average "objective" growth probabilities for a technology

$$P(\, g=0 \mid 1\,) = \sum_{i=0}^{I} P(\, g=0 \mid 1,i\,)\, P(\, i \mid 1\,)$$

are the same for the economy with and without insiders (this requires computing the solution for the economy with insiders first and then taking the results for the endogeneously determined probabilities P as input for the economy without insiders, see the discussion at the end of section V and in Appendix VI). Note the meaning of the word "average" here: $P(g=0 \mid 1)$ is the average growth probability before prices across the entire old capital stock on level l and not the average growth probability after prices across the entire new capital stock on level l, which would have to take into account the effect of i on the production of new capital and therefore should not be the same in the two economies.

a. Properties of a Specific Experiment.

To get an idea of how the results of a simulation look, we first plotted several results in figures 1.1.1 through 1.8.2 from one particular experiment, where we chose the spread at .4 and the signal quality at .7. We calculated an equilibrium to both the economy with and the economy without insiders. The figures with numbers ending in 1 (like 1.1.1, 1.2.1) are results from the with – economy, whereas the figures with numbers ending in 2 are the corresponding results from the without – economy. It turns out, that the increase of the total output in the economy with insiders over the economy without insiders is a little more than 1% and thus, the results between the two economies will not be too different. This is especially true for the capital distribution, since both economies use rather similar transition probabilities (remember that we enforced both economies to have the same transition probabilities $P(g|l)$): this is shown in figures 1.1.1 and 1.1.2. Differences arise only because there is an additional possibility in the insider – economy to invest relatively more in the technologies, which are relatively more promising as indicated by $P(g|i,l)$: this results in a 1 % increase of the level – 0 fraction of the capital in the with – economy compared to the without – economy: while this difference is barely visible in figures 1.1.1 and 1.1.2, it is more visible in figures 1.2.1. and 1.2.2, where we plotted the output produced by each level. Ultimately, it is this 1 % increase (and the induced increase in the level of the steady state capital), which is responsible for the 1 % increase of total output. Figures 1.3.1 and 1.3.2 show, how the agents are distributed across the different levels of the technologies, i.e. show the distribution $\pi(l)$. This distribution is calculated rather than a parameter and chosen so that the market demands by the insiders (which increase with decreasing prices, i.e. with increasing levels) end up being of the same size, see appendix VI. For the without – economy, the same formulas are applied, but the distribution of agents is of course without relevance there.

Differences between the two economies become better visible in figures 1.4.1 through 1.5.2, where we show the effect, information revelation (via i) has on prices $q_3(l,i)$ and on investment $x(l,i)$ respectively. The effect on investment in the highest level 0 is quite dramatic, resulting in a more than 100 % increase in investment as we move from $i = 1$ – capital to $i = 0$ – capital. This dramatic difference is translated into only a small difference in the distribution of capital on level 0, since the existing old capital and the CES – reproduction function have a smoothing effect.

Figure 1.6.1 and 1.6.2 plot the value functions: while figure 1.6.1 compares the insider and the outsider value function and shows how they cross at 12.2 in the with–economy, figure 1.6.2 shows the value function (which is the same as the outsider value function) in the without economy. The steady state distribution of assets is shown in figures 1.7.1 and 1.7.2 for the with and the with–out economy. The dotted line at .99 in figure 1.7.1 separates the insiders from the outsiders. Note, that the distribution in the with – economy is much more smeared out towards higher values of a beyond the cut – off point (at $\underline{a} = 12.2$) than in the without – economy: this is the result of the possibilities of "speculation" for the insiders. In fact, while they constitute less than 1 % of the population in the with – economy, their asset holdings as a fraction of the total assets of the economy is around 23 %! In other words, while the population of insiders is tiny in the entire economy, their wealth is certainly not. This point is demonstrated again in figure 1.8.1, where the maximal value of the value functions for the poorest x % of the population is plotted as x varies from 0 to 100. Again, there is a fat, sharp spike in figure 1.8.1 for the last percent or so of the population. We take this issue up again below when examining the welfare comparisons between the with– and the without–economies.

b. Variation of the Signal Quality and the Revealed Information.

To see how the results vary and to make the welfare comparisons to answer the question posed at the beginning, we show the results of all 66 experiments in 2.1 through 4.7. Here, figures with the same endings (such as 2.1, 3.1 and 4.1) show the same aspect of the comparison, but from different points of view.

In figures 2.1 through 2.7 we compared the results of the experiment directly, providing a "backwards–solving" view on the results. Six curves, one for each spread – value, are plotted against the quality of the signal. Higher spread values correspond to more information being revealed by the prices. Figure 2.1 shows the insider demands in %, if $i = g = 0$ as a fraction of the total supply of capital on a particular level (this fraction is independent of the level l by the construction of $\pi(l)$ in the numerical calculations). Note, that the fraction reaches sizable portions of the market for the higher signal qualities. Figure 2.1 shows clearly a monotonicity: the insider – demands are the higher, the better their signal and the lesser information is revealed by prices. This is quite intuitive: if the signal quality is better, more agent will become insiders and try to exploit their private information. They can do that the greater their informational advantage over the market is, that is, the lesser information is revealed through prices. This must then lead to lower returns on the portfolios for the outsiders, i.e. a lower interest rate R, and a higher return on the portfolios for the insiders. That this intuition is correct is demonstrated by figure 2.2 and figure 2.3: the interest rate R is the lower and the insider return is the higher, the better the signal quality and the lesser the information revealed by prices.

Figures 2.4 and 2.5 compare the relative aggregate wealths of the insiders and the relative aggregate sizes of the insider population. Both figures look rather similar (and similar in turn to figure 2.1), except that the fractions are different by an order of magnitude: while the insider population is typically quite small and less than 3 %, their relative wealth is not, reaching 50 % for the "worst" case. The interpretation of that is, that while the reduction of the labor force due to the presence of insiders is almost negligible even for high – quality signals, we have to expect huge redistributional effects in direction towards the group of insiders simply because they own a large part of the entire capital stock.

Figures 2.6 and 2.7 finally compare two aggregates to judge, which economy is better off. These figures deliver an answer to the question posed at the beginning, whether or not the economy is better off with the information acquirerers than without them.

In figure 2.6, we calculated the increase in % of total output, when moving from the steady state in the economy without insiders to the economy with insiders. Not surprisingly, the effect is mostly positive, since insiders most often improve the capital distribution and the reduction of the labor – force as the only off – setting effect is quite small. Only, when too little of their information is revealed through prices (as is the case for e.g. spread = .1) and when too many agents become insiders (i.e. for high signal qualities), too much of that return to information remains private and the output actually decreases through the foregone labor. For the majority of the cases, however, insiders actually increase the gross national product in the experiments. If there was a way to redistribute this increase across the entire population, everybody would support their presence in these cases.

However, we did not introduce such a redistributional scheme and introducing it would likely to have an effect on the information acquisition incentives, which could destroy the gains it was intended to distribute. In absence of a redistributional scheme, every agent has to judge such a switch to the insider – economy from his own evaluation of his current asset holdings. We thus calculated the average welfare in the with– and the without–economy and calculated the growth required in the steady state of the economy without insiders to reach the same average welfare level as the economy with insiders. This is the same as calculating by how many percent every consumption has to be increased from now on in all the future in the without–economy to make agents there as happy on average as in the with–economy. The algebra on how this is done is explained in appendix IV. Note then, that the decision for an economy with insiders, judging from this average welfare criterion, is no longer as clear – cut as from just looking at output: only, if the quality of the signal is rather small and only, if a lot of the information gets revealed through prices do we see that agents are better off on average with insiders. This contrasts quite sharply with the results above, where we examined aggregate output. Both figures, however, answer the question of whether the economy is better off with the insiders qualitatively in the same way: only if the information of these insiders is revealed sufficiently well to the market for everybody to share. This answer, of course, makes intuitively sense.

Figures 3.1 through 4.7 are now designed to provide a "forward – solving" view of the results. I.e. instead of fixing the information revealed by prices (the spread) and back – solving for fundamental ingredients such as the noise, we use a comparison between our experiments to proceed the other way around. What we do it to fix the level of the noise in some meaningful fashion and solve for the information revealed by prices. That way, a comparison is possible, if we regard the level of the noise and the signal quality as fundamental properties of our economy and the information revealed by prices as endogeneous.

To achieve this goal, we need to identify a sensible measure for the noise. Noise should be that part of the model, which covers up the insider trades, i.e. noise is that part of the model, which creates the inference problem, given e.g. a high price, whether that price is high because of knowledgeable insider demands or uninformative noise demand. Thus, noise is readily identified with the variations in the mutual fund demand: the mutual fund ends up buying the bad stocks more often than the good stocks through the biased wheel of selecting stocks, thereby partly covering up the presence of the insiders, given a particular price. Since market clear, noise is as well represented by the variation in the insider – demands, given the price. Since insiders are always long in stocks, if the receive the message $m = 0$ and short in stocks, if they receive the message $m = 1$, a reasonable measure of the noise is the insider demand for $m = i = 0$ as a fraction of the total supply as given in figure 2.1 (a potentially more accurate measure would take an average over all four possibilities for m and i).

Figures 3.1, 3.3 and 3.4 examine, whether this is indeed a sensible way of describing the noise (note, that there is no figure 3.2). In figure 3.1 we just inverted figure 2.1, plotting the signal quality as a function of the noise, rather than the noise as a function of the signal quality. To draw curves, a linear interpolation of the signal quality over the logarithm of the noise is used to take care of the fact that the curves seem to grow at geometric rates in figure 2.1. Figure 3.3 shows that our measure for noise is almost the same thing as a measure for the average return on the insider portfolios, regardless of the information revealed by prices. The intuition for that is, that the noise is the exploitable mistake by the mutual fund of buying the wrong stocks on average. Given a certain size of this mistake, i.e. given a certain level of noise, the advantage for the insiders from exploiting this error has nothing to do with the information eventually revealed in the prices or with the quality of their information, since this level of noise will be exploited "no matter what". Figure 3.4 demonstrates, that the fraction of assets owned by insiders is close to a linear function of our measure of the noise, where the slope depends somewhat on the information revealed in prices. We conclude therefore, that we have a sensible way of measuring noise in our model.

Having identified the noise, we can now develop the forward view by solving for the information contained in prices, given the signal quality and the noise: this is done in figure 4.1. Figure 4.1 is obtained from figure 2.1 by fitting, for each of our 11 grid points of the signal quality, a line per least squares through the 6 noise values from spread = .1 through spread = .6 at that signal grid point and then inverting this line to solve for the spread values corresponding to the four arbitrarily chosen noise levels 3, 4, 5 and 6.

Connecting these four solutions for each signal – grid point across the different signal – grid points delivers figure 4.1. Figure 4.1 thus provides a table to look up the spread corresponding to one of the four chosen noise levels and a signal quality (on the grid).

This table is used in the graphs 4.2 through 4.7, which are created in a similar way to figure 4.1 by fitting, for each signal quality grid point, a least – squares line through the six function values of the corresponding figure 2.2 through 2.7 to define a function of signal quality and spread and then calculating a corresponding function of signal quality and noise by looking up the spread corresponding to signal quality and noise in (the calculations for) figure 4.1. This delivers, for each signal grid point, four function values for the four chosen level of noises, which are then connected across signal quality values to generate the curves in pictures 4.2 through 4.7.

Figure 4.1 shows, that the more information is revealed by prices, the greater the quality of the signal of the insiders and the lesser the amount of noise there is to cover up the trading activities of these insiders. Consequently, with a fixed level of noise, the return on the mutual fund increases as the signal quality increases as demonstrated in figure 4.2. Figure 4.3 shows, that the the variation in the insider returns for different levels of noise decreases as the signal quality improves. More importantly, however, figure 4.3 shows that the insider return almost does not change at all for a fixed level of noise: even for the noise = 3 – curve, the variation of the insider return is less than 1 %. This corresponds well to the insight generated by figure 3.3 and the argument above, that the return on the insider portfolio should be a function of the noise alone.

Figures 4.4 and figures 4.5 demonstrate, that the fraction of the assets owned by insiders and the fraction of insiders in the population goes done for fixed levels of noise, as the signal quality increases, since the improvement in the private return to that information is more than offset by the higher degree of revelation of that information in the prices.

Finally, to answer again the question posed at the beginning, figures 4.6 and 4.7 show, that the output and the average welfare of the economy with insiders is the higher, the better the quality of the signal, given a fixed level of noise. This result is intuitive, since the cost for the information remains the same – it is the non – participation in the production process – whereas the rewards become higher with higher signal quality. Keeping the level of the noise constant ensures that the private gains and the social gains from that information move in the same direction.

c. Distributional Considerations.

Comparing average welfare in figures 2.7 and 4.7 has a few drawbacks, most notably that we compare between steady states and that we average across the entire population. Little can be done about judging, how the economy would move towards the new steady state, once insiders are allowed in or ruled out, since it is only possible to compute steady state versions of the model at this stage of the research. However, insights can be provided as to whether and how the average welfare criterion distorts the perspective of each individual agents. This is the purpose of figures 5.1 through 5.4.2, which are all derived from results for the example with spread = .4 and signal quality = .7 examined above. Figure 2.7 shows for these parameters, that the economy is better off without the insiders as measured by the average welfare criterion and that a growth rate of approximately .957 is required of all consumptions in the without – economy to make agents indifferent on average, although aggregate output is higher by about 1.2 % in the with – economy (cmp. figure 2.6).

Figures 5.1 through 5.4.2 now show the results of several variations of the same experiment. Suppose, we had two countries, both completely autonomous with no trade between them (there is some kind of very high mountain, say), both very similar with the parameters stated above, both in their steady state equilibrium, but one country has outlawed information acquirerers, whereas the other one allows them. Suppose now, an individual agent found a tiny tunnel, that would carry just him alone and some of his belongings to the other economy. Would he do so? And by what factor would his home – country have to change his consumption from now on into all the future to make him just indifferent between leaving or staying. Finally, if a poll was taken on how many people would leave, if given the chance, how would the results look like?

The answer to these questions depend somewhat on which economy it is, that the agent is leaving and what he can take with him when traversing the tunnel. In figure 5.1, the agent is leaving the without – economy to traverse to the with – economy. The two lines indicate his change in happiness by providing the "bribing factor" of the home economy to make the agent indifferent: a growth factor less than one, indicates that he is happier in his own country anyways. The solid line indicates his relative change in happiness if he can take along the consumption – good equivalent of his assets, whereas the dotted line indicates his relative change in happiness, if he can take along the productive capital he owns. Both curves are plotted against a logarithmic scale for the value of their asset holdings (cmp. figure 1.7.1). It should not surprise, of course, that a superrich agent would rather want to traverse to the with – economy. What is perhaps surprising at first, is that a reasonably rich agent finds it even more unattractive than a very poor agent to use the tunnel. The intuition for that result is that the interest rate on the mutual fund is lower in the with – economy than in the without economy (where it is just the average rate of return on capital): a relatively rich, but not superich agent would dislike this lower interest rate on his substantial asset holdings, since he is still poor enough as to be an outsider (facing this lower interest rate) with sufficiently high probability in the future.

Figures 5.2 contains the fraction of voters in favor of leaving the without – economy (or similarly, of transforming their economy to a with – economy by instantaneously adapting to the new steady state). They vote for leaving, if their growth factor is bigger than 1.0, which is the case for practically nobody in the economy. The poll changes slightly, if taken in the with – economy in figure 5.3 to find the fraction of people who would like to stay, even though a tunnel is available to them. While the vast majority still favors the without – economy, there is a visible, though extremely tiny fraction of agents (the superrich), who favor the economy which allows insider – trading. These superrich agents are simply almost not present in the steady – state without – economy.

Finally, we imagine, that our tunnel – traveler, traversing from the without– to the with–economy could take along a document that shows his relative status within his society instead of any assets and that the government of the other country awards him with an asset endowment that gives him the same relative status. That is, if exactly 3 % of the agents were richer than our agent before crossing the tunnel, exactly 3 % of the agents will be richer than him after he got endowed with these assets in the new country. This experiment possibly captures most closely of what would happen, if an economy slowly transforms itself after legalizing insiders, since with a given level of noise, it will be the top fraction of the population which can exploit this noise for their private insider – return (see also the argument above to explain figure 3.3). Figure 5.4.1 (where the last .1 % before 1.0 of the graph was cut off) indicates, that almost nobody favors the with – economy that way and actually, the aversion against a switch of that type is the bigger the wealthier an agent is. Only, if agents are superrich and belong to the very upper crest of the population will they favor traversing to the with – economy: this is demonstrated by figure 5.4.2.. Again the intuition for the result in figure 5.4.1, that the rather rich dislike to switch even more than the poor results from the fact, that the presence of insiders depresses the returns an outsider earns on his (mutual fund) portfolio: this is only important, if the agent has a portfolio to talk about to begin with.

Overall, these figures 5.1 through 5.4.2 show, that distributional effects cannot be ignored. While it is true in this experiment, that the gross national product is higher by 1.2 % and the wage is higher by 2.1 % in the economy, which permits insiders, it turns out that the interest rate in this economy is also depressed from $R = 1.094$ in the without – economy to 1.065 in the with – economy, inducing in particular the reasonably wealthy agents to favor outlawing insiders.

It probably is true, that for some of the experiments performed, the very poor and the very rich agents would join forces against the middle – class in a vote in favor of permitting insider trading, since the very poor are made happy by the higher wage and the very rich are made happy by the possibilities to speculate, whereas the middle – class faces the draw – back of a lowered return on their portfolio in the absence of special stock market information.

VIII. Conclusions

We demonstrated that it is possible and feasible to present a model in which the questions of the welfare effects of information acquirerers (called insiders) on the stock market can be adressed meaningfully and in a computationally tractable way. The results one obtaines depend on the particular parameters chosen for the numerical calculations.

Three theoretical results are worth highlighting. First, a simple Bayesian formula is given which allows agents to combine different sources of information to form their final beliefs. This formula might be of practical use. Secondly, equilibria will not exist unless a certain consistency condition is met, that ensures that the signal extraction problem agents face is well posed at the same time that markets clear. I.e. the distribution of noise in the economy cannot be chosen "independently" from the activities by information acquirerers.

Thirdly, noise in this model can only arise if agents are prevented from perfect diversification. In particular, if agents can hold a mutual fund, whose share holdings mirror the structure of capital in the economy (i.e. a "perfect" market portfolio), insiders can not earn a higher than average return and in equilibrium there will be no noise left that insiders can "exploit". Thus, the costly acquisition of information ceases to exist.

66 numerical experiments have been performed. The results of this experiment support the intuition that insiders perform a welfare–increasing

role only as long as their information is revealed sufficiently well to the market. It was also shown, that distributional effects across the population are important in judging the welfare effects of insiders: it is possible, that although wages and aggregate output increase due to their presence, the vast majority of the population, facing a depressed return on their portfolio due to their imperfect asset selections, will be in favor of outlawing insider trading.

There are many interesting possibilities for extensions of this work. The importance of the consistency condition for other classes of models with costly information acquisition and incomplete information aggregation through prices needs to be examined. It will be interesting to analyze whether the existence of a market portfolio mutual fund rules out acquisition of information even in models with aggregate uncertainty. Thirdly, the introduction of e.g. treasury bills in a model of this type will lead to different available portfolio structures and thus has an impact on the relationship of returns paid on equity versus the return paid on government bonds. Then, there is the question of better arrangements for generating the information about future possibilities of technologies. Simple changes of the arrangement include various forms of taxation or trading restrictions in this model, more complex arrangements will substitute new forms of contracts in place of the stock market altogether. Finally, the quest for a general equilibrium model which can be fit to data to adress the question of the welfare effects of information acquirers and to answer more detailed question regarding policy has not yet come to an end. While the model in this paper takes a step in that direction, incorporating aggregate uncertainty is necessary before one can be serious about estimation. Incorporating endogeneous growth and long time lags before the private information becomes public will increase the importance of insiders in these models.

The hope is, that this research can lead to a better understanding of the interesting question on how to design an economic mechanism that will optimally generate and transmit information to agents in the economy.

F. Figures.

The following figures 1.1.1 through 5.4.2 are explained in part VII, describing the results of the numerical experiments. Figures 6.1.1 through 6.8.2 are referred to in appendix VI, which contains details on the way the numerical calculations were performed.

Figure 1.1.1: Distribution of Capital across Levels

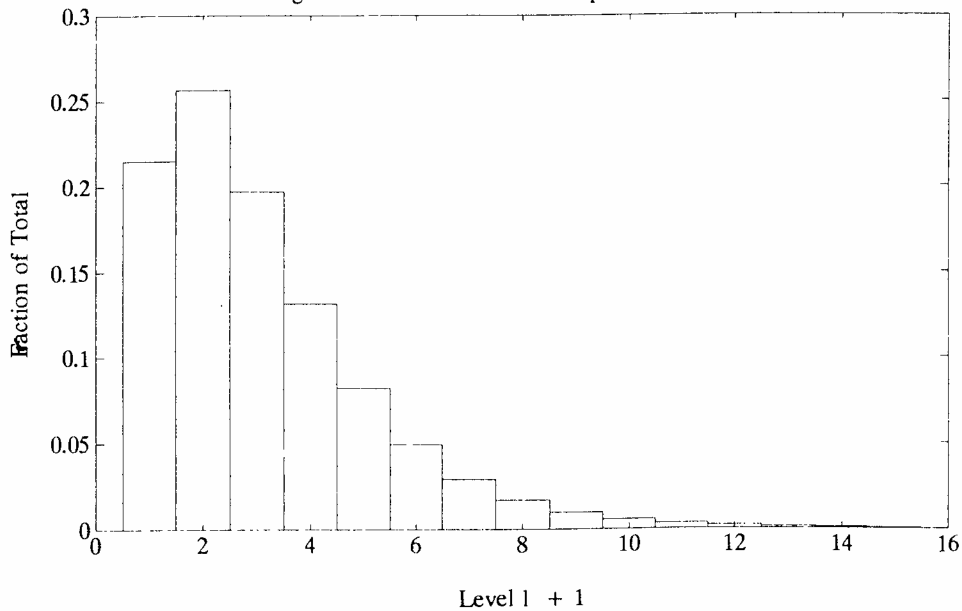


Figure 1.1.2: Distribution of Capital across Levels

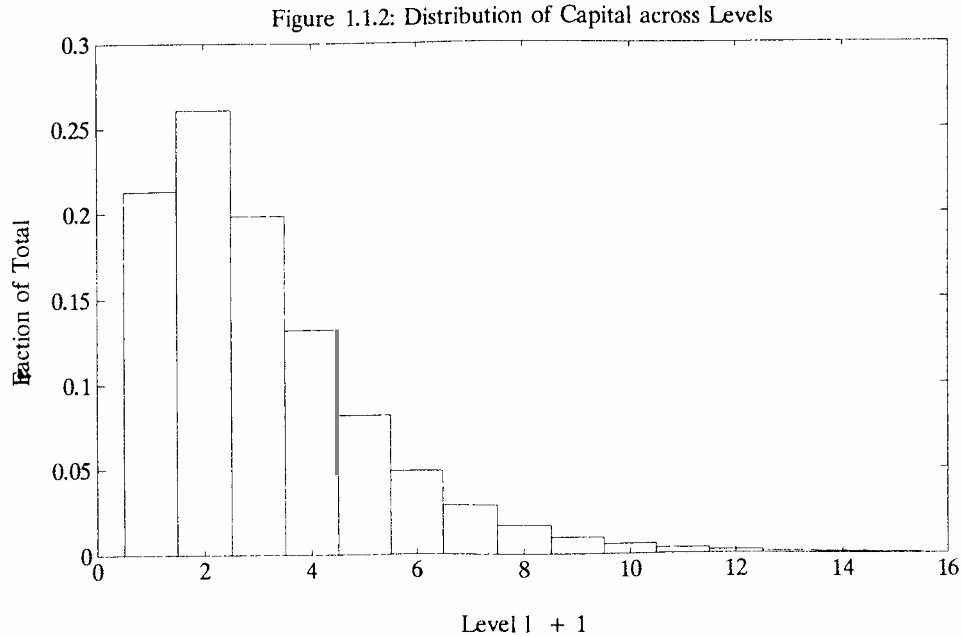


Figure 1.2.1: Output Produced by each Level

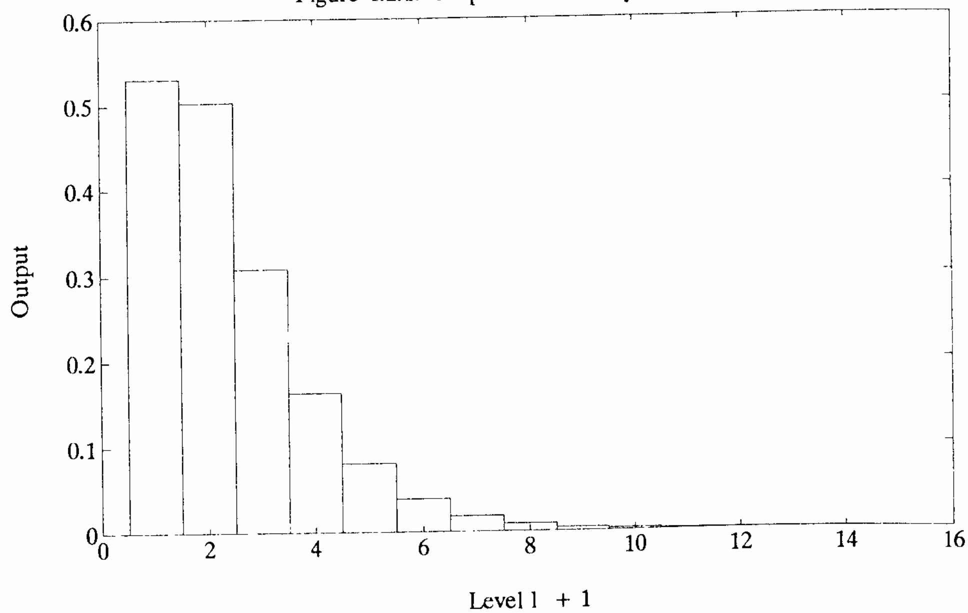


Figure 1.2.2: Output Produced by each Level

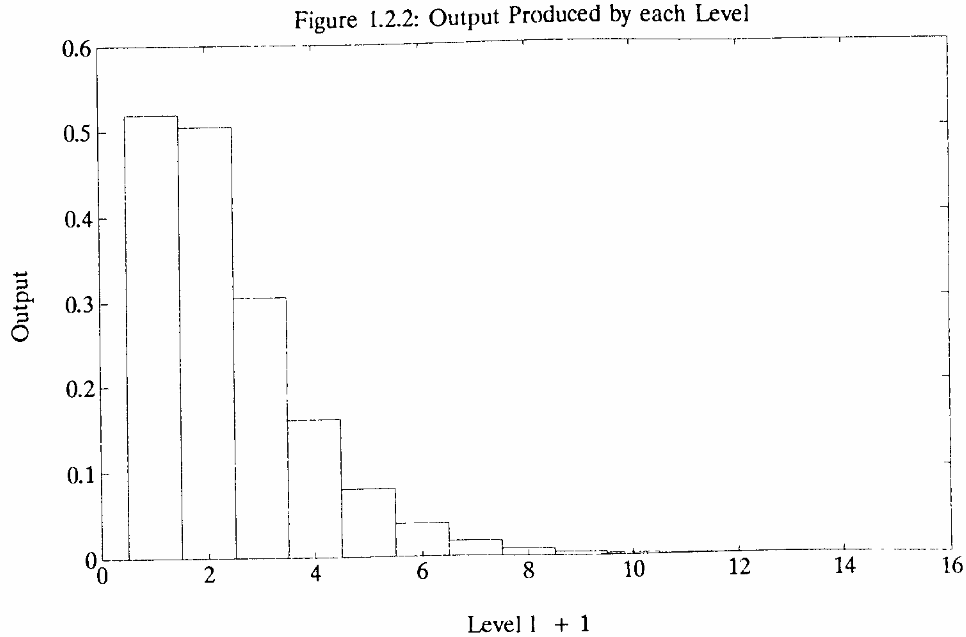


Figure 1.3.1: Distribution of Agents across Levels

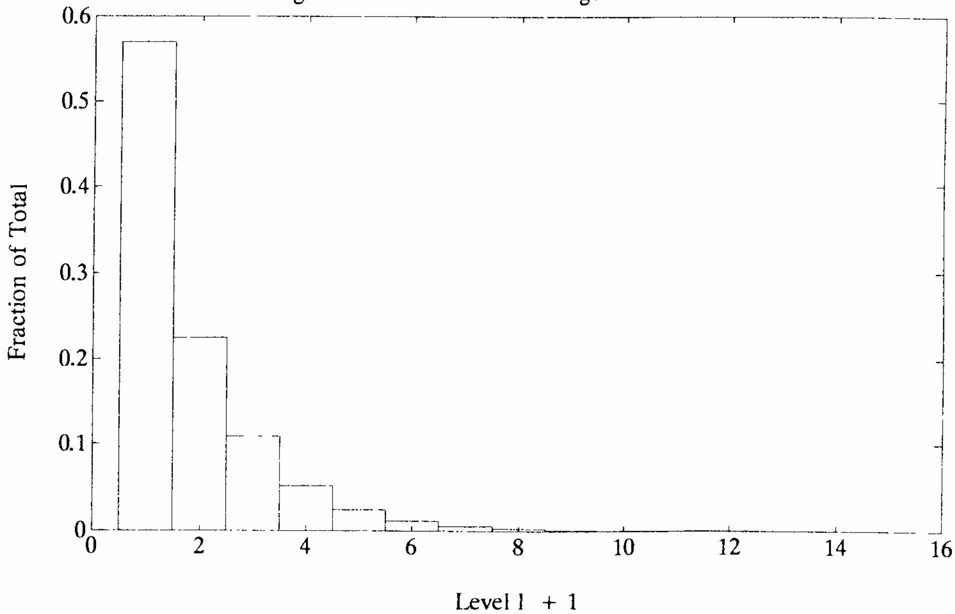


Figure 1.3.2: Distribution of Agents across Levels

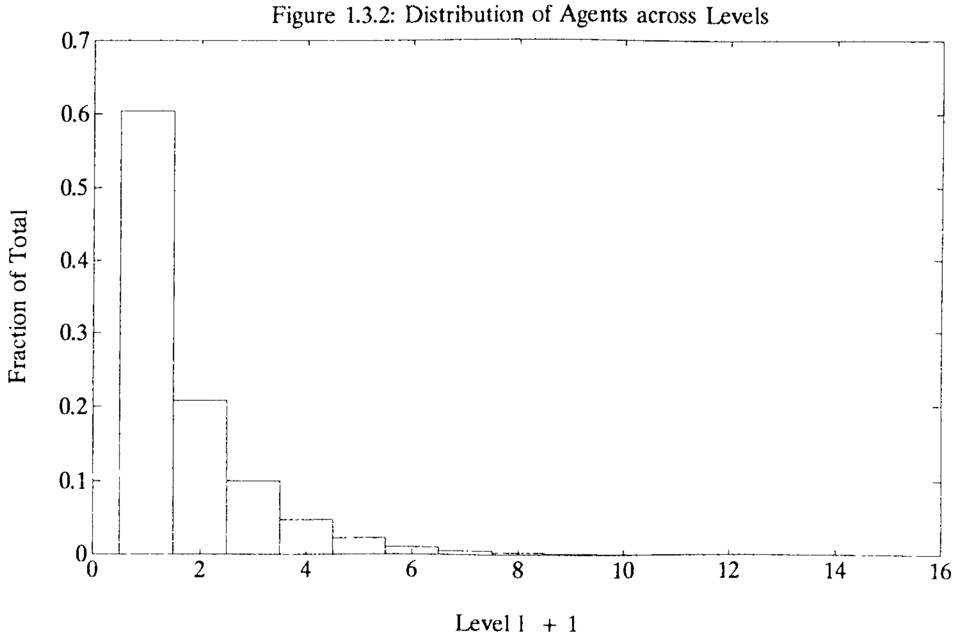


Figure 1.4.1: Prices of One Unit of New Capital

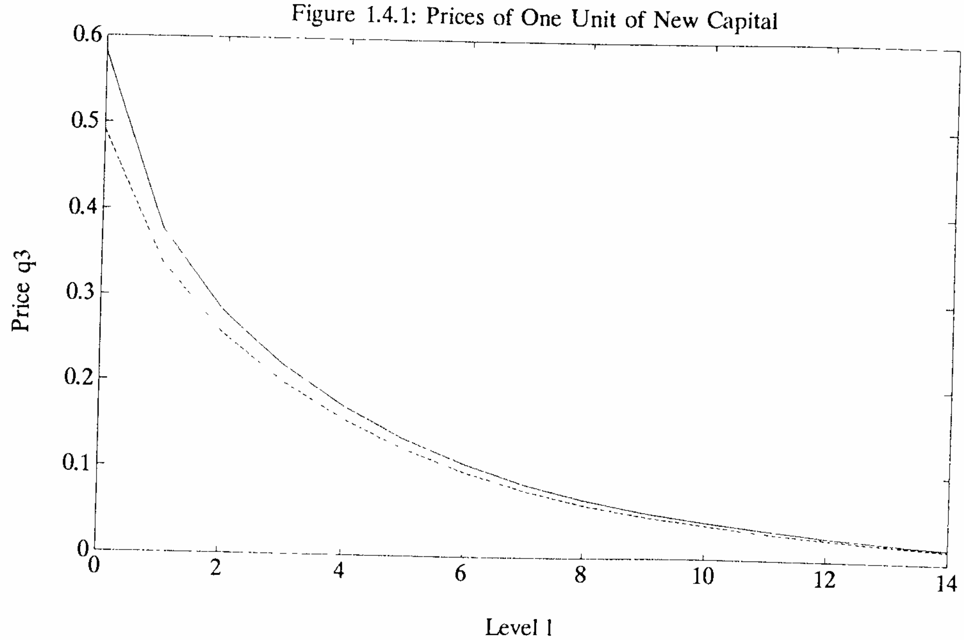


Figure 1.4.2: Prices of One Unit of New Capital

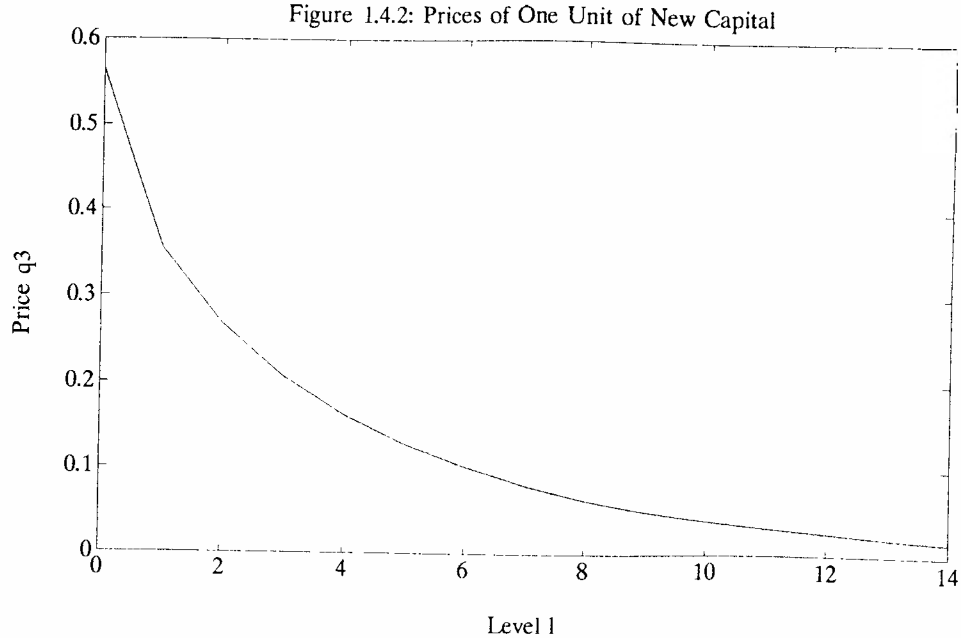


Figure 1.5.1: Investment per Unit of Old Capital

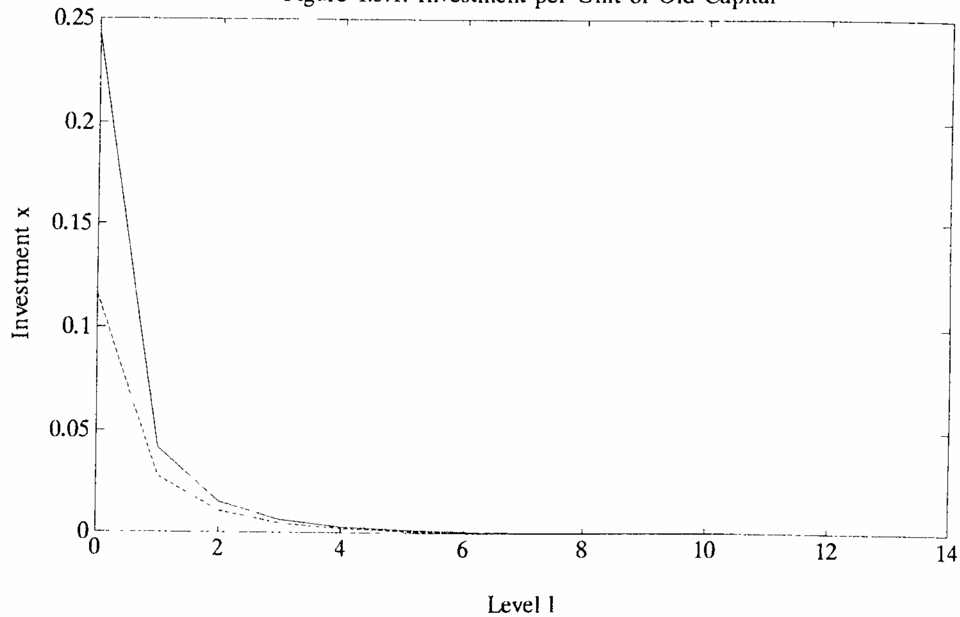


Figure 1.5.2: Investment per Unit of Old Capital

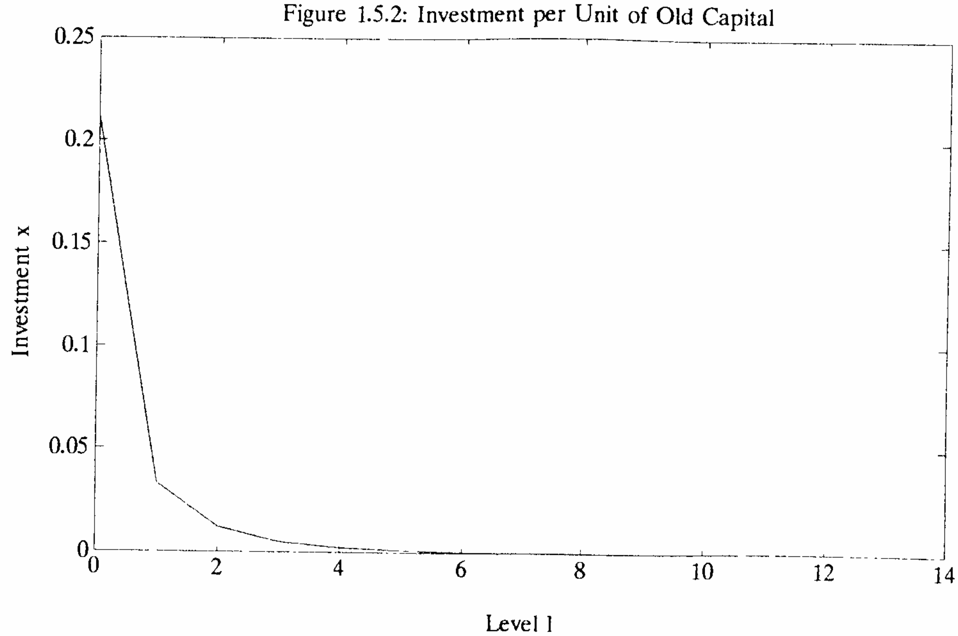


Figure 1.6.1: Value Functions

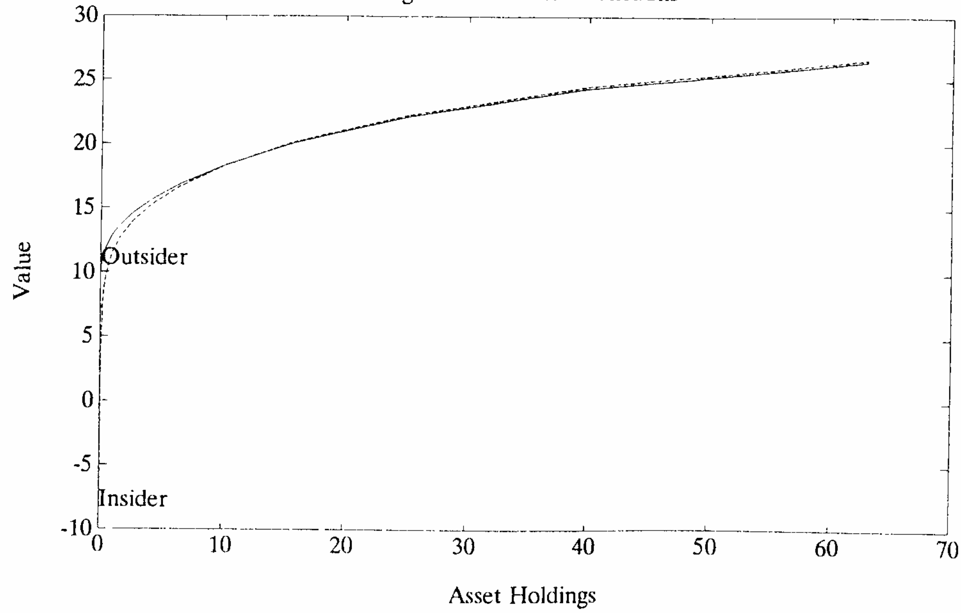


Figure 1.6.2: Value Function

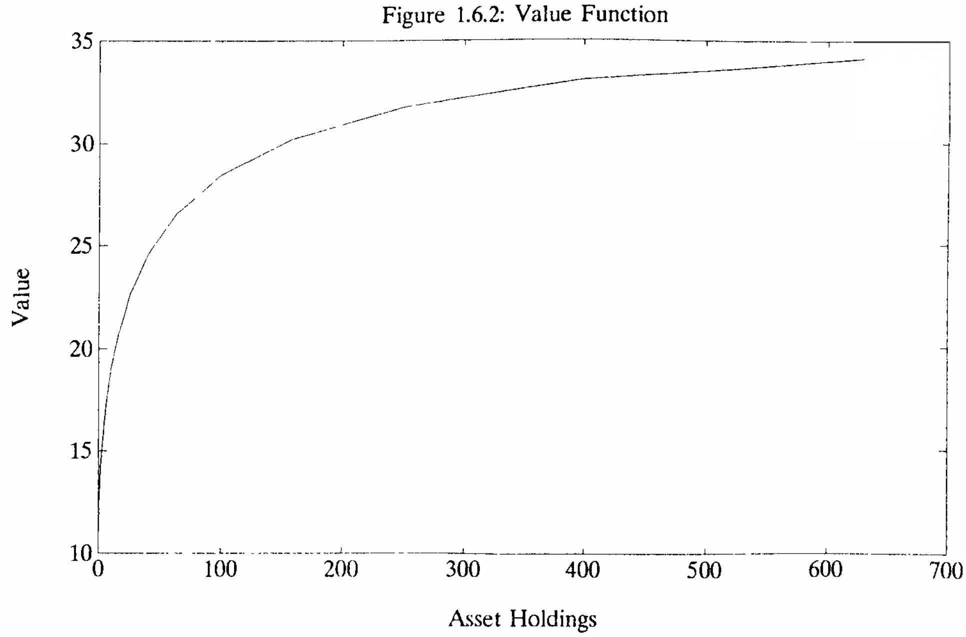


Figure 1.7.1: Distribution over Assets (Line at F_a_cut)

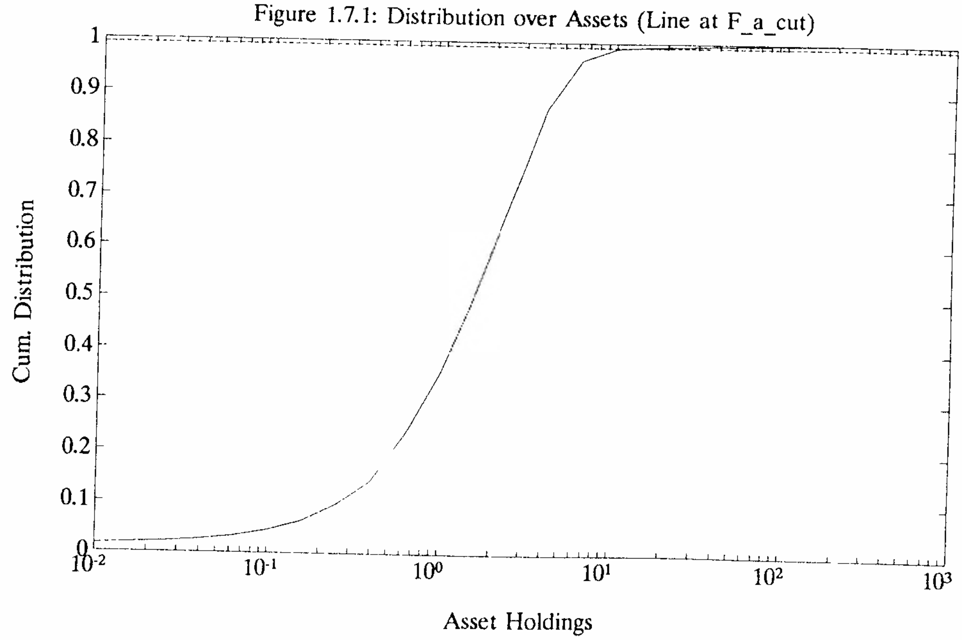


Figure 1.7.2: Distribution over Assets

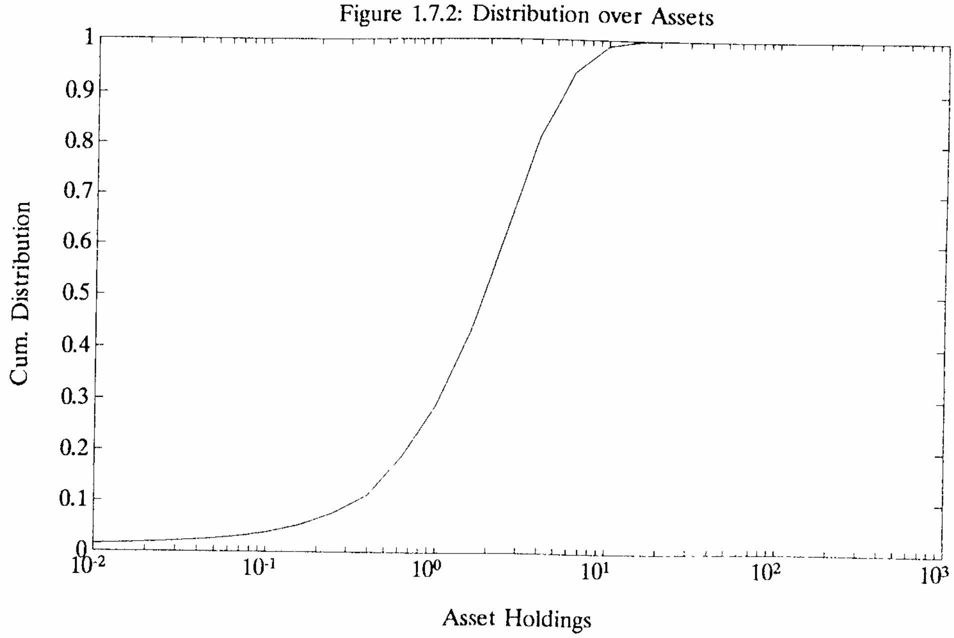


Figure 1.8.1: Max. Value in Fraction of Population

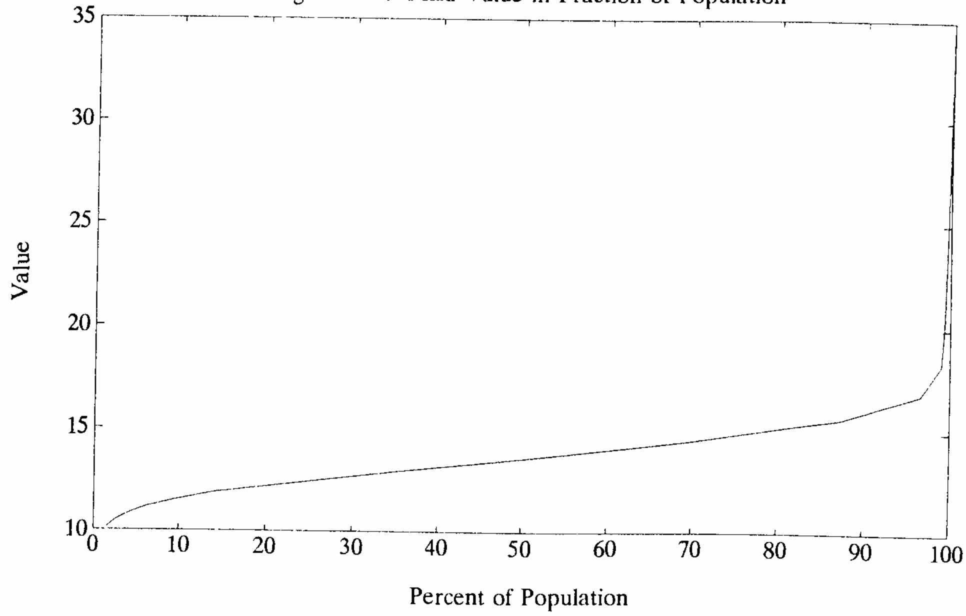


Figure 1.8.2: Max. Value in Fraction of Population

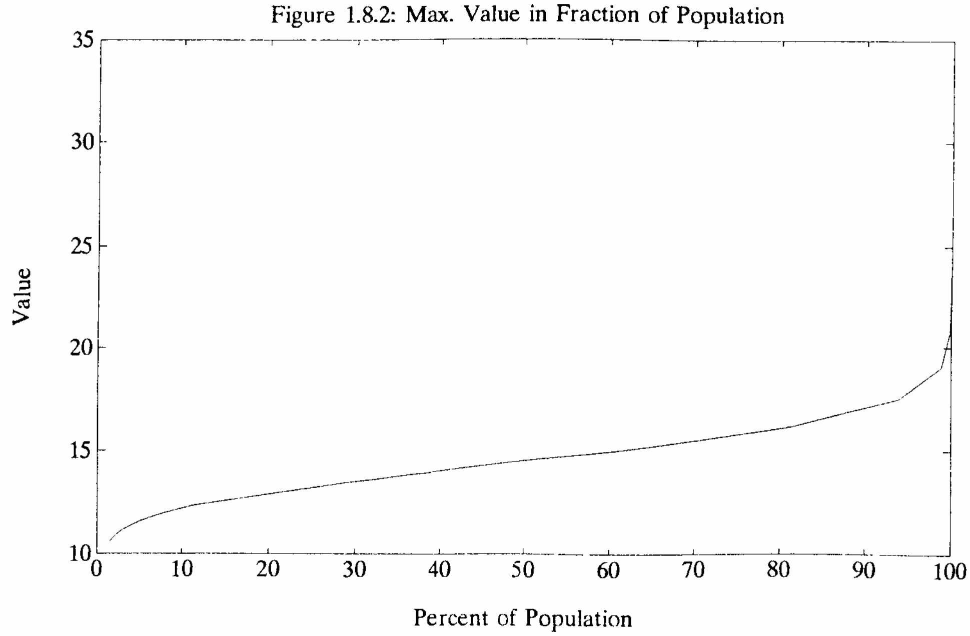


Figure 2.1: Insider Demands in % if i = g = 0

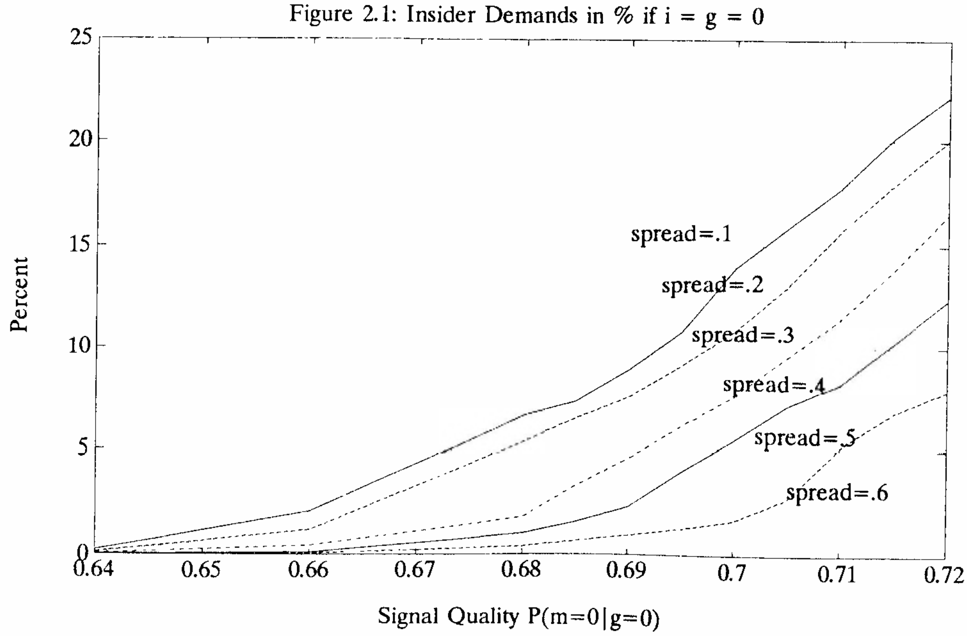


Figure 2.2: Interest Rate

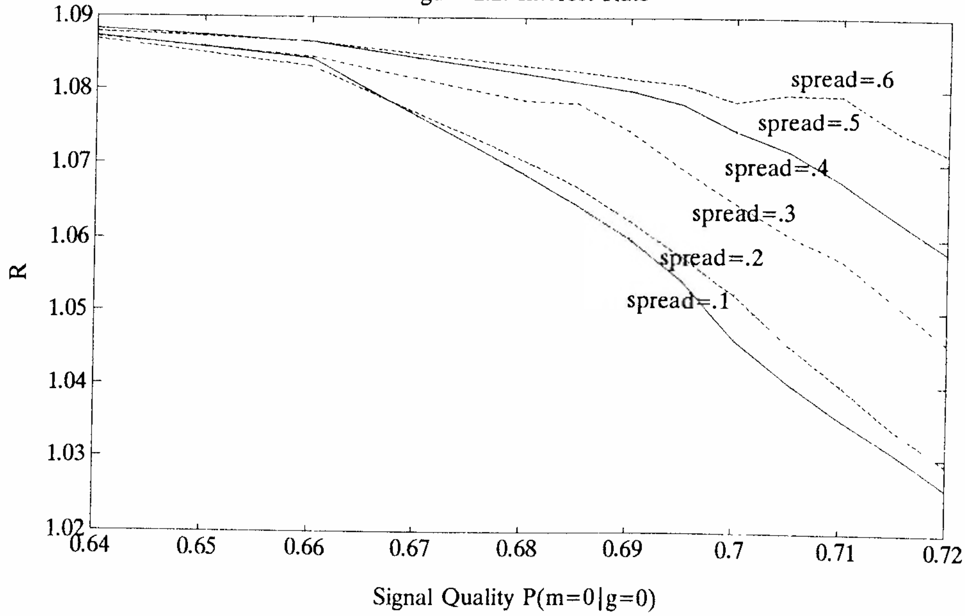


Figure 2.3: Insider Return

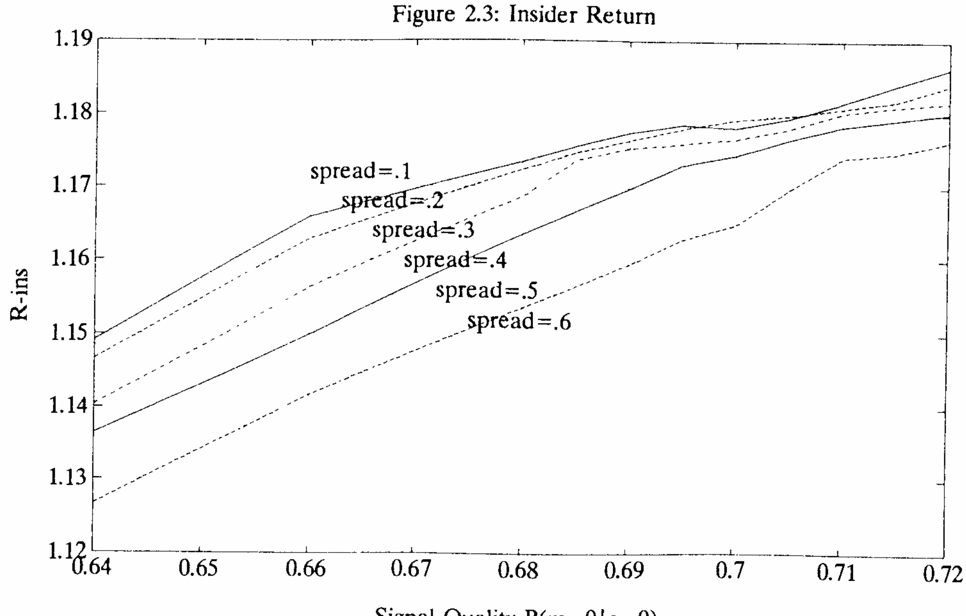


Figure 2.4: Fraction of Assets Owned by Insiders

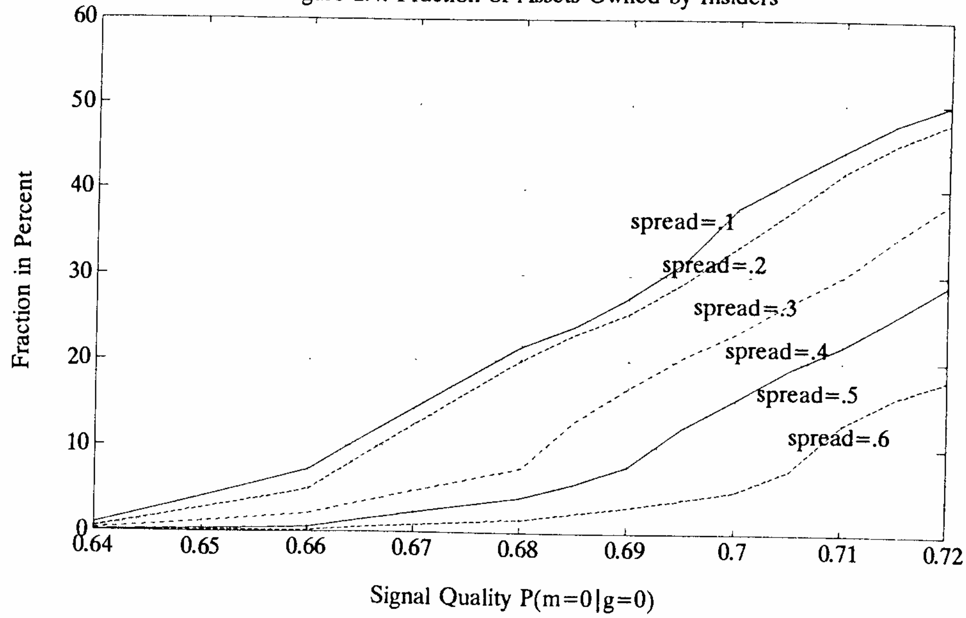


Figure 2.5: Fraction of Insiders in Population

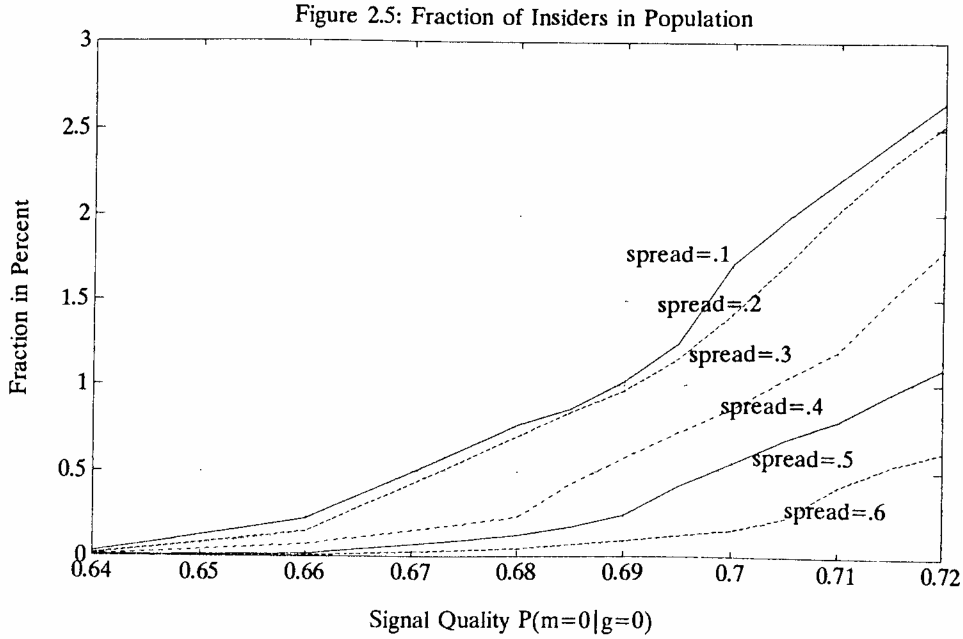
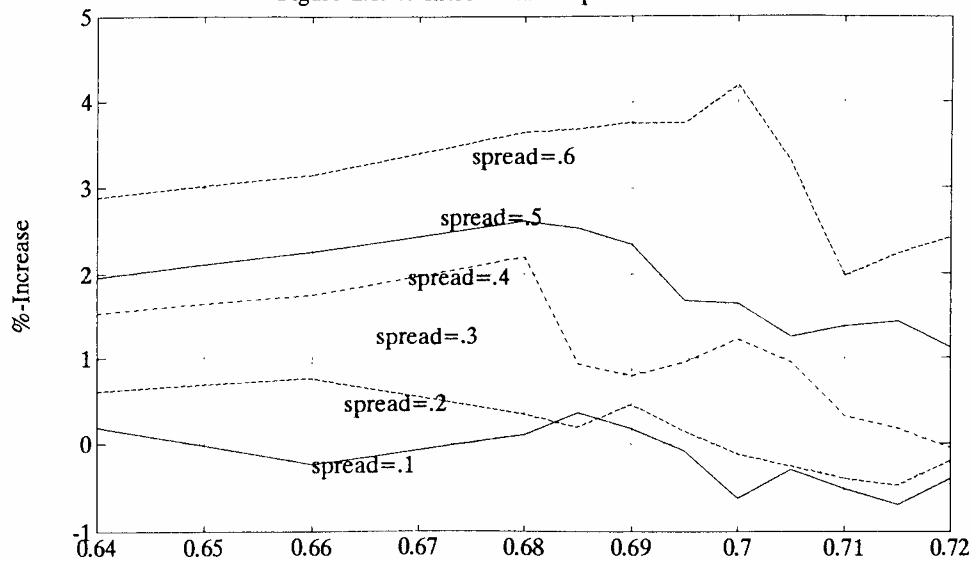
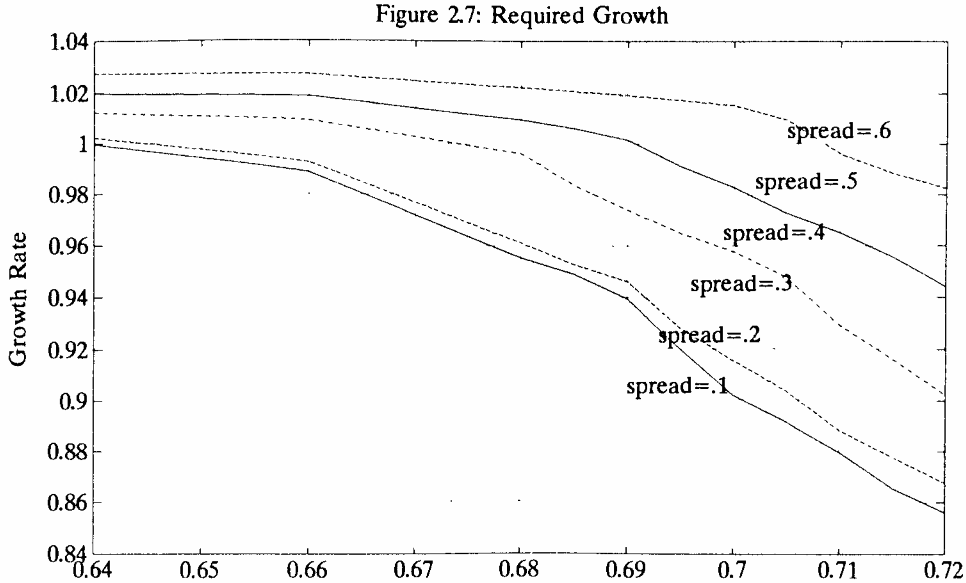


Figure 2.6: %-Increase in Output due to Insiders



Signal Quality P(m=0|g=0)

Figure 2.7: Required Growth



Signal Quality P(m=0|g=0)

Figure 3.1: Signal Quality as Function of Noise

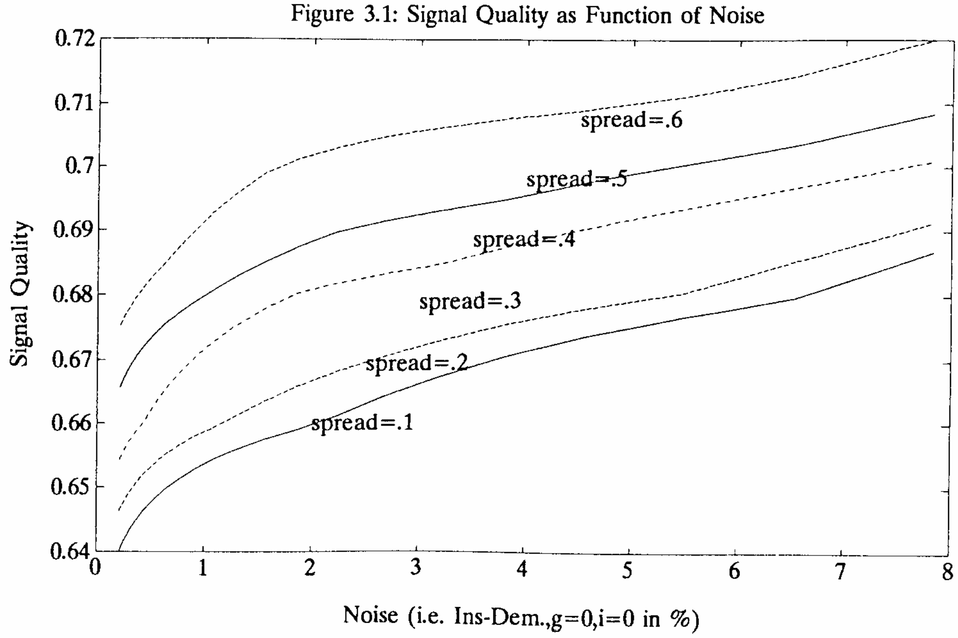


Figure 3.3: Insider Return

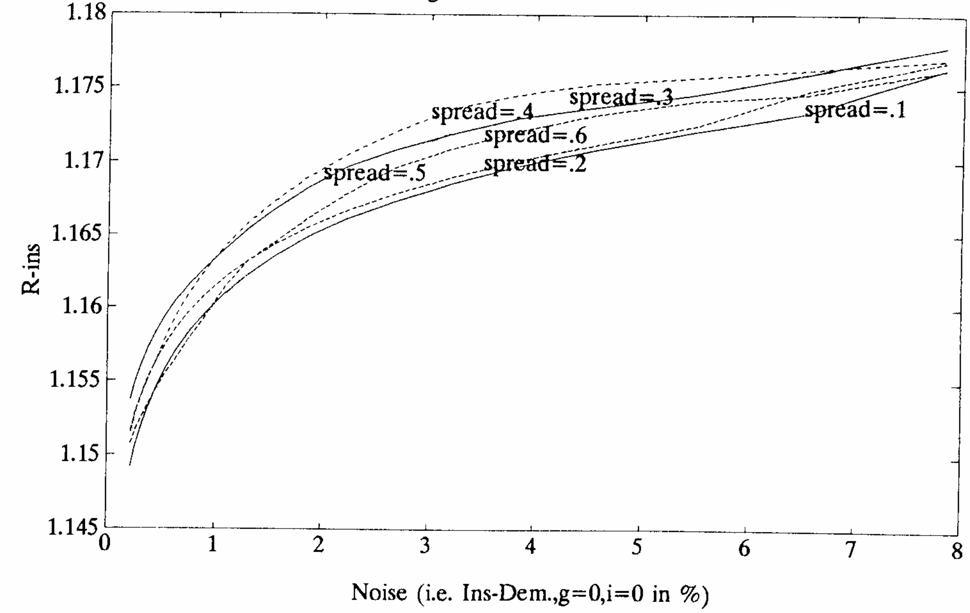


Figure 3.4: Fraction of Assets Owned by Insiders

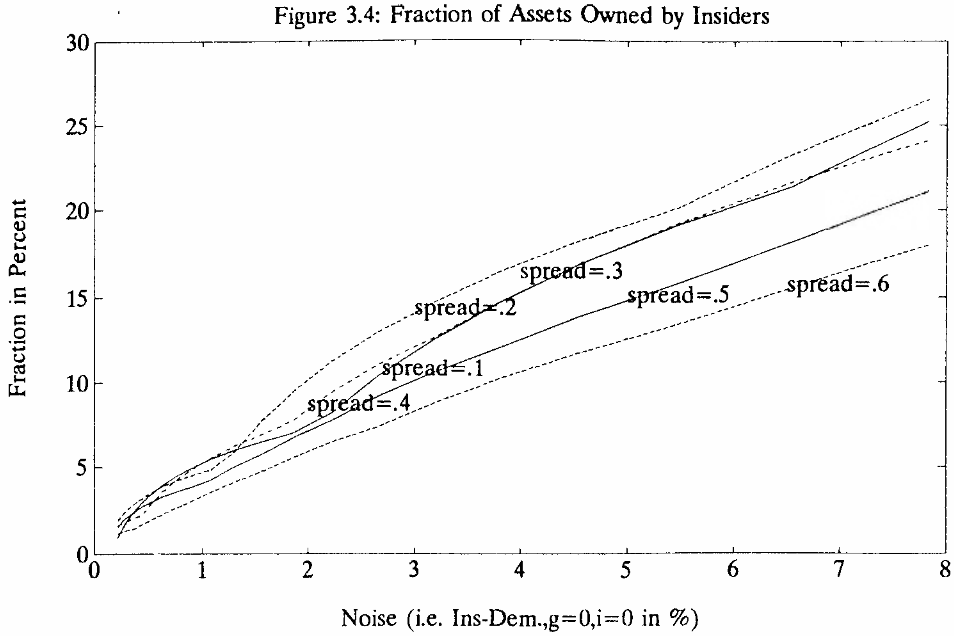


Figure 4.1: Information Revealed in Prices

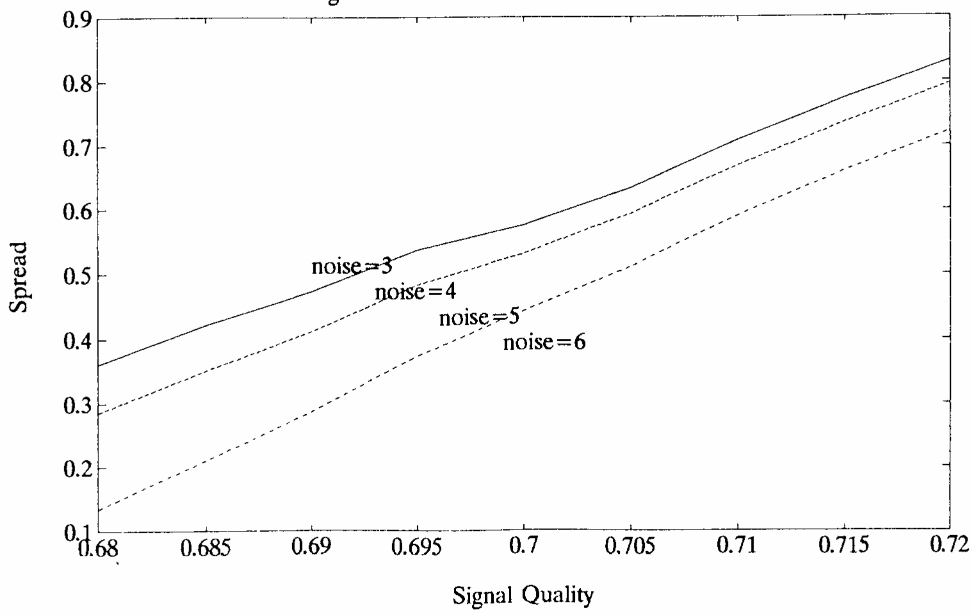


Figure 4.2: Interest Rate

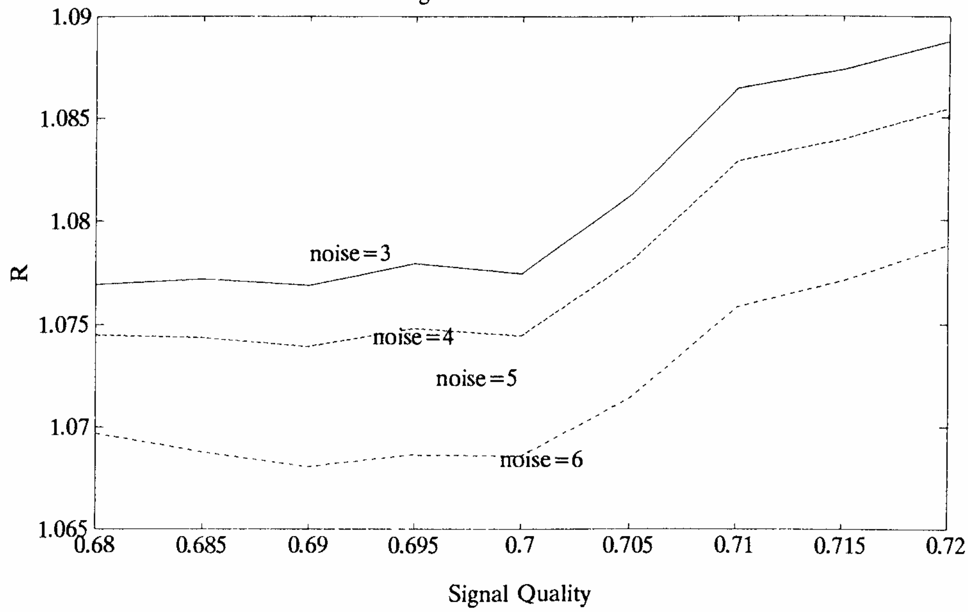


Figure 4.3: Insider Return

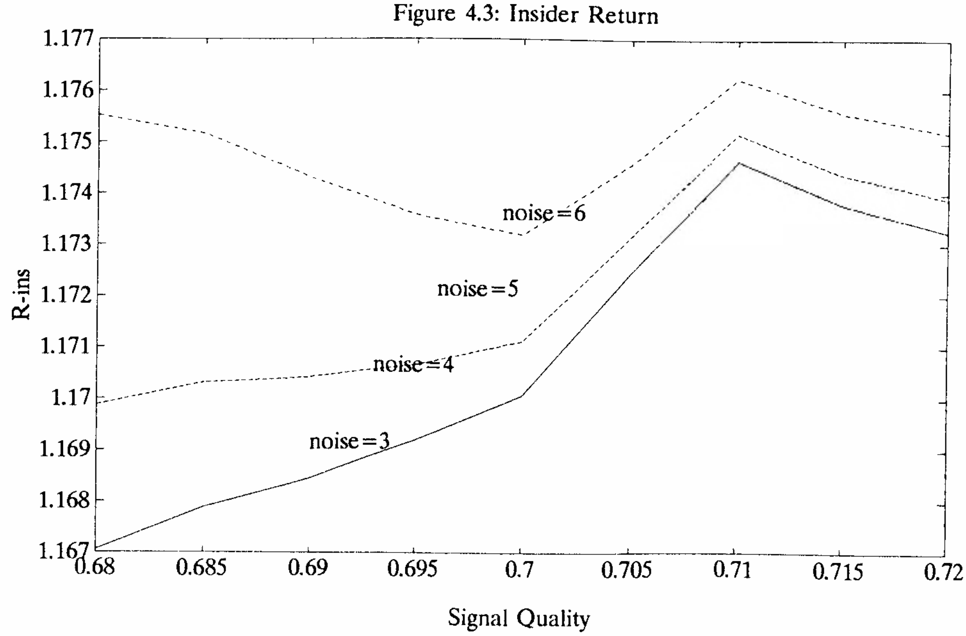


Figure 4.4: Fraction of Assets Owned by Insiders

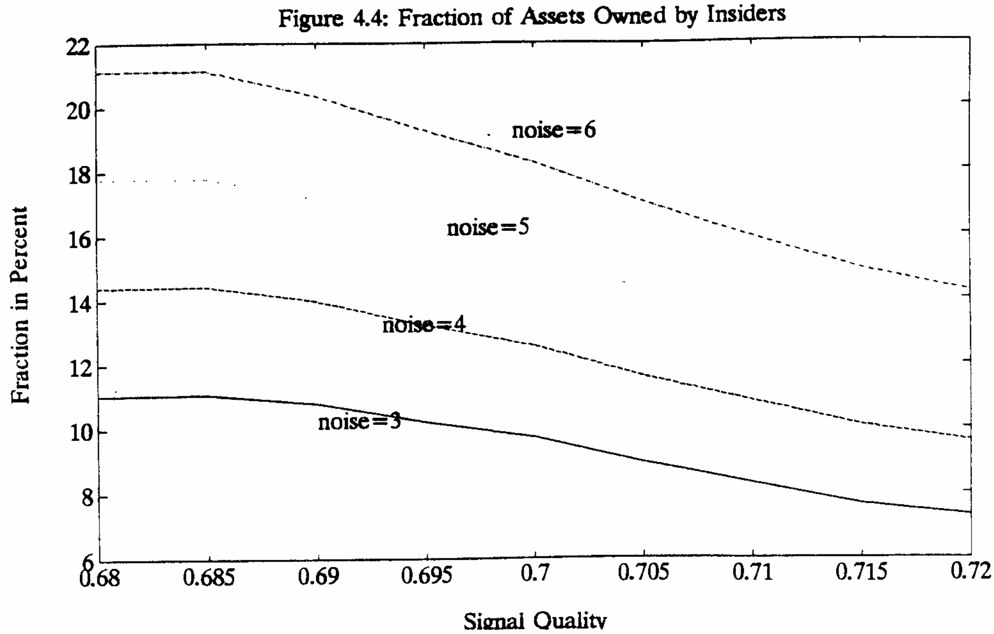


Figure 4.5: Fraction of Insiders in Population

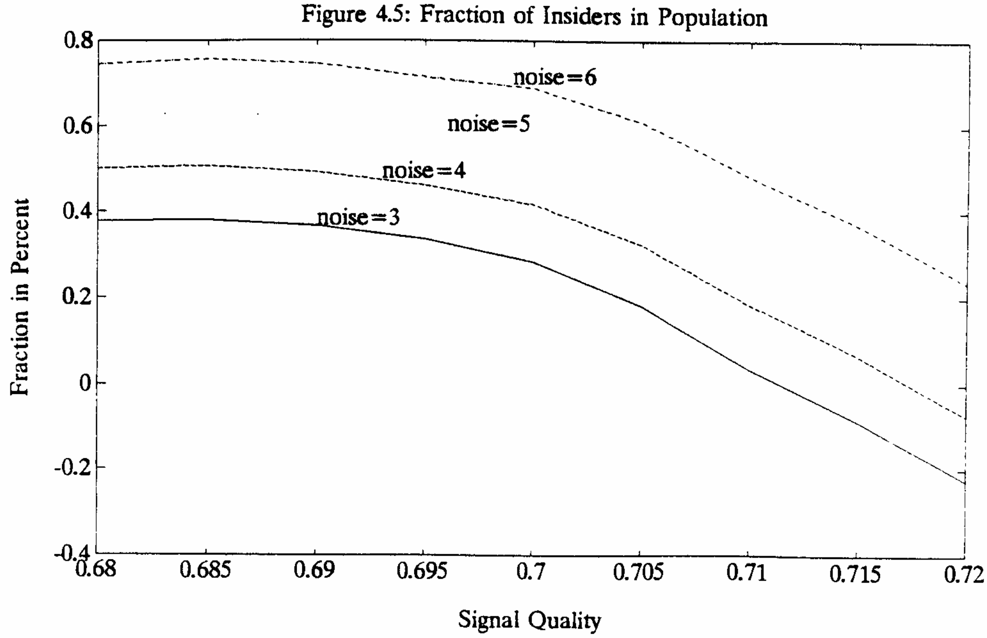


Figure 4.6: Increase of Total Output due to Insiders in %

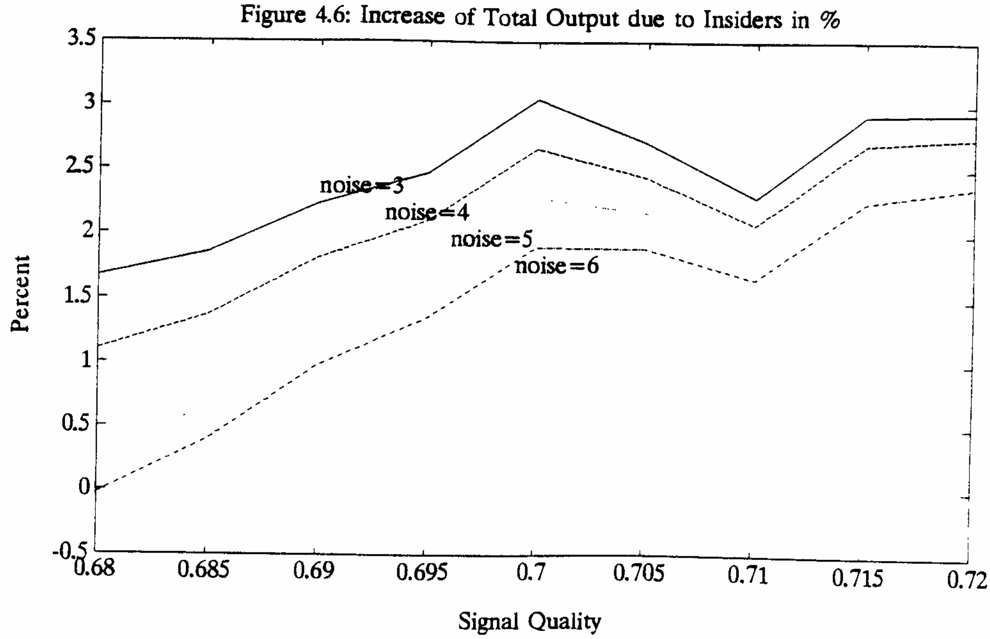


Figure 4.7: Required Growth

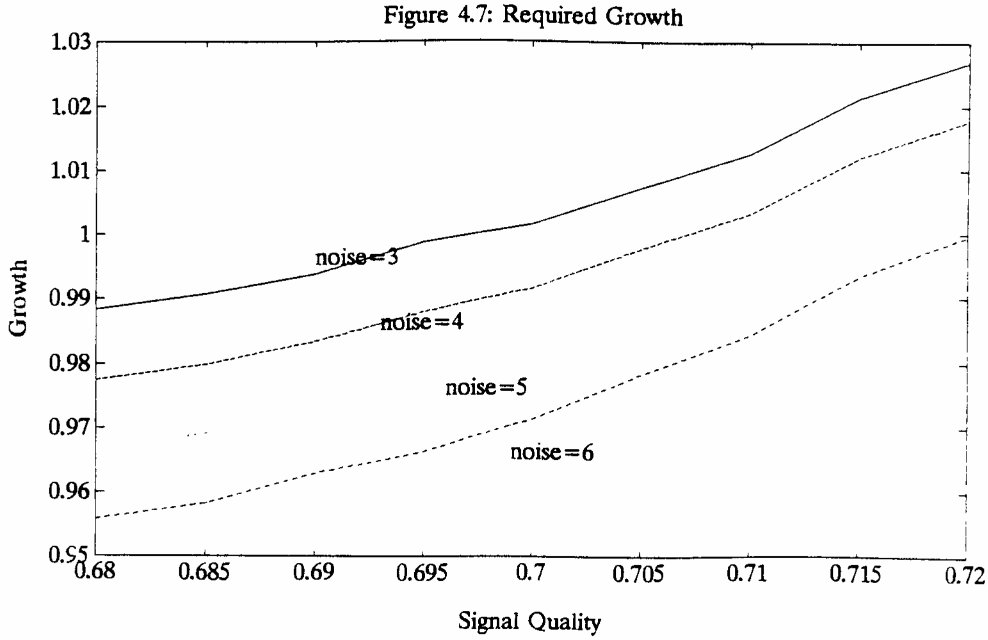


Figure 5.1: Individually Required Growth

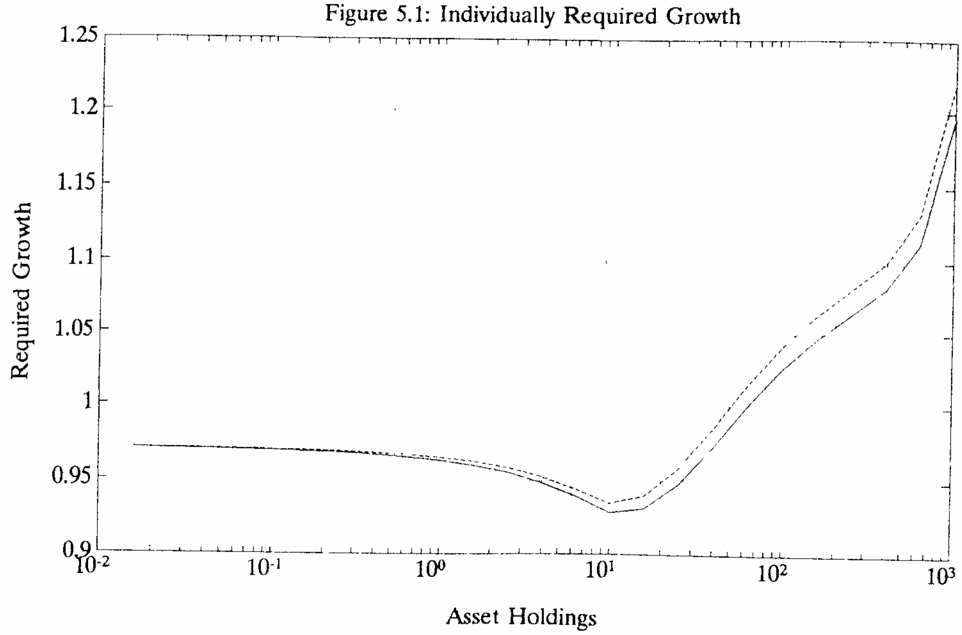


Figure 5.2: Voters in without-economy

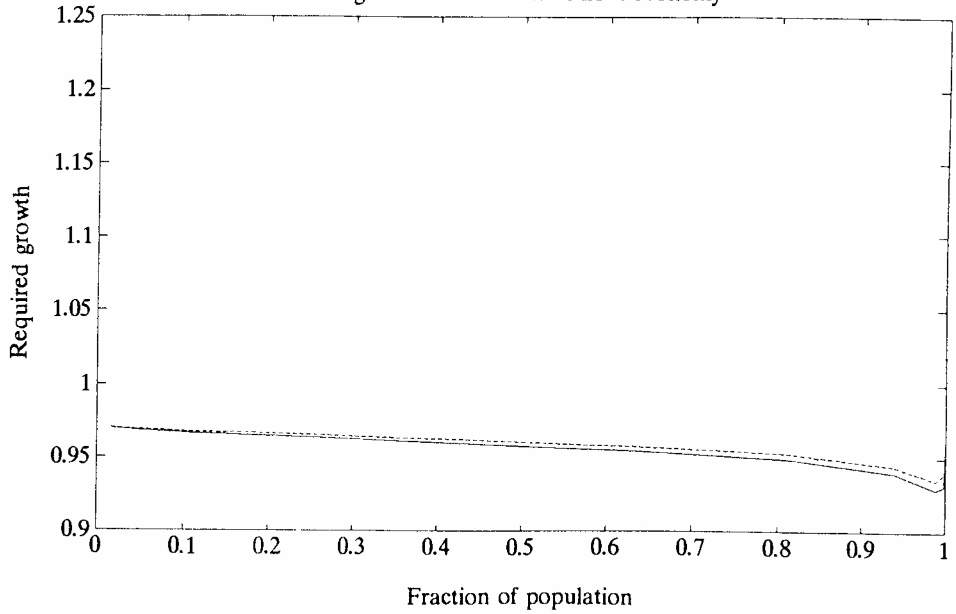


Figure 5.3: Voters in with-economy

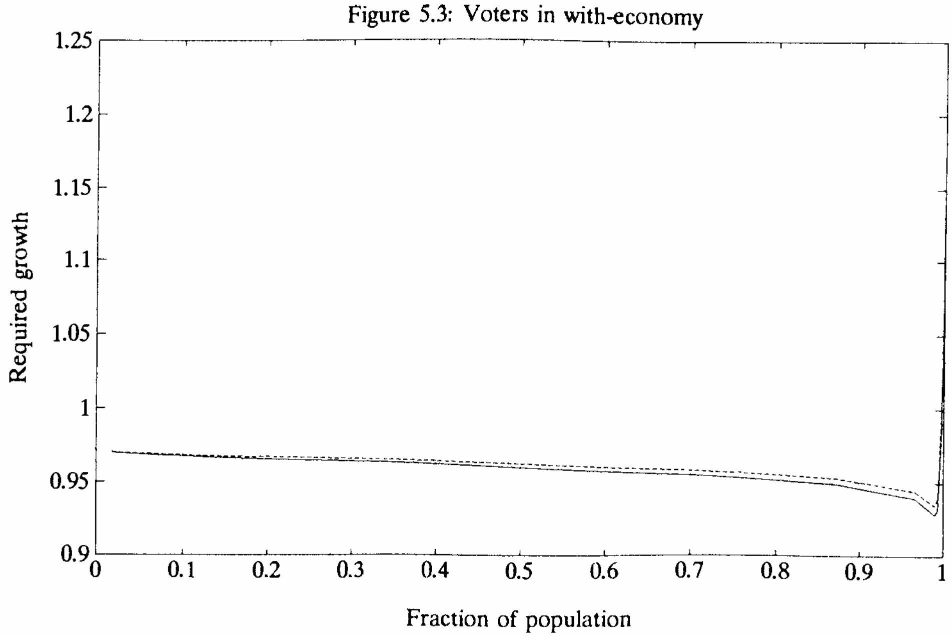


Figure 5.4.1: Individually required growth, same rel. position

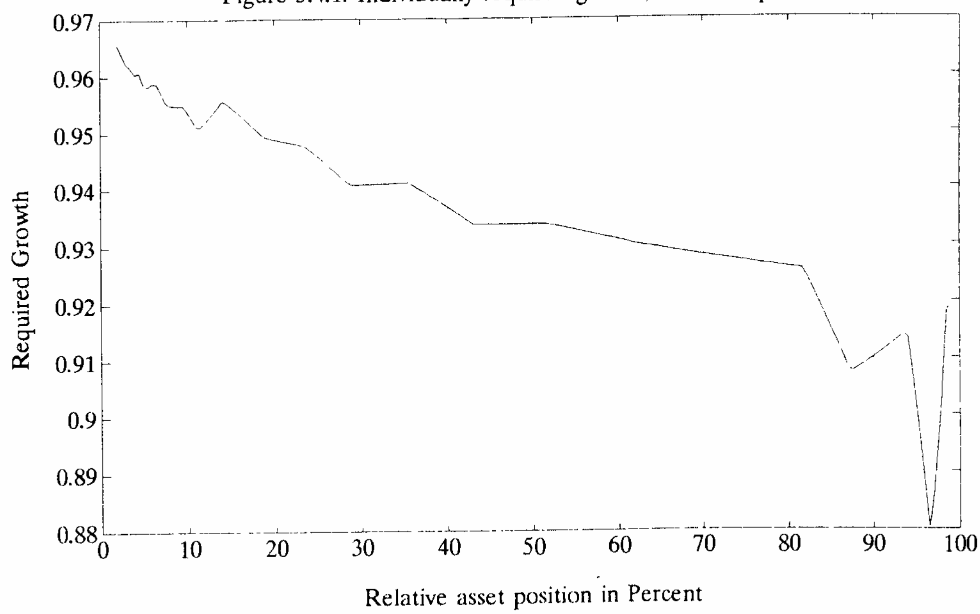


Figure 5.4.2: Individually required growth, same rel. position

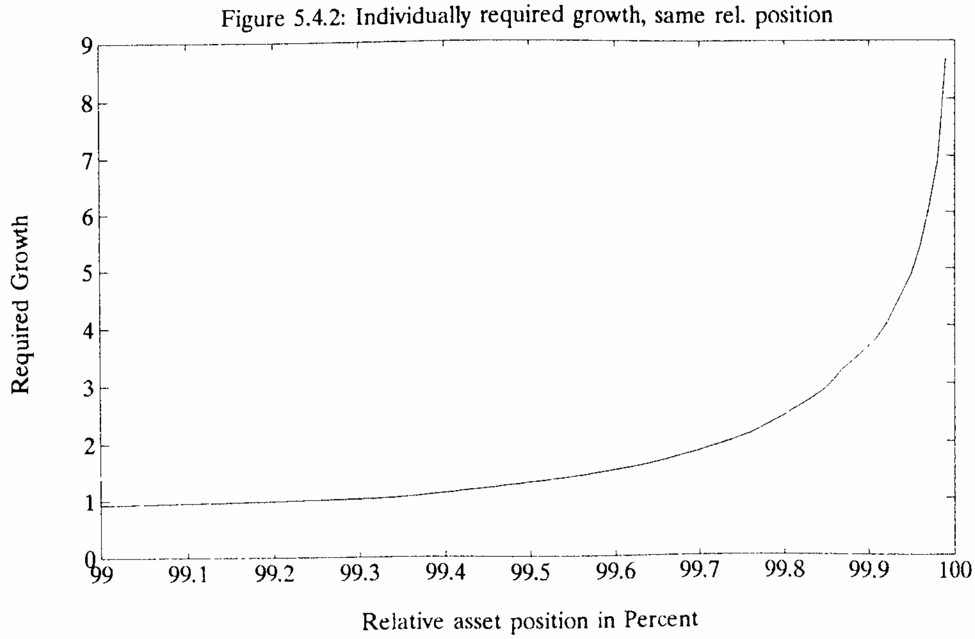


Figure 6.1.1: Concavity Test for Value Functions

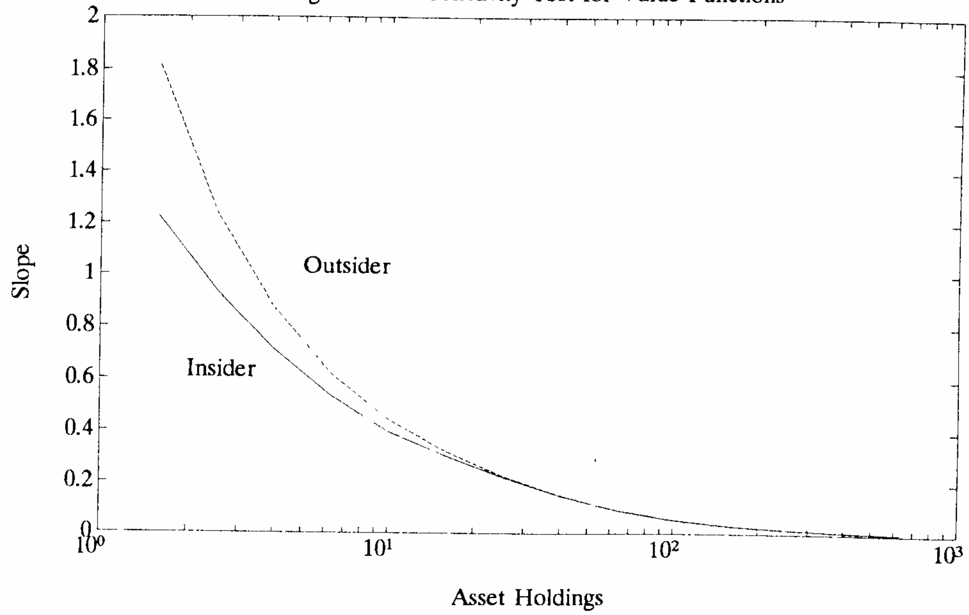


Figure 6.1.2: Concavity Test for Value Function

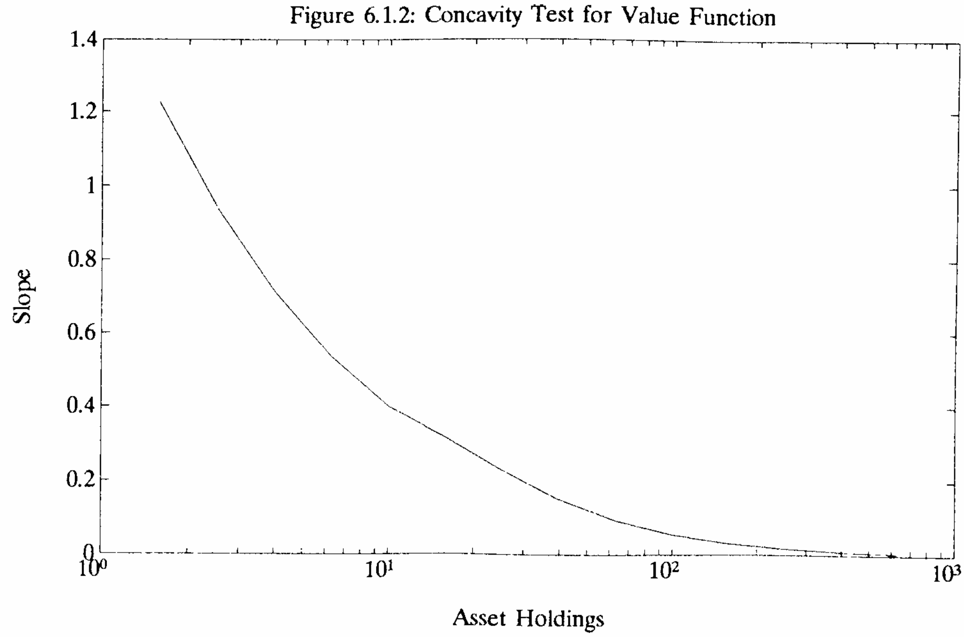


Figure 6.1.3: Difference of the Value Functions

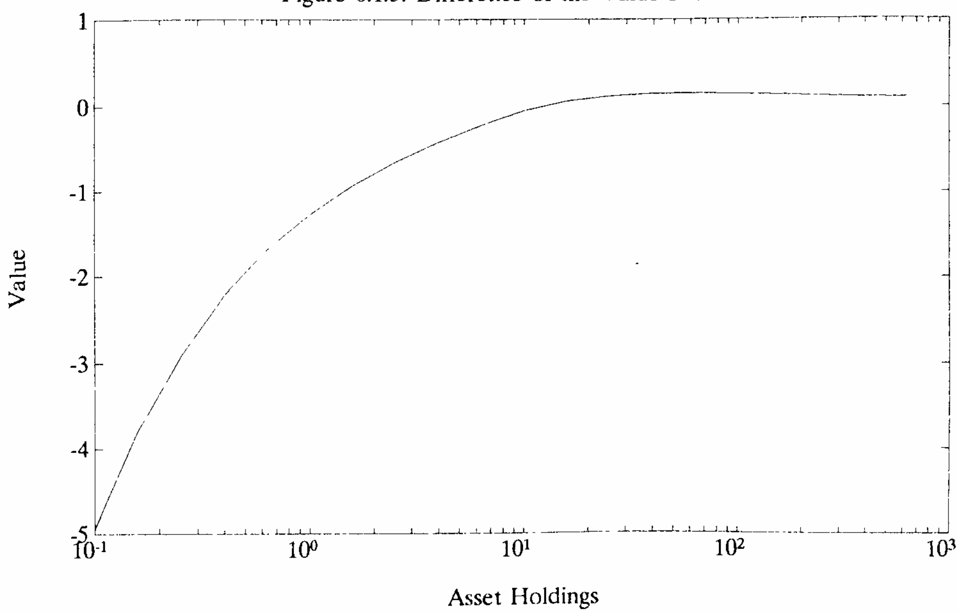


Figure 6.2.1: The Last Ten Value-Function Iterations

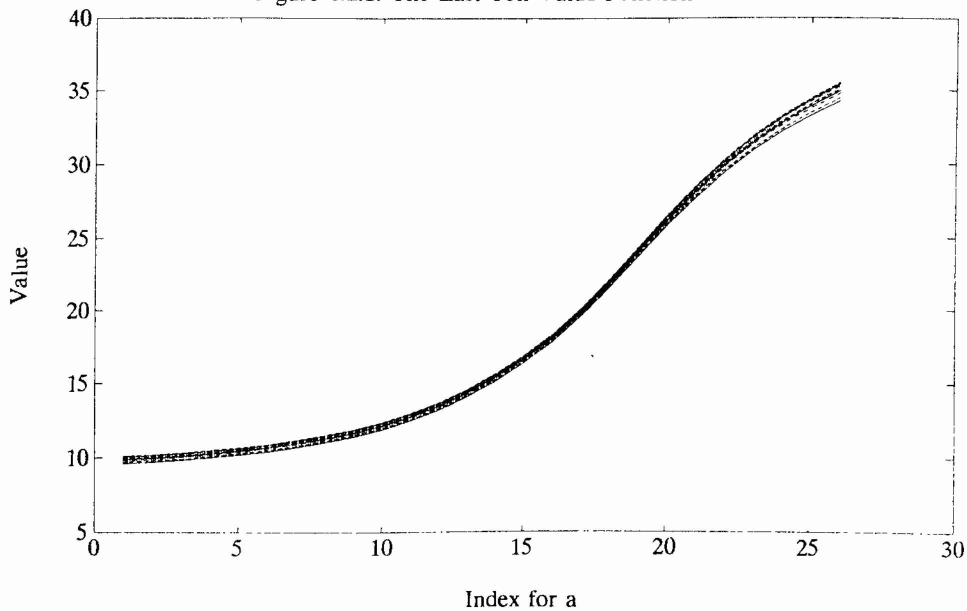


Figure 6.2.2: Extrapolated Value-Function Iterations

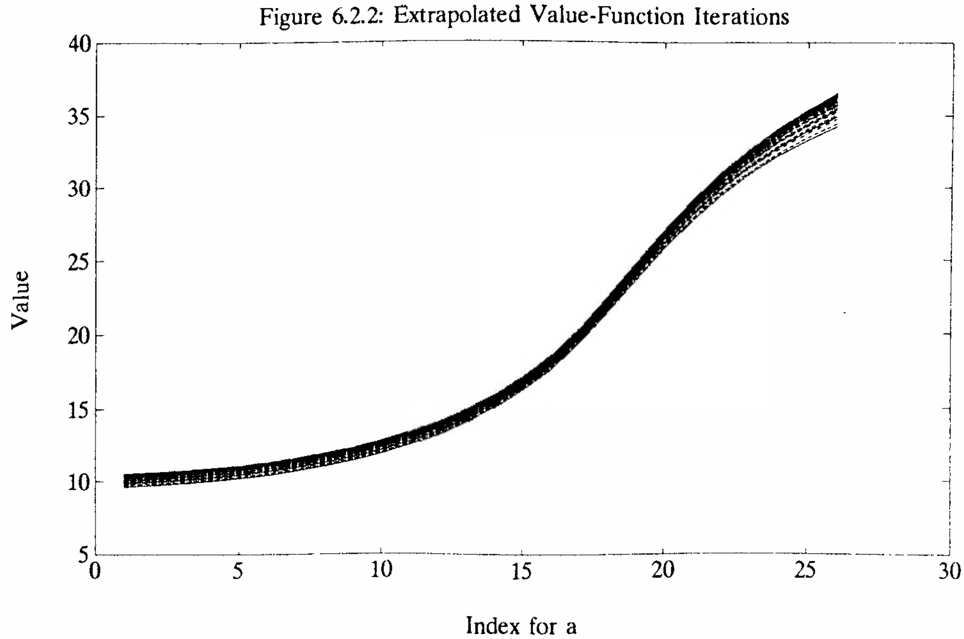


Figure 6.3.1: The Last Ten Stock-Investment(m=0,i=0)-Iterations

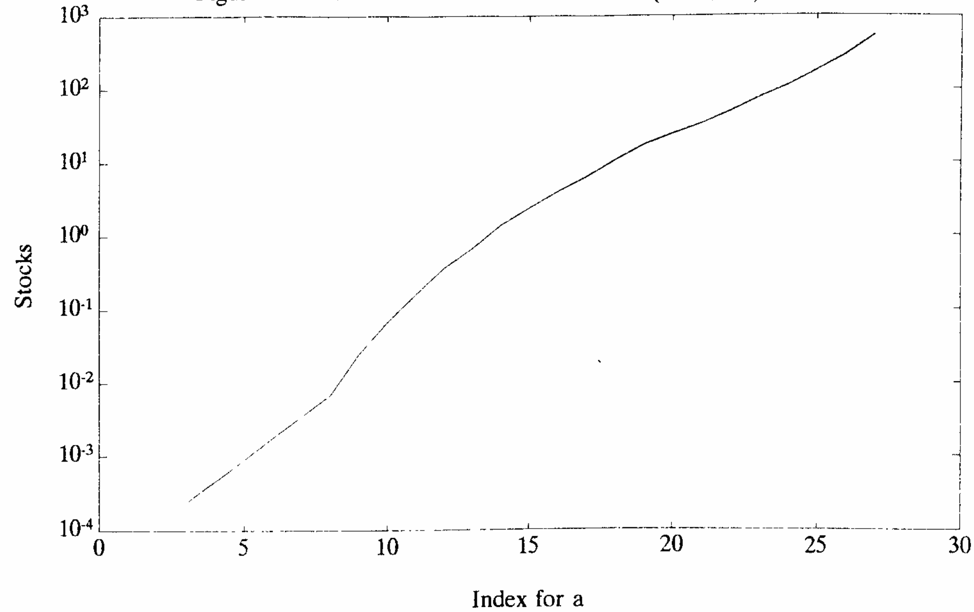


Figure 6.3.2: Extrapolated Relative Differences, Stock-Investment

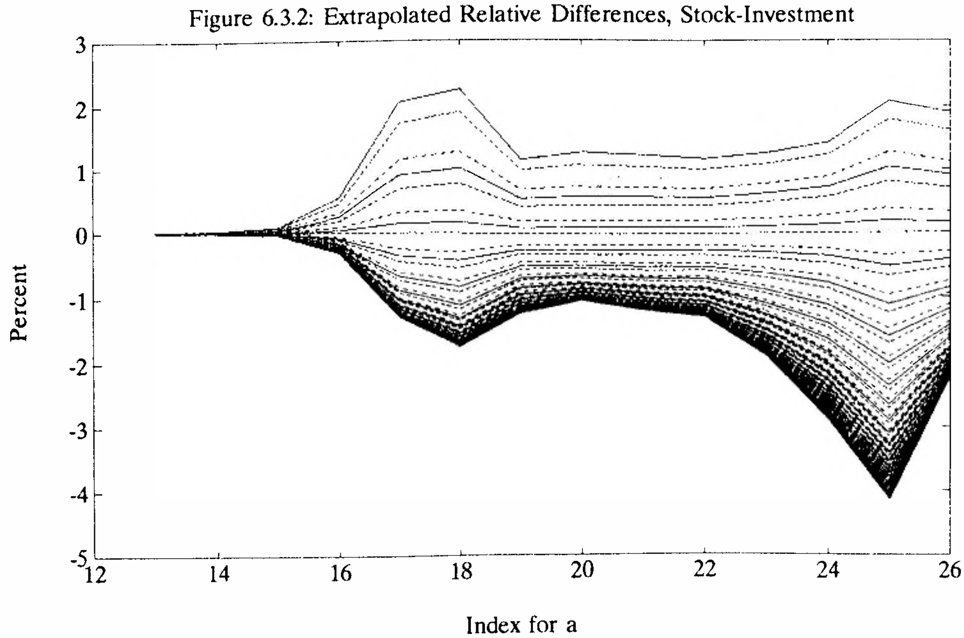


Figure 6.4.1: Difference in the Last Ten F_A-Iterations

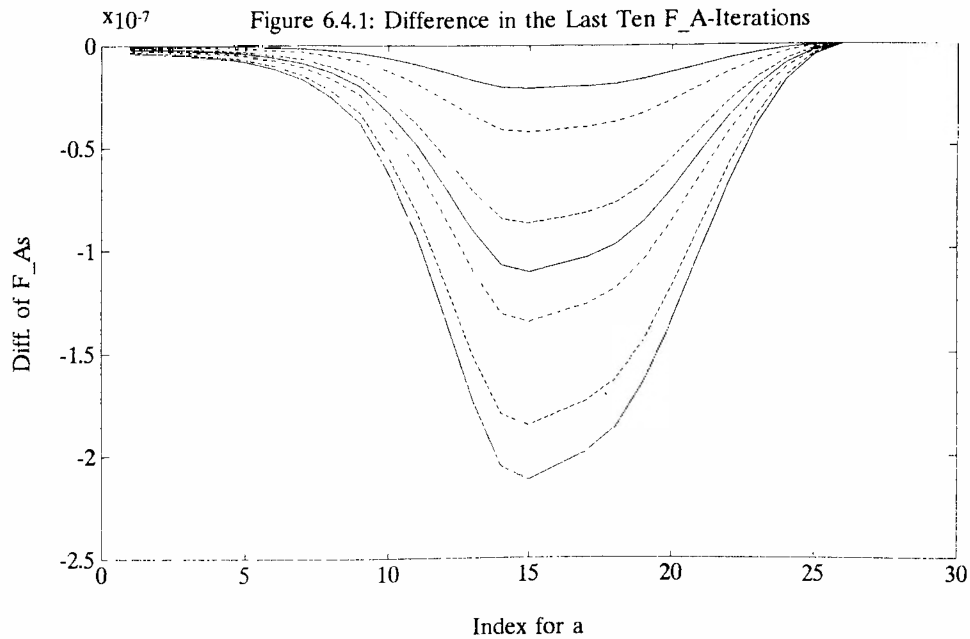


Figure 6.4.2: Extrapolated Differences, F_A-Iterations

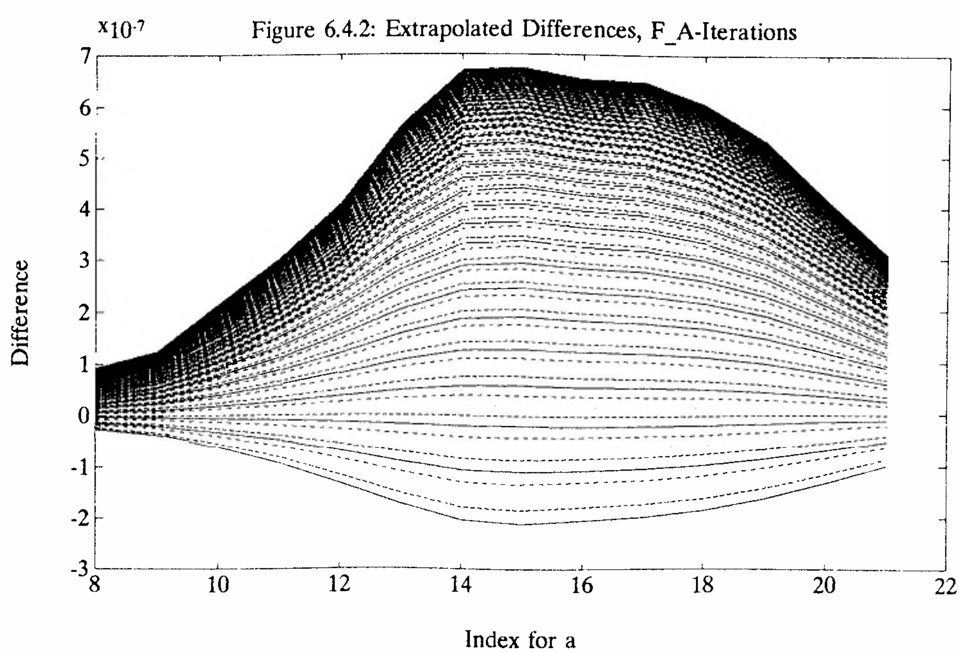


Figure 6.5: Differences of all P(i=0|l)-Iterations

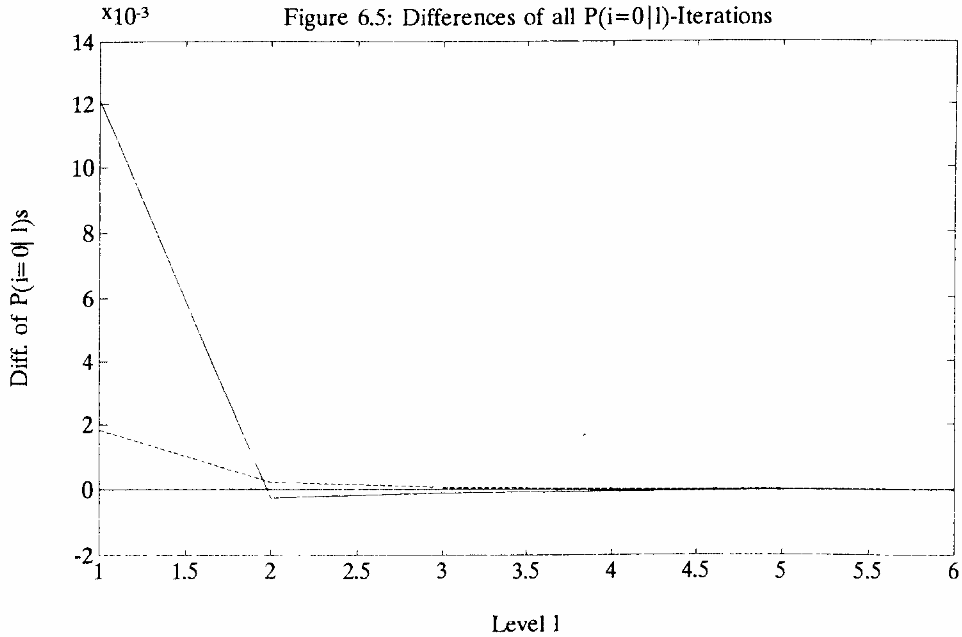


Figure 6.6: All Five P(g=0|l,i=0)-Iterations

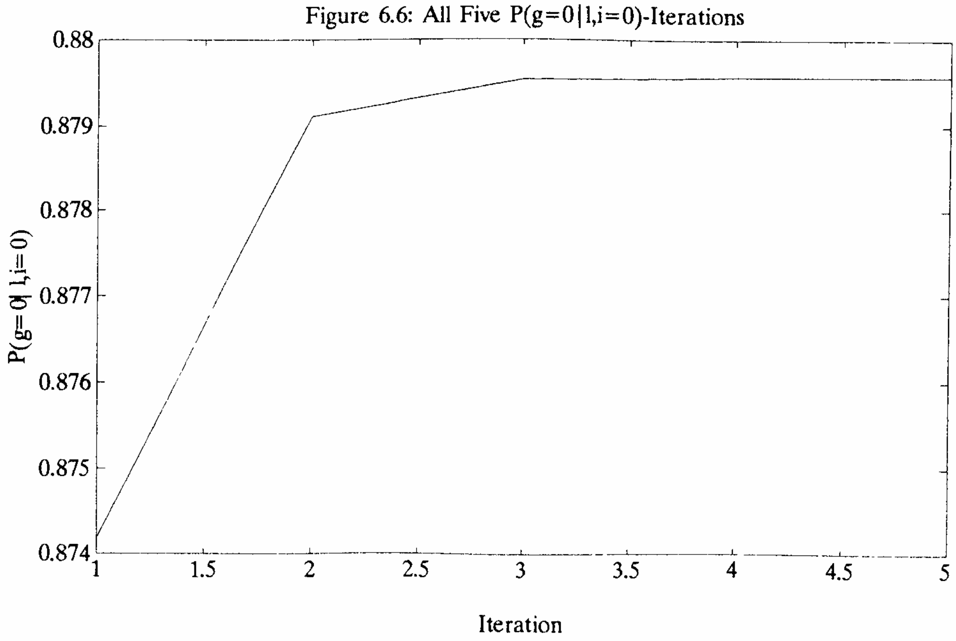


Figure 6.7.1: Differences of the Last Ten q1[0]-Iterations

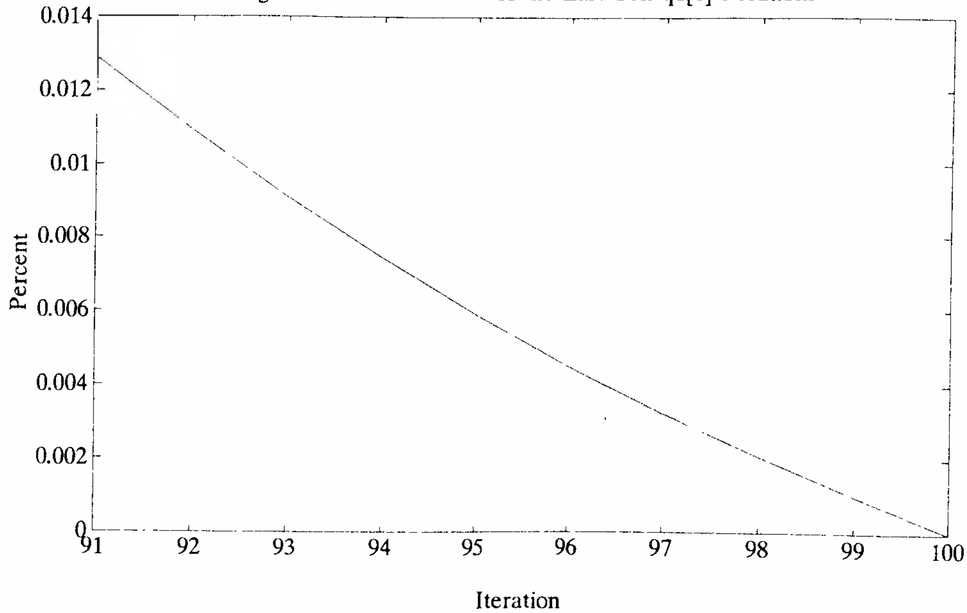


Figure 6.7.2: Exponential Extrapolation of q1[0]-Differences

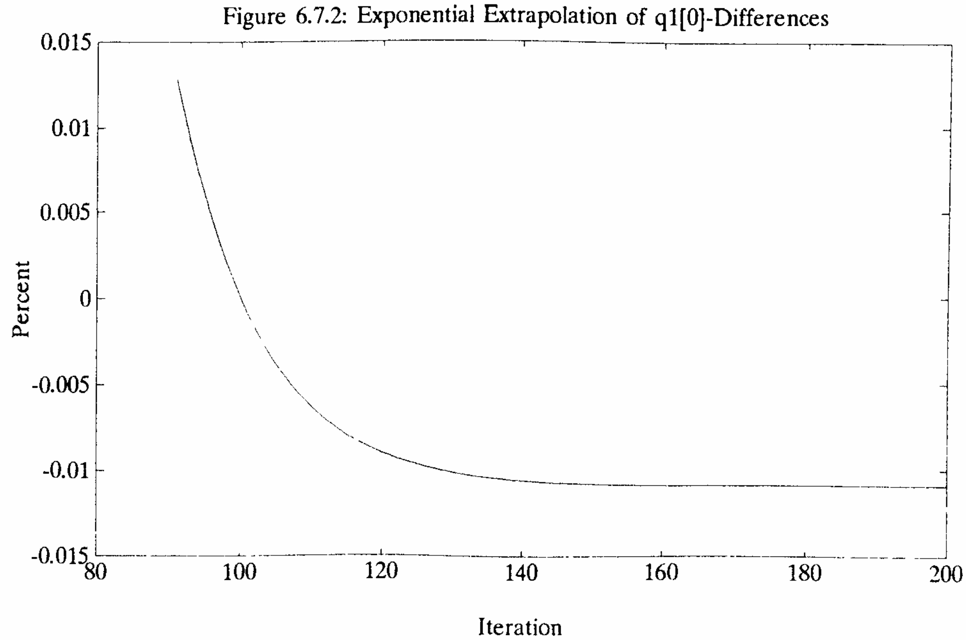


Figure 6.8.1: Differences of the Last Ten x[0]-Iterations

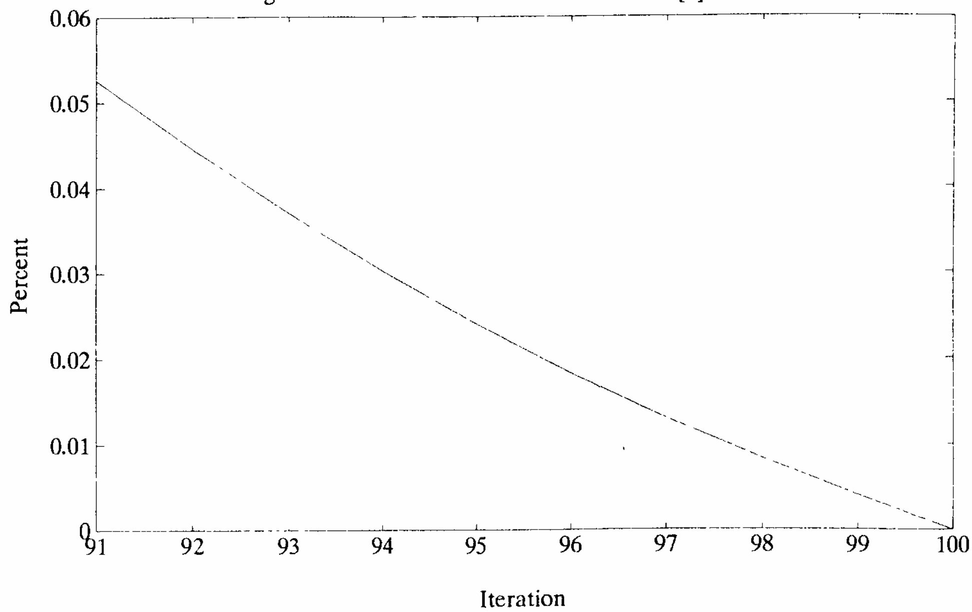
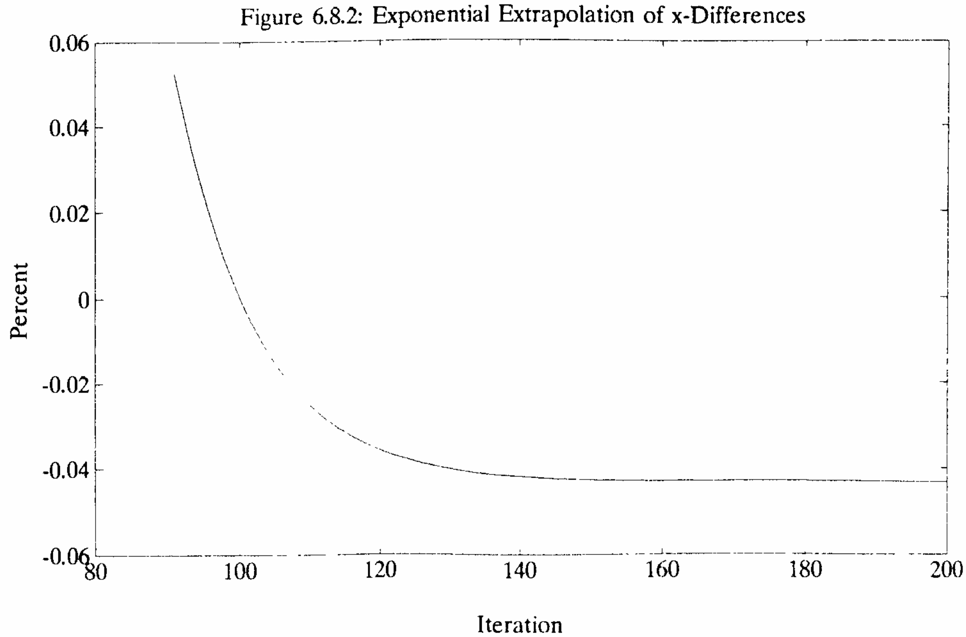


Figure 6.8.2: Exponential Extrapolation of x-Differences



Appendix I: The law of large numbers

In the model the following technical question arises: let $(X_l)_{l\in[0,1]}$ be a collection of independent and identically distributed random variables with mean μ and variance σ^2. Is there a meaningful way of stating a law of large numbers which states

$$\int X_l \, dl = \mu \quad ?$$

We will state and prove a simple version of this law of large numbers below, generalizations of which are needed in this paper. For its original source, generalizations and an extended discussion for how to interpret the integral in a mathematically more elegant way as a Pettis–integral, see Uhlig (1987).

As in Calculus, let

$$\Gamma = \{ (n, l_0, l_1, .., l_n, \psi_1, .., \psi_n) \mid n \in \{1,2,..\}, \\ 0 = l_0 < l_1 .. < l_n = 1, \; l_{j-1} \leq \psi_j \leq l_j, \; j=1,..,n \}$$

be the set of all partitions T of the interval [0,1]. For $T \in \Gamma$, we define the mesh

$$\zeta(T) := \max \{ l_j - l_{j-1} \mid j \in \{1,..,n\} \}.$$

In order to define the integral, we need a convergence concept for random variables. We chose the mean square as a measure of distance. What we want to define is the Riemann–type integral of a vector–valued function:

Definition:

Let $(X_l)_{l\in[0,1]}$ be a collection of random variables, defined on the probability space (Ω,Σ,P). If there is a random variable Y, such that

$$\lim_{\zeta(T)\to 0} E[\,(Y - \sum_{j=1}^{n} X_{\psi_j}(l_j - l_{j-1}))^2\,] = 0,$$

we write

$$Y = \int X_l \, dl$$

and call Y the integral of $(X_l)_{l\in[0,1]}$.

We call (X_l) **Riemann–type integrable.**

THEOREM A.1. (**The law of large numbers for a large economy**)

Let $(X_l)_{l\in[0,1]}$ be a collection of pairwise uncorrelated random variables with common finite mean μ and variance σ^2. Then (X_l) is Riemann–type integrable and we have

$$\mu = \int X_l \, dl.$$

PROOF:

Calculate

$$
\begin{aligned}
E[\,(\mu - \sum_{j=1}^{n} X_{\psi_j}(l_j - l_{j-1}))^2\,] &\\
&= \sum_{j=1}^{n} E[\,((X_{\psi_j} - \mu)(l_j - l_{j-1}))^2\,] \\
&= \sum_{j=1}^{n} (l_j - l_{j-1})^2 \sigma^2 \\
&\leq \zeta(T)\sigma^2 \sum_{j=1}^{n} (l_j - l_{j-1}) \\
&= \zeta(T)\sigma^2
\end{aligned}
$$

converging to zero as $\zeta(T)$ converges to zero. This completes the proof. •

Appendix II

Table 1: Notation:

$1_{\{.\}}$	indicator function, = 1, if {.} is true,
a	individual total asset holdings at the beginning of the period or after the lottery,
α	parameter of the CES–function $f(k,x) = (\kappa_1 k^\alpha + \kappa_2 x^\alpha)^{1/\alpha}$,
$\underline{a}$	cutoff point for insider–outsider decision,
b	individual mutual fund holdings, paying a sure return R,
β	discount factor,
c	c_j is individual consumption, $\bar{c}$ is aggregate consumption,
d	d(l) is the dividends (capital share/rental rates) for one unit of ("old") capital on level l,
D	D(l,i,g) is the aggregate stock demand by individuals for all technologies of category (l,i,g) in units of capital,
dom_Z	domain of the random variable Z,
$E[\cdot]$	Expectation operator. $E_X[f(x,y,z) \mid Y]$ denotes the conditional expectation of f(x,y,z) for a fixed z and randomly varying x, given the random variable Y.
f	k' = f(k,x) is the production function for new capital k' from old capital k and investment x,
$F_{\hat{a}}$	distribution function of assets before the lottery,
F_a	distribution function of assets after the lottery,
F_k	$F_k(l)$ is the fraction of all capital (in physical units) on level l,
F_N	distribution function for N,

F_Z distribution function for Z,

φ φ(l,i,g) is the fraction the mutual fund holds of the aggregate capital stock of category (l,i,g), $\varphi(l,i,g) = \varphi(l,i)\ \pi(l,i,g)$.

g index for the random **growth** rate Γ_g, $\gamma_{\tau t+1} = \Gamma_{g_{\tau t}}\ \gamma_{\tau t}$,

γ $\gamma_{\tau t}$ is the productivity parameter of an individual technology, $\bar{\gamma}_t$ is the maximal productivity parameter in period t,

Γ $\Gamma_0 > \Gamma_1 > 0$ are the random growth rates $\gamma_{\tau t+1} = \Gamma_{g_{\tau t}}\ \gamma_{\tau t}$.

h the function $h(q) = q\ f_k(\ 1,\ X(q)\ )$, where defined,

η coefficient of relative risk aversion,

i $i_{\tau t}$ is the random index for the **information** revealed by prices about technology τ at time t,

ins abbreviation for insider,

I,II,III,IV,V,VI parts of a period,

j index of an individual, $j \in [0,1]^2$,

ψ scaling factors for the calculation of probabilities (see the consistency–theorem),

k $k_{\tau t}$ is the total ("old") capital for technology τ, $\bar{k}_t$ is total aggregate capital (in physical units), produced in period t and productive in period t+1,

k_{-1} is the initial capital for each technology,

κ κ_1,κ_2 are parameters of the CES–function $f(k,x) = (\kappa_1 k^{\alpha} + \kappa_2 x^{\alpha})^{1/\alpha}$,

l $l_{\tau t}$ is the **level** of a technology τ in period t, l = 0,1,2,.. ,

$(l_{\tau t}, i_{\tau t})$ is the **type** of technology τ at time t, $(l_{\tau t}, i_{\tau t}, g_{\tau t})$ is the category of technology τ at time t.

λ Lebesgue measure,

m,M m=1,..,M is the random **message** index

μ $\mu_{\hat{a}}$ is the actuarily fair lottery an agent with beginning–of–the–period asset holdings $\hat{a}$ buys,

n $\bar{n}$ is the aggregate labor input, n(l) is the labor ratio for technologies on level l,

N individual endowment with productive labor,

outs abbreviation for outsider,

P denotes "probability":

$P(\Gamma_g)$ is the unconditional probability for growth rate Γ_g for a particular technology,

P(g,i|l) is the probability for growth rate Γ_g and information index i, conditional on the level l of a technology τ,

P(m | g) is the conditional probability for receiving message if the growth rate of a technology is Γ_g,

π π(l,i,g) is the assignment probability for category (l,i,g) due to the "imperfect" random selection of stocks by agents, etc.

q $q_1(l)$, $q_2(l,i)$ and $q_3(l,i)$ are prices for one unit of capital (one stock) of level l or type (l,i) during the period. q_1 and q_2 price "old" capital (before investment) and q_3 prices "new" capital,

θ $\theta = (P,\pi)$ collects the probability parameters of an economy

R gross return on the mutual fund between periods,

ρ exponent of capital in the Cobb–Douglas production function for output, s stock holdings, e.g. $s^{ins}(a,l,i,m)$ are the insider holdings of the risky stock, given his assets a, a stock of type (l,i) and the private message m,

t time index,

τ technology index. τ_{jt} is the technology assigned to agent j at time t,

U sum of all expected, discounted utilities,

u period per period utility for consumption,

v value function,

w wage,

$\aleph$ $\aleph_t$ denotes the (aggregate) state of the economy in period t,

x x(l,i) is investment per unit of old capital of type (l,i),

$\bar{x}_t$ is aggregate investment,

X X(q) (where defined) is the investment per unit of capital if the price per unit of new capital is q, $q\, f_x(\, 1\, ,\, X(q)\,) = 1$,

ξ $\xi_l = (\, \Gamma_1 \,/\, \Gamma_0\,)^l$. $\gamma_{\tau t} = \xi_l \bar{\gamma}_t$, if technology τ is on level l,

y $y_{\tau t}$ is output per individual technology,

$\bar{y}_t$ is aggregate output,

Z $Z_{\tau t}$ is the random variable drawn for technology τ at time t,

ζ steady state growth rate of output.

Appendix III.

Table 2: Structure of a period

Sequence of events.

part	variables	events and actions,
I	$\Gamma_{\tau t-1}, q_1, \hat{a}$	ante–signal trading (stock market 1), $\gamma_{\tau t}$ is now public information. We restrict agents to $\hat{a}_{jt} \geq 0$,
II	$\mu_{\hat{a}}, a$	lottery. Agents decide wether to become an "outsider" or an "insider". The agent specific labor–productivity N_{jt} is not known.
III	N,y,d,n,w	production of output. Wages $w\,N_{jt}$ are paid to outsiders. Insiders are busy acquiring information.
IV	Z, τ_{jt}, m	Agents "pick" a technology index τ_{jt}. technology shocks/ technology signals are realized,
V	q_2, x, q_3, i, φ	physical investment and post–signal trading (stock market 2). Mutual fund portfolio decisions. Noise trades/ informed trades.
VI		consume c_i.

Appendix IV

We need to show that a solution to (3.6), (3.7) and the equation preceeding (3.8), which we denote by (3.8) in the sequel, yields a solution to (3.1) to (3.3) in steady state. Let $\aleph$ be equal to the list of state variables $\aleph_0$ at time 0 in steady state. We denote by $\alpha\aleph$ the list of steady state variables after growth by factor α, i.e. wages, aggregate output, aggregate capital and so on are multiplied by factor α, whereas stock prices q_1, q_2, q_3, investment per unit of capital x and so on remain the same. Note that in steady state, $\aleph_t = \zeta \aleph_{t-1}$, $t \geq 1$. Let $\tilde{v}^{ins}$, $\tilde{v}^{outs}$ and $\tilde{v}$ denote a solution to (3.6) to (3.8). Define

$$v(a, \alpha\aleph) = \alpha^{1-\eta} \tilde{v}(a/\alpha) + \frac{1}{1-\beta} \frac{\alpha^{1-\eta} - 1}{1-\eta}, \tag{A.4.1}$$

likewise for v^{ins} and v^{outs}.

Theorem A.4.1.

(A.4.1) delivers a solution to (3.1) to (3.3).

Note that (A.4.1) implies in particular, that $v(a, \aleph_0) = \tilde{v}(a)$ at the time–0 steady state list $\aleph_0$. Furthermore, (A.4.1) allows for a simple way of comparing steady states for different parameters by calculating how much an economy had to grow to reach the same welfare level as the other economy. For suppose, economy 1 reaches the average welfare level v_1 and economy 2 reaches the average welfare level v_2. (A.4.1) then implies, that economy 2 would have to grow by the factor

$$\left(\frac{v_1 + \frac{1}{1-\beta}\frac{1}{1-\eta}}{v_2 + \frac{1}{1-\beta}\frac{1}{1-\eta}} \right)^{1/1-\eta} \tag{A.4.2}$$

to reach the same average welfare level.

PROOF:

The proof for our claim now follows from the following simple calculation, which has to be repeated likewise for v^{ins} and v. We have to check wether the value functions defined by (A.4.1) satisfy the dynamic programming problem equations (3.1) to (3.3). I.e. for v^{outs}, we have to show that

$$\begin{aligned} v^{outs}(a,\alpha\aleph) = \\ & E_{(N,l,i)}[\max_{c,b,s} \{ \frac{c^{1-\eta}-1}{1-\eta} + \beta E_e[v(a',\alpha\zeta\aleph) \mid i] \mid \\ & c + q_3(l,i)s + b \leq a + \alpha w N, \\ & 0 \leq a' = Rb + q_1(1+g)s \ \}]. \end{aligned}$$

Now, working from inside and using (A.4.1), the right hand side equals

$$\text{r.h.s.} = E_{(N,l,i)}[\max_{c\,,b,s} \{ \frac{c^{1-\eta}-1}{1-\eta} +$$
$$(\zeta\alpha)^{1-\eta}\beta E_e[\,\tilde{v}(a'/(\alpha\zeta))|\ i\,] + \frac{\beta}{1-\beta}\frac{(\zeta\alpha)^{1-\eta}-1}{1-\eta}\ |$$
$$c + q_3(l,i)s + b \le a + \alpha w\ N,$$
$$0 \le a' = Rb + q_1(1+g)s\ \}\ ]$$

(substituting $\tilde{a}' = a/(\alpha\zeta)$, $\tilde{a} = a/\alpha$, $\tilde{b} = b/\alpha$, $\tilde{c} = c/\alpha$ and $\tilde{s} = s/\alpha$)

$$\text{r.h.s.} = E_{(N,l,i)}[\max_{\tilde{c}\,,\tilde{b},\tilde{s}} \{ \frac{\tilde{c}^{1-\eta}-1}{1-\eta}\,\alpha^{1-\eta} + \frac{\alpha^{1-\eta}-1}{1-\eta} +$$
$$\alpha^{1-\eta}\zeta^{1-\eta}\beta E_e[\tilde{v}(\tilde{a}')|i] +$$
$$\frac{\beta}{1-\beta}\,\frac{\zeta^{1-\eta}-1}{1-\eta}\,\alpha^{1-\eta} + \frac{\beta}{1-\beta}\,\frac{\alpha^{1-\eta}-1}{1-\eta}\ |$$
$$\tilde{c} + q_3(l,i)\tilde{s} + \tilde{b} \le \tilde{a} + w\ N,$$
$$0 \le \tilde{a}' = R\tilde{b}\ /\ \zeta + q_1(1+g)\tilde{s}\ /\ \zeta\ \}\ ]$$

$$= \alpha^{1-\eta}\, E_{(N,l,i)}[\max_{\tilde{c}\,,\tilde{b},\tilde{s}} \{ \frac{\tilde{c}^{1-\eta}-1}{1-\eta}\,\alpha^{1-\eta} +$$
$$\zeta^{1-\eta}\beta E_e[\tilde{v}(\tilde{a}')|i] + \frac{\beta}{1-\beta}\,\frac{\zeta^{1-\eta}-1}{1-\eta}\ |$$
$$\tilde{c} + q_3(l,i)\tilde{s} + \tilde{b} \le \tilde{a} + w\ N,$$
$$0 \le \tilde{a}' = R\tilde{b}\ /\ \zeta + q_1(1+g)\tilde{s}\ /\ \zeta\ \}\ ]$$
$$+ \frac{1}{1-\beta}\,\frac{\alpha^{1-\eta}-1}{1-\eta}$$

$$= \alpha^{1-\eta}\,\tilde{v}^{outs}(\tilde{a}) + \frac{1}{1-\beta}\,\frac{\alpha^{1-\eta}-1}{1-\eta}$$
$$= v^{outs}(\,a\,,\,\alpha\aleph\,).$$

This finishes the calculation and the proof. •

Appendix V.

In this appendix, we discuss how to analyze the model by examining its individual parts. The key insight is that the analysis falls into two separate parts: the analysis of the production side of the economy and the analysis of the decision problem. At the ends, both parts have to be put together.

We denote with (P,π) the probability parameters of our economy. We assume that (P,π) is "rich", i.e. that no probability equals 0. Equilibria can only exist, if a certain consistency condition for the probability structure (P,π) is satisfied: we will derive this condition below. We will concentrate on equilibria in which the mutual fund holds shares of all types in his portfolio. A direct consequence of this is equation (5.1):

$$Rq_3(l,j) = \pi(g=0|i)\, q_1(l) + \pi(g=1|i)\, q_1(l+1).$$

We will prove below in Lemma A.V.2, that q_1 is strictly decreasing in l. Thus, the identifiability condition (vi) in the definition of a steady state equilibrium is met iff either $i \neq i'$ implies $\pi(g=0|i) \neq \pi(g=0|i')$ or $\pi(g=0|i)$ does not depend on i at all. We make the assumption, that this is true for the probability structure (P,π).

We now proceed to "dissect" the model. The first step regards the production sector.

V.a. The Production Side.

In order to prove our results below, we will restrict the investment function f to satisfy some further requirements:

ASSUMPTION A.V.1:

The investment function $k' = f(k,x)$ satisfies the following list of assumptions:

- f is homogeneous of degree 1,
- f is continous on $\mathbb{R}_+$,
- f is twice continuously differentiable on $\mathbb{R}_{++}$,
- $f(1,0) > 0$,
- $f_{xx}(1,x) < 0$ for all $x > 0$,
- $\lim_{x \to 0} f_x(1,x) = \infty$ and $\lim_{x \to \infty} f_x(1,x) \geq 0$,
- $\lim_{q \to 0} X(q) = 0$, where $X(q)$ is the (unique) solution to $f_x(1,x) = 1/q$ for $q > 0$.
- $\frac{d}{dq} f(1, X(q)) \to 0 \ (q \to 0)$.
- The function

$$h(q) = q\, f_k(1,X(q)),\ h(0) = 0$$

  is well defined and continuously differentiable on some interval $[0, q_{max})$, $0 < q_{max} \leq \infty$,
- $h(0) = 0$, $h'(0) < 1$,
- $h'(q) > 0$ for all $q \in (0, q_{max})$.

— $h(q) \to \infty$ as $q \to q_{max}$

This assumption can be justified by concidering its economic interpretations. E.g. it requires that the marginal product of investment decreases, given a constant level of old capital, and that old capital will be valued the higher the higher new capital will be valued. The assumption is violated by the usual linear investment function[1], but there is a fairly common class of functions that satisfies the assumption as the next lemma shows:

LEMMA A.V.1:

Let

$$f(k,x) = (\kappa_1 k^{\alpha} + \kappa_2 x^{\alpha})^{1/\alpha}$$

be the CES – investment function with parameters $\kappa_1 > 0$, $\kappa_1 < 1$, $\kappa_2 > 0$ and $1/2 < \alpha < 1$. Then f satisfies Assumption V.1.

---

[1]Actually, for a linear investment function of the type (2.10), the model becomes even simpler. It turns out, that investment will only be undertaken in the type (0,i) of technologies for which $\pi(g = 0|i)$ is maximal. In particular, all technologies with level $l \geq 1$ will be without investment forever. Since this is a somewhat odd feature and also since this feature creates great problems for versions of the model, where i is taken from a continuum rather than a discrete set, we decided rather to use e.g. CES–investment functions as described in Lemma V.1. For Lucas–trees (see 2.11), a steady state equilibrium typically does not exist since there will be no steady state distribution of capital unless we allow $\bar{\gamma}_0 = \infty$. Both cases can probably be analyzed as limiting cases with the CES–function described in Lemma V.1, however, if we let $\alpha \to 1.0$ (for the usual linear investment technology) and / or $\kappa_2 \to 0$ (for Lucas–trees).

PROOF:

Calculate that

$$X(q) = (\kappa_1 \,/\, ((\kappa_2 q)^{\alpha/(\alpha-1)} - \kappa_2))^{1/\alpha}$$

and

$$h(q) = q\, \kappa_1^{1/\alpha} (1+((\kappa_2^{1/\alpha} q)^{\alpha/(\alpha-1)} - 1)^{-1})^{(1-\alpha)/\alpha}.$$

The rest is algebra as well. •

THEOREM A.V.1:

Given a wage w > 0, the following rules solve the maximization problem of the production firms:

$$n(l) = (\xi_l\, \gamma_0\, (1-\rho) \,/\, w)^{1/\rho},$$

$$y(l) = (\xi_l\, \gamma_0)^{1/\rho} (1-\rho)^{(1-\rho)/\rho}\, w^{1 - 1/\rho}$$

and

$$d(l) = \rho\, y(l)$$

PROOF:

Calculate. •

THEOREM A.V.2

Given the probability structure (P,π) and a gross interest rate $R \geq 1$, there exists a wage $w(R)$ such that for every wage $w > w(R)$ and with the dividend rule above, there are strictly positive prices q_1, q_2, and q_3 as well as an investment rule x, solving the maximization problem of the investment firm and the mutual fund.

Furthermore, this solution is unique (given the wage w) in the sense that there is no other solution q_1', q_2', q_3' and x with the property that $0 < q_1'(l) < q^*$ and $0 < q_1(l) < q^*$ for some q^* and all l.

PROOF:

For an interior solution, the two pricing equations resulting from the mutual fund decision as well as

$$q_3(l,i)\, f_x(\, 1,\, x(l,i)\,) = 1$$

and

$$q_2(l,i) = q_3(l,i)\, f_k(\, 1,\, x(l,i)\,)$$

from the first–order condition of the investment firms problem have to be satisfied. These last two equations imply

$$q_2(l,i) = h(\, q_3(l,i)\,).$$

We proceed to prove that a solution exists to these equations for any $w > w(R)$, where $w(R)$ is defined below.

Given w, let $d(l,w)$ be $d(l)$ as calculated from the rule in Theorem A.V.1. Observe that we have $d(0,w) > d(l,w) > 0$ for all $l \in \{ 0,1,2, \dots \}$. Define

$$\bar{q}(w) = \qquad \inf \{ q > 0 \mid q R - h(q) > d(0,w) \},$$

where h is the function defined in Assumption A.V.1 and where we let $\bar{q}(w) = \infty$, if the set is empty. Observe that $\bar{q}(w)$ is monotonously decreasing in w and $\lim_{w \to \infty} \bar{q}(w) = 0$, since $d(w,0)$ is monotonously decreasing in w and $\lim_{w \to \infty} d(w,0) = 0$. Define

$$\bar{\bar{q}} = \qquad \sup \{ \tilde{q} \mid h'(q) < R \text{ for all } q \in [0, \tilde{q}] \}.$$

It follows from Assumption A.V.1, that $\bar{\bar{q}} > 0$. Hence, let

$$w(R) = \qquad \inf \{ w > 0 \mid \bar{q}(w) < \bar{\bar{q}} \}.$$

It follows that $w(R) < \infty$. Pick $w > w(R)$. By construction, there is a q^* with $\bar{q}(w) < q^* < \bar{\bar{q}}$, such that

$$d(0,w) + h(q^*) < R\, q^*.$$

and

$$\max \{ h'(q) \mid 0 \leq q \leq q^* \} < R.$$

Let ℓ^∞ be the space of all bounded sequences of real numbers $q = (q(l))_{l=0}^{\infty}$ with the norm $\| q \| = \sup_{l} | q(l) |$ and the usual order structure, and let ℓ^∞_{I+1} be the space of all bounded sequences of (I+1)–dimensional vectors $q = (q(l,i))_{l=0,i=0}^{\infty \quad I}$ with the norm $\| q \| = \sup_{l,i} | q(l,i) |$ and the usual order structure. Define operators Q_1, Q_2 and Q_3 according to

$$Q_1 \colon D_1 \to \ell^\infty,$$
$$(Q_1(q))(l) = d(l) + \sum_{i=0}^{I} P(i|l)\, q(l,i),$$
$$Q_2 \colon D_2 \to \ell^\infty_{I+1},$$
$$(Q_2(q))(l,i) = h(q(l,i)) \text{ and}$$
$$Q_3 \colon D_3 \to \ell^\infty,$$
$$(Q_3(q))(l) = (\pi(g{=}0|i)q_1(l) + \pi(g{=}1|i)q_1(l+1))/R,$$

where

$$D_1 = \{ q \in \ell^\infty_{I+1} \mid 0 \leq q(l,i) \leq h(q^*) \},$$
$$D_2 = \{ q \in \ell^\infty_{I+1} \mid 0 \leq q(l,i) \leq q^* \}, \text{ and}$$
$$D_3 = \{ q \in \ell^\infty \mid 0 \leq q(l) \leq R\, q^* \}.$$

It follows that all operators are well defined and that $Q_1(D_1) \subseteq D_3$, $Q_2(D_2) \subseteq D_1$ and $Q_3(D_3) \subseteq D_2$. Furthermore,

$$\| Q_1(q') - Q_1(q'') \| \leq \| q' - q'' \|,$$
$$\| Q_2(q') - Q_2(q'') \| \leq \max_{0 \leq q \leq q^*} h'(q),$$

by the mean–value theorem and

$$\| Q_3(q') - Q_3(q'') \| \leq \| q' - q'' \| / R.$$

Thus define

$$Q : D_3 \to D_3, \quad Q = Q_1 \circ Q_2 \circ Q_3.$$

It follows from the construction of q^*, that Q is a contraction mapping with contraction factor $\nu = \max_{0 \leq q \leq q^*} | h'(q) | / R < 1$. It follows from the contraction mapping theorem that Q has a unique fixed point $q_1 \in D_3$. Define $q_3 = Q_3(q_1)$, $q_2 = Q_2(q_3)$ and $x(l,i) = X(q_3(l,i))$, where X is defined in Assumption A.V.1. Observe, that prices are strictly positive and that $q_1(l) < q^*$. It follows from the first order conditions of the investment firms problem and the equilibrium conditions for the mutual fund that q_1, q_2, q_3 and x are a solution. The uniqueness claim follows from the uniqueness of the fixed point of Q and the properties of the first order conditions of the investment firms problem. This concludes the proof of Theorem A.V.2. •

LEMMA A.V.2:

Given θ and R, let $q_1^{(w)}$, $q_2^{(w)}$, $q_3^{(w)}$ and $x^{(w)}$ be the solutions found in Theorem A.V.2 for $w > w(R)$. Then

(i) the mappings $w \mapsto q_1^{(w)}$, $w \mapsto q_2^{(w)}$, $w \mapsto q_3^{(w)}$, $w \mapsto x^{(w)}$ are continuous and strictly monotonously[2] decreasing,

(ii) $\lim_{w \to \infty} x^{(w)} = \lim_{w \to \infty} q_2^{(w)} = \lim_{w \to \infty} q_3^{(w)} = 0$, $\lim_{w \to \infty} q_1^{(w)} = 0$,

(iii) Fix i and $w > w(R)$. Then $q_1^{(w)}$, $q_2^{(w)}(l,i)$, $q_3^{(w)}(l,i)$ and $x^{(w)}(l,i)$ are strictly decreasing in l in the sense that e.g. $x^{(w)}(l+1,i) < x^{(w)}(l,\iota)$, all l, all i, all ι. Furthermore, $\lim_{l \to \infty} q_1^{(w)}(l) = \lim_{l \to \infty} q_2^{(w)}(l,i) = \lim_{l \to \infty} q_3^{(w)}(l,i) = \lim_{l \to \infty} x^{(w)}(l,i) = 0$.

PROOF:

For $w > w(R)$, let $Q_1^{(w)}$ and $Q^{(w)}$ be the operators Q_1 and Q introduced in the proof to Theorem A.V.2. Observe that the operators Q_2 and Q_3 do not depend on w. Note, that all operators are monotone, i.e. that

$$q' \geq q'' \Rightarrow Q^{(w)}q' \geq Q^{(w)}q'',$$

similarly for "$\gg$". It follows immediately from this equation and the fact that

---

[2]We introduce the notation $q \gg q'$ to mean $q(l) > q'(l)$ for every l, likewise for $q,q' \in \ell^{\infty}_{I+1}$. Strict monotonicity then means strict monotonicity in every entry, i.e. $w > w' > w(R)$ implies $q_1^{(w')} \gg q_1^{(w)}$, etc.

Q is a contraction that

$$q \gg Q^{(w)}q \Longrightarrow q \gg q_1^{(w)},$$

since then

$$q \gg Q^{(w)}q \geq Q^{(w)}(Q^{(w)}q) \geq Q^{(w)}(Q^{(w)}Q^{(w)}q) \ldots \to q.$$

Pick $w > w' > w(R)$. Following the notation in the previous proof, choose the same q^* for both, w and w' by choosing the q^* that works for w', since it will also work for w: this way, the operators associated with w and w' and any w'' in between operate on the same domains and the operators $Q^{(w'')}$ are contractions for the same contraction factor $\nu = \max_{0 \leq q \leq q^*} |h'(q)|/R$. By definition of $Q_1^{(w)}$ and the monotonicity of the dividend rule in Theorem A.V.1, we have

$$Q^{(w')}q \gg Q^{(w)}q \text{ for all } q \in D_3.$$

Thus, we get the monotonicity claimed in (i), since we have $q_1^{(w')} = Q^{(w')}q_1^{(w')} \gg Q^{(w)}q_1^{(w')}$, hence $q_1^{(w')} \gg q_1^{(w)}$. For the continuity, choose an arbitrary, but sufficiently small $\epsilon > 0$. It is clear by the definition of Q_1 and Q and the continuity of d(l,w) in w, that we can find $\delta > 0$, so that for any w'' with $w' \leq w'' < w + \delta$, we have

$$Q^{(w'')}q_1^{(w')} + \epsilon \geq Q^{(w')}q_1^{(w')} = q_1^{(w')}.$$

Define $\dot{q}^{(0)} = q_1^{(w')}$ and $\dot{q}^{(n+1)} = Q^{(w'')}\dot{q}^{(n)} + \epsilon$. Define $\ddot{q}^{(0)} = q_1^{(w')}$ and $\ddot{q}^{(n+1)} = Q^{(w'')}\ddot{q}^{(n)}$. Observe that $\dot{q}^{(n)} \in D_3$ for all n, if ϵ is sufficiently small. It follows the last equation and an analogue to the second inequality above that

$$\dot{q}^{(n)} \geq q_1^{(w')} \geq \ddot{q}^{(n)}.$$

On the other hand, it follows from the contraction property and an induction argument that

$$\ddot{q}^{(n)} \rightarrow q_1^{(w'')}$$

and

$$\| q'^{(n)} - q_1^{(w'')} \| \leq \epsilon \,/\, (1 - \nu).$$

Thus,

$$\| q_1^{(w')} - q_1^{(w'')} \| \leq \epsilon \,/\, (1 - \nu),$$

proving continuity from above. Continuity from below is proved similarly. This completes the proof for claim (i) of the lemma for q_1, the properties for q_2, q_3 and x follow easily.

For part (ii), fix some $\overline{w} > w(R)$. Fix the q^* corresponding to $\overline{w}$ and contraction parameter ν, which we are going to use for all $w \geq \overline{w}$. Pick any $\epsilon > 0$ and find $w_\epsilon \geq \overline{w}$, such that $d(0,w_\epsilon) \leq (1-\nu)\,\epsilon$. We claim that $\| q_1^{(w)} \| \leq \epsilon$ for all $w \geq w_\epsilon$, thus proving (ii). To prove the claim, pick any $w \geq w_\epsilon$. Define $q^{(0)} \equiv 0$ and $q^{(n+1)} = Q^{(w)} q^{(n)}$. By definition of $Q^{(w)}$ and the monotonicity properties of d, we have $q^{(1)} \leq (1-\nu)\,\epsilon$ or $\| Q^{(w)} q^{(0)} - q^{(0)} \| \leq (1-\nu)\,\epsilon$. Hence,

$$\| Q^{(w)} q^{(n)} - q^{(n)} \| \leq \nu^n (1-\nu)\,\epsilon$$

for all n by induction. The triangle inequality together with $q^{(n)} \to q^{(w)}$ now deliver the claim.

For the claims on limits in (iii), fix w and find the contraction parameter ν. Pick $\epsilon > 0$ and find l_ϵ such that $d(l,w) \leq \epsilon\,(1-\nu)$ for all $l \geq l_\epsilon$. Define $\tilde{d}(l,w) = 0$ for $l < l_\epsilon$ and $\tilde{d}(l,w) = d(l,w)$ for $l \geq l_\epsilon$. Define $\tilde{Q}_1$ and $\tilde{Q}$ analogously and note, that $\tilde{Q}$ is a contraction mapping on D_3 with contraction parameter ν and a fixed point $\tilde{q}_1$. Note further that by definition of Q and $\tilde{Q}$, $\tilde{q}_1(l) = q_1(l)$ for all $l \geq l_\epsilon$. Define as in the proof of (ii) the vectors $\tilde{q}^{(0)} = 0$, $\tilde{q}^{(n+1)} = \tilde{Q}\,\tilde{q}^{(n)}$. Note that $\| \tilde{Q}\,\tilde{q}^{(0)} - \tilde{q}^{(0)} \| \leq \epsilon\,/\,(1-\nu)$ and thus $\| \tilde{q}_1 \| \leq \epsilon$. This proves the result, that $\lim_{l\to\infty} q_1(l) = 0$, the other limiting results follow.

We finally derive the result about the strict monotonicity in (iii).

Fix w. Note that Q has the following property:

$$q(l)\geq q(l+1) \text{ for all } l \Longrightarrow Qq(l)>Qq(l+1) \text{ for all } l$$

Since $q^{(n)} \to q_1^{(w)}$, where $q^{(0)} \equiv 0$, and $q^{(n+1)} = Q\, q^{(n)}$, (iii) follows. This concludes the proof of Lemma A.V.2. •

THEOREM A.V.3:

The steady state growth rate ζ has to satisfy

$$\zeta = \Gamma_0^{1/(1-\rho)}.$$

PROOF:

This follows directly from from the formula for y(l) in Theorem A.V.1 and the definition of ζ in the the definition of the steady state equilibrium •

We now need to impose a second technical assumption about our investment function f and the probabilities involved.

ASSUMPTION A.V.2:

Given (P,π) and given any $R \in (R_{min},R_{max}) \neq \emptyset$ for some R_{min}, R_{max}, the investment function f satisfies the following list of conditions:

– $$\sum_{i=0}^{I} P(\, g=0, i \mid l=0\,)\, f(\,1\,,0\,) < \zeta,$$

- $$\lim_{w \downarrow w(R)} \sum_{i=0}^{I} P(g=0,i|l=0)\ f(1,x^{(w)}(0,i)) > \zeta$$
- If for some $w > w(R)$

  $$\sum_{i=0}^{I} P(g=0,i|l=0)\ f(1,x^{(w)}(0,i)) = \zeta,$$

  then for $l \geq 1$, we have

  $$\sum_{i=0}^{I} P(g=0,i|l)\ f(1,x^{(w)}(l,i)) < \zeta$$

This assumption is not very restrictive. The first inequality is satisfied if $\Gamma_0 > 1$ and $f(1,0) \leq 1$, i.e. if there is no appreciation of capital (e.g. $\kappa_1 \leq 1.0$ for the CES–investment function). The second inequality is satisfied for the CES–investment function if κ_1 is sufficiently close to 1 and the probability $P(g=0 \mid l=0)$ is sufficiently close to 1, because x is monotone in w by Lemma A.V.2 (i). Likewise, by Lemma A.V.2 (iii), the third part in Assumption A.V.2 is satisfied if $P(g=0,i|l)$ is independent of l. However in general we do not want to make this latter assumption up front (see the consistency theorem below).

Theorem A.V.4:

If $\Gamma_0 > 1.0$ and given (P,π), R and aggregate labor supply $\bar{n} > 0$, there is a unique[3] level of aggregate capital $\bar{k}$, a wage w, a

[3]The phrase "unique" here means that we restrict ourselves to the prices and investment rules from Theorem V.2, which are unique in the sense explained there.

distribution[4] F_k, prices q_1, q_2 and q_3 and an investment rule x, such that the equilibrium conditions for the production side of the economy are satisfied:

- F_k is stationary,
- firms maximize profit and
- the labor market clears.

PROOF:

By the continuity and strict monotonicity of $x^{(w)}$ in w and by $x^{(w)} \to 0 \; (w \to \infty)$ from Lemma A.V.2 and by the first two equations of Assumption A.V.2, there is a unique wage w, such that

$$\sum_{i=0}^{I} P(\, g = 0, i \mid l = 0 \,) \, f(\, 1 \,, x^{w}(0,i) \,) = \zeta.$$

Observe that this is necessary for the stationarity of F_k. Now define $\tilde{F}_k(0) = 1.0$. For $l \geq 1$, define

$$\tilde{F}_k(l) = \psi(l) \, \tilde{F}_k(l-1),$$

where

[4]The term "distribution" always refers to "probability distribution", i.e. a distribution yields nonnegative weights for measurable sets and integrates out to 1.0 over the whole set.

$$\psi(l) = \frac{\sum_{i=0}^{I} P(g=1,i \mid l\text{-}1)\ f(\ l\ ,\ x^{W}(l\text{-}1,i)\ )}{\zeta - \sum_{i=0}^{I} P(g=0,i\ \mid l\ )\ f(\ l\ ,\ x^{W}(\ l\ ,i)\ )}.$$

Observe, that $\tilde{F}_k(l)$ is well defined and strictly positive because of third part of Assumption A.V.2. Define $\overline{\psi} = \lim\sup_{l \to \infty} \psi(l)$. With the help of Lemma A.V.2 (iii), it is easy to see, that $\overline{\psi} \leq 1\ /\ \zeta < 1$. It follows that

$$0 < \overline{\tilde{F}}_k = \sum_{l=0}^{\infty} \tilde{F}_k(l) < \infty,$$

since the tail is dominated by a geometric sum. Hence let

$$F_k(l) = \tilde{F}_k(l)\ /\ \overline{\tilde{F}}_k.$$

It is easily verified that the stationarity condition in the definition of the steady state equilibrium is satisfied. Also, $F_k(\cdot)$ is the only distribution compatible with the formulas for stationarity at the beginning of this proof and in the definition of a steady state equilibrium. Finally, let

$$\bar{k} = \bar{n}\ /\ \sum_{l=0}^{\infty} F_k(l)\ n(l),$$

where n(l) is given by Theorem A.V.1. This completes the proof of Theorem

A.V.4. •

We now come to the second part of our analysis.

A.V.b. Analyzing the decision problems of the agents

We now want to analyze the decision problem as given by equations (3.6) through (3.8), which we call problem (A). For that, we write the decision problem in a simpler way.

First note, that the wage is simply a scale factor for everything, i.e. denote by $v(\hat{a};\tilde{w})$, $v^{ins}(a;\tilde{w})$, $v^{outs}(a;\tilde{w})$, $c^{ins}(a,l,i,m;\tilde{w})$ and so on the solutions to problem (A) for some wage $\tilde{w}$ and let a different wage w be given. Let $\omega := w \,/\, \tilde{w}$ be the ratio of the two wages. It is then easy to see that a solution for $\tilde{w}$ can be obtained via

$$
\begin{aligned}
v(\hat{a};w) &:= \omega^{1-\eta}\, v(\hat{a}/\omega;\tilde{w}) + \frac{\omega^{1-\eta} - 1}{1 - \eta}\frac{1}{1 - \beta}, \\
v^{ins}(a;w) &:= \omega^{1-\eta}\, v^{ins}(a/\omega;\tilde{w}) + \frac{\omega^{1-\eta} - 1}{1 - \eta}\frac{1}{1 - \beta}, \\
c^{ins}(a,l,i,m;w) &:= \omega\, c^{ins}(a,l,i,m;\tilde{w}),
\end{aligned}
$$

etc.. Heuristically, suppose $\tilde{c}_t$ is a time – path for consumption for the wage $\tilde{w}_t = \tilde{w}\, \zeta^t$ and $c_t := \omega\, \tilde{c}_t$ a time – path for consumption for the wage $w_t = w\, \zeta^t = \omega\, \tilde{w}_t$. Calculate, that

$$\sum_{t=0}^{\infty} \beta^t \frac{(\omega\tilde{c}_t)^{1-\eta}-1}{1-\eta} =$$

$$\omega^{1-\eta}\sum_{t=0}^{\infty} \beta^t \frac{\tilde{c}_t^{\,1-\eta}-1}{1-\eta} + \sum_{t=0}^{\infty} \beta^t \frac{\omega^{1-\eta}-1}{1-\eta}.$$

Thus we may assume w.l.o.g. that $\tilde{w} = 1$: for the problem in the text with a given wage w simply use the translation rule above.

Secondly, we eliminate the possibility for outsiders to buy stocks (and thus the uncertainty and expectation – formation that goes along with it), keeping in mind, that we have to prove below at the appropriate point, that the decision problem for the outsider does not look different, if we include the stock – buying possibility again.

Next, observe that we can write the decision problem for the insider simply in terms of probabilities: let s and b a solution to his original problem. Define a new security with price $\pi($ g=0 | i), which pays R, if g=0 happens and pays 0, if g=1 occurs. Let

$$\tilde{s} = \frac{q_1(l) - q_1(l+1)}{R} s$$

be the investment in this new security and

$$\tilde{b} = b + \frac{q_1(l+1)}{R} s$$

the corresponding new holdings of the mutual fund. Observe that by virtue of the pricing equation (A.5.1), we have

$$c + \pi(0 \mid i) \tilde{s} + \tilde{b} = c + q_3(l,i)\, s + b,$$
$$R(\tilde{b} + (1-g)\tilde{s}) \neq R b/\zeta + q_1(l+g)s/\zeta,\ g = 0,1,$$

i.e. we can rewrite the insiders problem in terms of the new security and use the following translation rule to get the asset– and bond–holdings for the original problem:

$$s = \frac{R}{q_1(l) - q_1(l+1)}\, \tilde{s}$$
$$b = \tilde{b} - \frac{q_1(l+1)}{q_1(l) - q_1(l+1)}\, \tilde{s}.$$

We formulate the implicit indifference of agents regarding the level l of the stock that was picked for them in the following

THEOREM A.V.5:

In part IV of the period, agents are indifferent between technologies of different levels.

PROOF:

Consider two types (l,i) and (l',i) of technologies, which differ only in their level. Given the type (l,i), suppose an agent chooses consumption c_l, stock holding s_l and holdings of mutual fund shares b_l. Given type (l',i), define $c_{l'} = c_l$ as well as

$$s_{l'} = \frac{q_1(l) - q_1(l+1)}{q_1(l') - q_1(l'+1)} s_l \text{ and}$$

$$b_{l'} = b_l + \frac{s_l}{R} \frac{q_1(l')\, q_1(l+1) - q_1(l)\; q_1(l'+1)}{q_1(l') - q_1(l'+1)}.$$

It is now easy to verify that

$$c_l + b_l + q_3(l,i)s_l = c_{l'} + b_{l'} + q_3(l',i)s_{l'}$$

$$Rb_l/\zeta + q_1(l)s_l/\zeta = Rb_{l'}/\zeta + q_1(l')s_{l'}/\zeta = \tilde{a}_0' \text{ and}$$

$$Rb_l/\zeta + q_1(l+1)s_l/\zeta = Rb_{l'}/\zeta + q_1(l'+1)s_{l'}/\zeta = \tilde{a}_1'.$$

I.e. both portfolios (s_l ,b_l) and ($s_{l'}$,$b_{l'}$) satisfy the same budget constraint with the same level of consumption, and both portfolios deliver the same next–period asset positions $\tilde{a}_0'$ with probability $\pi(g=0 \mid i)$ (resp. $\pi(g=0 \mid i,m)$ for an insider) and $\tilde{a}_1'$ with probability $\pi(g=1 \mid i)$ (resp. $\pi(g=1 \mid i,m)$ for an insider). Thus, the agent must be indifferent. •

Furthermore, the choice of the lottery μ will be particularly simple and (in plain terms) of the form:

- choose and accept a fair gamble for a, which delivers one of two values $a_o \geq 0$ or $a_i \geq 0$.
- If a_o is delivered, become an outsider with a_o.

— If a_i is delivered, become an insider with a_i.

Gambles which deliver $a_o = a$ or $a_i = a$ for sure are legitimate. In formulating the decision problem, we can restrict agents to lotteries of this simple type (as we already formulated in (3.8)), keeping in mind that we have to prove that the availability of more general lotteries does not improve the situation of the agent.

Finally, it is possible to eliminate the growth rate and the constant term in the value function formulation. I.e. define

$$\tilde{\beta} = \beta\, \zeta^{1-\eta},$$
$$\tilde{R} = R \,/\, \zeta, \text{ and}$$
$$\tilde{\zeta} = 1.0\ .$$

If $\tilde{v}$, $\tilde{v}^{outs}$, $\tilde{v}^{ins}$ is a solution to problem A with $\tilde{\beta}$,$\tilde{R}$ and $\tilde{\zeta}$ replacing β,R and ζ, we will find a solution to the original problem by using the same decision rules and

$$v(a) = \tilde{v}(a) + \frac{1}{1 - \zeta^{1-\eta}\beta} \frac{\beta}{1 - \beta} \frac{\zeta^{1-\eta} - 1}{1 - \eta} \text{ or}$$
$$v(a) = \tilde{v}(a) + \frac{1}{1 - \eta} \left(\frac{1}{1 - \zeta^{1-\eta}\beta} - \frac{1}{1 - \beta} \right),$$

likewise for v^{outs}, v^{ins}.

Thus, instead of analyzing the problem A, we analyze the following problem B. Problem B will yield a "tilda" – solution (with $\tilde{w} = 1$) which translates back into a solution to problem A via the rules above. Of course, in order to formulate problem B, we leave the tildas away for the sake of a more convenient notation. Denote the per – period utility from consumption by u:

$$u(c) = \frac{c^{1-\eta} - 1}{1 - \eta}$$

with the convention that $u(0) = -\infty$ for $\eta \geq 1$.

Then we have:

<u>Problem B:</u>

$$v^{ins}(a) = \sum_{i=0}^{I} \sum_{m=0}^{M} \pi(i)\ \pi(\ m \mid i\ )\ v_{i,m}^{in\,s}(a),$$

$$v_{i,m}^{in\,s}(a) = \max_{c\ ,b,s} \{\ u(c) + \beta \sum_{g=0}^{1} \pi(g|i,m)\ v(\ a_g'\ ) \mid$$

$$c + \pi(\ g=0 \mid i\ )\ s + b \leq a,\ c \geq 0,$$

$$a_g' = R(\ b + (1-g)s\ ) \geq 0,\ g=0,1\ \},$$

where the minimum – variance portfolio is selected, if several choices of c,b,s deliver the maximum,

$$v^{outs}(a) = \int v_e^{outs}(a + N)\ dF_N,$$

$$v_e^{outs}(y) = \max_{c,b} \{ u(c) + \beta v(a') \mid$$

$$c+b \leq y, c\geq 0, b\geq 0, a'=Rb \},$$

$$v(a) = \max_{a_i, a_o} \{ P_o v^{outs}(a_o) + P_i v^{ins}(a_i) \mid$$

either $0 \leq a_o \leq a \leq a_i$ or $0 \leq a_i \leq a \leq a_o$,

and $P_o = \frac{a_i - a}{a_i - a_o}$, $P_i = \frac{a - a_o}{a_i - a_o}$ with the convention for $\frac{0}{0}$, that P_o=1.0, P_i=0.0 if $v^{outs}(a) \geq v^{ins}(a)$ and P_o=0.0, P_i=1.0 else },

where the gamble with the minimal distance $|a_i - a_o|$ is selected, if several choices of a_i and a_o deliver the maximum.

We will have reasons to compare the solutions to solutions of the following problem C, which is a generalization of the standard consumption – savings – problem (see Stokey – Lucas, with Prescott (1989), p. 126) in that there is uncertainty with respect to the labor – income: this is essentially problem B without insiders. This problem C is an important benchmark to prove equilibrium in the full model.

Problem C:

$$\check{v}(a) = \int \check{v}_e(a + N)\, dF_N,$$

$$\check{v}_e(y) = \max_{c,b} \{ u(c) + \beta \check{v}(a') \mid$$

$$c + b \leq y,\ c \geq 0,\ b \geq 0, a' = Rb \}.$$

LEMMA A.V.3:

Define a return $\bar{R}$ via

$$\bar{R} := \tilde{R} \max_{i,g,m} \frac{\pi(g \mid i,m)}{\pi(g \mid i)}.$$

Then $\bar{R}$ is the maximal expected return, an insider can earn on his portfolio in problem B.

PROOF:

Let assets a be given and fix the consumption c for the insider, $0 < c \leq a$. Let b and s be any portfolio choice in the insiders decision problem, which finances c, i.e. we have

$$-\frac{a - c}{\pi(g=1|i)} \leq s \leq \frac{a - c}{\pi(g=0|i)}$$

and

$$b = a - c - \pi(g=0|i)s.$$

The expected return on that portfolio is then calculated to be

$$\text{Ret}(b,s) = \sum_{g=0}^{1} \pi(g\,|\,i,m)\,\frac{R(b + (1 - g)s)}{b + \pi(g=0\,|\,i)s}$$
$$= R\,\frac{b + \pi(g=0\,|\,i,m)s}{b + \pi(g=0\,|\,i)\,s}.$$

If $\pi(g=0\,|\,i,m) \geq \pi(g=0\,|\,i)$, then

$$\text{Ret}(b,s) \leq \text{Ret}\left(0, \frac{a - c}{\pi(g=0\,|\,i)}\right) = R\,\frac{\pi(g=0\,|\,i,m)}{\pi(g=0\,|\,i)} \leq \overline{R}$$

by definition of $\overline{R}$. If $\pi(g=0\,|\,i,m) \leq \pi(g=0\,|\,i)$, then likewise

$$\text{Ret}(b,s) \leq \text{Ret}\left(\frac{a - c}{\pi(g=1\,|\,i)}, \frac{c - a}{\pi(g=1\,|\,i)}\right)$$
$$= R\,\frac{\pi(g=1\,|\,i,m)}{\pi(g=1\,|\,i)} \leq \overline{R},$$

concluding the proof. •

We make the following general assumptions:

<u>**Assumption (A.V.3):**</u>

(i) $0 < \beta < 1$, $1 \leq R$,

(ii) $\beta\,\overline{R} < 1$,

(iii) $\int N\,dF_N = 1$, $\int N^2\,dF_N < \infty$, $\int N^{1-\eta}\,dF_N < \infty$, $\int N^{-\eta}\,dF_N = \infty$ and F_N has a density $F_N'(N)$ with respect to the Lebesgue measure which is strictly positive for all $N > 0$,

(iv) $\eta > 1$.

Observe, that (i) and (ii) are equivalent to the following assumptions about our original problem via our translation rules above:

(i') $0 < \beta\zeta^{1-\eta} < 1$, $1 \leq R$,

(ii') The maximal expected return $R \max_{i,g} \frac{\pi(g \mid i,m)}{\pi(g \mid i)}$, an insider can earn on his portfolio in the original problem, is strictly smaller than $R^\star := \zeta^\eta/\beta$, the benchmark interest rate of the standard neoclassical growth model.

To analyze problem B, we define two operators T_e^{outs} and $T_{i,m}^{in\,s}$, which will map future value functions into present solution to the after – randomness problems of the outsider or the insider. Let

$$W = \{ w : \mathbb{R}_+ \to \mathbb{R} \mid w \text{ is continuous, increasing, concave, } MIN \leq w(a) \leq MAX \text{ for all } a \},$$

be the set of "admissable" value functions, where

$$MIN = \min\{ \frac{\int N^{1-\eta} dF_N - 1}{(1-\beta)(1-\eta)}, 0 \}$$

and

$$\text{MAX} = \frac{1}{(1 - \beta)(\eta - 1)} > 0.$$

Note for the proofs below that $\text{MAX} > \frac{c^{1-\eta} - 1}{1 - \eta} + \beta\,\text{MAX}$ for all $c \in \mathbb{R}$. Define operators T_e^{outs} and T_e^{ins} from W into the set of all functions from $\mathbb{R}_+$ into $\{-\infty\} \cup \mathbb{R}$ via

$$(T_e^{outs} w)(y) := \max_{c\,,b} \{\, u(c) + \beta w(a') \mid c+b \le y,\ c \ge 0, b \ge 0,\ a' = Rb \,\}$$

and

$$(T_{i,m}^{ins} w)(y) := \max_{c\,,b,s} \{\, u(c) + \beta \sum_{g=0}^{1} \pi(g \mid i,m)\, w(a_g') \mid c + \pi(\, g=0 \mid i\,)\, s + b \le a,\ c \ge 0,\ a_g' = R(\, b + (1-g)s\,) \ge 0,\ g=0,1 \,\}.$$

I.e. these operators deliver the function v_e^{outs} and $v_{i,m}^{ins}$ if we plug in the solution v of problem B for w.

Lemma A.V.4:

Let f be concave, increasing. Let $x < y$ and $\Delta > 0$. Then

$$f(\, y + \Delta\,) - f(\, y\,) \le f(\, x + \Delta\,) - f(\, x\,).$$

If f is strictly concave, we have "$<$".

PROOF:

By concavity, we have

$$f(x+\Delta) \geq \lambda f(x) + (1-\lambda) f(y+\Delta)$$

or

$$f(x+\Delta) - f(x) \geq (1-\lambda)\,(f(y+\Delta) - f(x))$$

and likewise

$$f(y+\Delta) - f(y) \leq \mu\,(f(y+\Delta) - f(x)),$$

where $1-\lambda = \dfrac{\Delta}{y+\Delta-x} = \mu$. •

LEMMA A.V.5:

Let $w \in W$ and $w^{outs} := T_e^{outs} w$. Then

(i) w^{outs} is welldefined, $w^{outs}(y) \geq \dfrac{y^{1-\eta}-1}{1-\eta} + \beta\, w(0)$,

(ii) w^{outs} is concave and strictly increasing,

(iii) $w^{outs}(y) \leq MAX$, all $y \in \mathbb{R}_+$,

(iv) the decision rules c,b are unique, (not necessarily strictly) increasing and Lipschitz continuous.

(v) w^{outs} is continuous,

PROOF:

Let $f(y,b) := u(y-b) + \beta w(Rb)$ and observe, that

$$w^{outs}(y) = \max_{b \in [0,y]} f(y,b).$$

(i) For the existence of w^{outs}, observe that $f(y,\cdot)$ is a continuous function (into $\mathbb{R} \cup \{-\infty\}$) and $[0,y]$ a compact interval. The inequality follows from evaluating $f(y,0)$.

(ii) Let b_1 be an optimum for y_1 and let $y_2 > y_1$. Monotonicity follows from $f(y_1,b_1) < f(y_2,b_1) \leq w^{outs}(y_2)$. For concavity, let b_2 be an optimum for y_2 and let $\lambda \in [0;1]$. Then

$$\begin{aligned} &\lambda\, w^{outs}(y_1) + (1-\lambda)\, w^{outs}(y_2) \\ &= \lambda\, f(y_1,b_1) + (1-\lambda)\, f(y_2,b_2) \\ &\leq f(\lambda y_1 + (1-\lambda)y_2 \,,\, \lambda b_1 + (1-\lambda)b_2) \\ &\leq w^{outs}(\lambda y_1 + (1-\lambda)y_2). \end{aligned}$$

(iii) is clear from the definition of MAX.

(iv) Observe, that f is a strictly concave function in b since it is the sum of a strictly concave and a concave function in b. Thus, the maximizing b for a given y is unique. To prove that the decision rule b is increasing, choose some $y_2 > y_1$ and assume to the contrary, that $b_2 < b_1$. In particular, it has to hold that

$$u(y_2-b_2) + \beta w(Rb_2) > u(y_2-b_1) + \beta w(Rb_1).$$

Since by the previous lemma (with $\Delta = b_1 - b_2$, etc)

$$u(y_2-b_2) - u(y_2-b_1) < u(y_1-b_2) - u(y_1-b_1),$$

we get

$$u(y_1-b_2) + \beta w(Rb_2) > u(y_1-b_1) + \beta w(Rb_1),$$

contradicting that b_1 was the maximizing choice for y_1. Thus, the decision rule for b must be increasing. The proof for the decision rule in c is similar except that the roles of u and w are reversed and the lemma above is applied to w instead of u.

By monotonicity of both rules, it follows furthermore that $0 \le b_2 - b_1 \le y_2 - y_1$, implying Lipschitz continuity. The same holds true for the decision rule in c.

(v) Continuity for w^{outs} follows immediately from (iv) and the continuity of f. •

LEMMA A.V.6:

Let $w \in W$ and $w^{ins} := T^{ins}_{i,m}w$. Then

(i) w^{ins} is welldefined, $w^{ins}(a) \geq \frac{a^{1-\eta} - 1}{1 - \eta} + \beta\, w(0)$, and $w^{ins}(0) = \lim_{a \searrow 0} \frac{a^{1-\eta} - 1}{1 - \eta} + \beta\, w(0) \in \mathbb{R} \cup \{-\infty\}$,

(ii) w^{ins} is concave and strictly increasing,

(iii) $w^{ins}(a) \leq MAX$, all $a \in \mathbb{R}_+$,

(iv) the decision rules c,b,s are unique. The decision rules for consumption c and for the total amount invested into the portfolio $inv(b,s) = b + \pi(g{=}0|i)s$ as well as the revenues the portfolio delivers next period $rev(b,s,g) = R(b + (1-g)s)$, conditional on the true state g, are monotonously increasing. The decision rules c,b,s, the amount invested and the next–period revenues are Lipschitz continuous functions of a.

(v) w^{ins} is continuous.

PROOF:

Define

$$f(a,b,s) := u(a-b-\pi(g{=}0|i)s) + \beta \sum_{g=0}^{1} \pi(g|i,m) w(R(b+(1-g)s)),$$

where implicitely $u(0) := \lim_{a \searrow 0} u(a)$ and $f(a,b,s) \in \mathbb{R} \cup \{ -\infty \}$. Observe, that

$$w^{ins}(a) = \max_{b,s \;\in S(a)} f(a,b,s),$$

where

$$\begin{aligned} S(a) &= \{ (b,s) \mid 0 \le s+b,\ 0 \le b,\ b+\pi(g=0|i)s \le a \} \\ &= \{ (b,s) \mid 0 \le c \le a,\ -\frac{a-c}{\pi(g=1|i)} \le s \le \frac{a-c}{\pi(g=0|i)}, \\ &\qquad b = a - c - \pi(g=0|i)s \} \end{aligned}$$

is the set of admissable (b,s) – pairs. S(a) is compact because of the borrowing constraint that the portfolio must have nonnegative value under all circumstances at the beginning of the next period: this puts a bound on speculation. Given a, it is easy to see that f is a concave function in (b,s).

For some arguments below, it will be easier to consider the maximization problem in two stages as follows:

$$w^{ins}(a) = \max_{inv \in [0,y]} u(y-inv) + g(inv)$$

where

$$g(inv) = \max \{ \beta \sum_{g=0}^{1} \pi(g|i,m)\, w(R(b+(1-g)s)) \mid$$
$$b+\pi(g=0|i)s = inv,\ b \ge 0,\ b+s \ge 0 \}$$

is a concave function.

(i) For the existence of w^{ins}, observe that $f(a,\cdot,\cdot)$ is a continuous function (into $\mathbb{R} \cup \{-\infty\}$) and $S(a)$ a compact set. The inequality follows from evaluating $f(a,0,0)$. The equality for $a = 0$ follows from noting that $S(0) = \{ (0,0) \}$ and evaluating $f(0,0,0)$.

(ii) Let (b_1,s_1) be an optimum for y_1 and let $a_2 > a_1$. Monotonicity follows from $f(a_1,b_1,s_1) < f(a_2,b_1,s_1) \leq w^{ins}(a_2)$. For concavity, let (b_2,s_2) be an optimum for a_2 and let $\lambda \in [0;1]$. Then

$$\begin{aligned}
&\lambda\, w^{ins}(a_1) + (1-\lambda)\, w^{ins}(a_2) \\
&= \lambda\, f(a_1,b_1,s_1) + (1-\lambda)\, f(a_2,b_2,s_2) \\
&\leq f(\lambda a_1+(1-\lambda)a_2,\ \lambda b_1+(1-\lambda)b_2,\ \lambda s_1+(1-\lambda)s_2) \\
&\leq w^{ins}(\lambda a_1 + (1-\lambda)a_2).
\end{aligned}$$

(iii) is clear from the definition of MAX.

(iv) Since u is strictly concave, all maximizing portfolios must deliver the same amount of consumption, i.e. cost the same. If $\pi(g=0|i) = \pi(g=0|i,m)$, the rest of the proof is identical to the proof for the operator T_e^{outs}, because we assumed the tie – breaking rule that agents choose the portfolio with the minimum amount of variance if they are indifferent between several portfolios, i.e. they would choose to only hold shares of the riskless mutual fund here.

Thus, assume w.l.o.g., that $\pi(g=0|i,m) < \pi(g=0|i)$. It is then easy to see that the optimal portfolio has $s \leq 0$, i.e. there is short – selling. Assume otherwise, i.e. for some a and let $(b,s) \in S(a)$ be some portfolio, where $s \geq 0$. Define $b' = b + \pi(g=0|i)s$, $s' = 0$, observing that (b',s') finances the same amount of consumption as (b,s), but has a sure rate of return. It follows from Jensens inequality, that

$$f(a,b,s) \leq w(b+\pi(g=0|i,m)s) < f(a,b',s'),$$

i.e. (b,s) cannot be the optimal portfolio (It is equally easy to see that we must have $s \geq 0$, if $\pi(g=0|i,m) > \pi(g=0|i)$). It now follows furthermore from the tie – breaking rule that the maximizing portfolio with minimum variance is unique.

Uniqueness, monotonicity and Lipschitz continuity for the amount invested as well as for the amount consumed follows exactly as in the previous proof (with g replacing w), using the "two–step" maximization via the function g introduced above.

We now continue to prove, that the revenue functions rev(b(a),s(a),g) are monotonously increasing in a. Suppose not, i.e. for some $a_2 > a_1$, suppose to the contrary that (w.l.o.g.) $rev(b_2,s_2,0) < rev(b_1,s_1,0)$. Find a difference portfolio (b_0,s_0) with the price $b_0 + \pi(g=0|i)s_0 = inv(b_2,s_2) - inv(b_1,s_1) > 0$, which pays $rev(b_0,s_0,0) = 0$. Since (b_2,s_2) is the portfolio choice for a_2, there can only be two cases: either we have

$$u(c_2) + \beta\sum_{g=0}^{1} \pi(g|m,i)\, w(rev(b_2,s_2,g))$$
$$> u(c_2)+\beta\sum_{g=0}^{1} \pi(g|m,i)w(rev(b_1+b_0,s_1+s_0,g)),$$

or there is equality, but the variance for (b_2,s_2) is smaller, i.e. $s_1+s_0 < s_2 \leq 0$ with our assumption from above, that $\pi(g=0|i,m) < \pi(g=0|i)$ (the proof for $\pi(g=0|i,m) > \pi(g=0|i)$ and $s_1+s_0 > s_2 \geq 0$ is very similar). Suppose first, that the inequality holds. Since $rev(b_2,s_2,g=1) > rev(b_1+b_0,s_1+s_0,g=1)$, we have by our lemma about concave functions, that

$$w(rev(b_2,s_2,g=1)) - w(rev(b_2-b_0,s_2-s_0,g=1))$$
$$\leq w(rev(b_1+b_0,s_1+s_0,g=1)) - w(rev(b_1,s_1,g=1))$$

and thus

$$u(c_1) + \beta\sum_{g=0}^{1} \pi(g|m,i)w(rev(b_2-b_0,s_2-s_0,g))$$
$$> u(c_2) + \beta\sum_{g=0}^{1} \pi(g|m,i)\, w(rev(b_1,s_1,g)),$$

a contradiction to (b_1,s_1) being a maximizing portfolio. Hence, suppose secondly, that we have equality in both equations and that $s_1+s_0 < s_2 \leq 0$. But then $s_1 < s_2 - s_0$ as well as $s_2 - s_0 \leq 0$, since there could not have been equality in the last equation otherwise (recall the argument of the beginning of the paragraph which showed that a portfolio with $s > 0$ could be dominated by a riskless portfolio, if $\pi(g=0|i,m) < \pi(g=0|i)$). This contradicts, that (b_1,s_1) is the maximizing portfolio with minimum variance for a_1. This proves that the revenue functions are monotonously increasing in a.

Let $a_2 > a_1$ and (b_1,s_1), (b_2,s_2) be the associated portfolio choices. It now follows from the monotonicity of both revenue functions that

$$0 \leq rev(b_2,s_2,g) - rev(b_1,s_1,g)$$
$$\leq \frac{inv(b_2,s_2) - inv(b_1,s_1)}{\pi(g|i)} \leq \frac{a_2 - a_1}{\pi(g|i)},$$

implying Lipschitz continuity of the revenue functions as well as of the decision rules for b and s via

$$b(a)=rev(b(a),s(a),g=1),$$
$$s(a)=rev(b(a),s(a),g=0)-rev(b(a),s(a),g=1).$$

(v) Continuity for w^{ins} follows immediately from (iv) and the continuity of f. •

Now let T^{outs} and T^{ins} be the operators "before expectation", i.e.

$$(T^{outs}w)(a) = E_N \left[(T_y^{outs}w)(a + N) \right]$$

and

$$(T^{ins}w)(a) = \sum_{i=0}^{I} \sum_{m=0}^{M} \pi(i)\pi(m|i)(T_{m,i}^{ins}w)(a).$$

Let finally T be the transition operator from choosing a_1, a_2, that is

$$(Tw)(a) = \max_{a_i, a_o} \{P_o(T^{outs}w)(a_o) + P_i\,(T^{ins}w)(a_i) \mid$$

either $0 \le a_o \le a \le a_i$ or $0 \le a_i \le a \le a_o$,

and $P_o = \frac{a_i - a}{a_i - a_o}$, $P_i = \frac{a - a_o}{a_i - a_o}$ with the convention for $\frac{0}{0}$, that P_o=1.0, P_i=0.0 if $v^{outs}(a) \ge v^{ins}(a)$ and P_o=0.0, P_i=1.0 else },

where the gamble with the minimal distance $|a_i - a_o|$ is selected, if several choices of a_i and a_o deliver the maximum.

LEMMA A.V.7:

Let $\tilde{w} \in W$, $w^{outs} = T^{outs}\tilde{w}$, $w^{ins} = T^{ins}\tilde{w}$ and $w = T\tilde{w}$. Then

(i) w^{outs} and w^{ins} are well-defined, $w^{outs} \le \bar{v}$, $w^{ins} \le \bar{v}$ and both functions, w^{outs} and w^{ins}, are increasing, concave and continuous.

(ii) $$w(0) \ge \frac{\int N^{1-\eta}\, dF_N - 1}{1 - \eta} + \beta\tilde{w}(0) > w^{ins}(0),$$

(iii) The decision rules $a_i(a)$ and $a_o(a)$ are welldefined, unique and increasing.

(iv) If $a_i(a)<a<a_o(a)$, then $a_i(a')=a_i(a)$ and $a_o(a')=a_o(a)$ for all $a' \in (a_i(a),a_o(a))$ as well as $a_i(a_i(a))=a_i(a)$, $a_o(a_o(a))=a_o(a)$ and $a_i(a') \ge a_o(a)$ for all $a'>a$. If $a_o(a)<a<a_i(a)$, the same holds with the roles of a_i and a_o interchanged.

(v) $w \in W$ and w is strictly increasing.

PROOF:

(i) Let $w_e^{outs} = T_e^{outs}\tilde{w}$. Note, that

$$\frac{N^{1-\eta} - 1}{1 - \eta} + \beta\tilde{w}(0) \le w_e^{outs}(a+N) \le MAX$$

and that

$$w^{outs}(a) = \int w_e^{outs}(a + N)\, dF_N.$$

This integral exists by our assumptions about F_N and by the continuity of w_e^{outs}. For the continuity of w^{outs}, use Lebesgues dominated convergence theorem with MAX as upper bound for $w_e^{outs}(a+N)$. All other properties are trivial.

(ii) Observe, that by the lemma about T_e^{outs} and by the assumption about F_N having a strictly positive density for $N > 0$,

$$w^{outs}(0) \geq \frac{\int N^{1-\eta}\, dF_N - 1}{1 - \eta} + \beta\tilde{w}(0)$$
$$> \lim_{a \to 0} \frac{a^{1-\eta} - 1}{1 - \eta} + \beta\tilde{w}(0) = w^{ins}(0).$$

For $a = 0$, consider $a_o = 0$ and $a_i = 1$ with $P_o = 1$ and the inequality in (ii) follows.

(iii) We now show, that $a_o(a)$ and $a_i(a)$ are well – defined. To that end, we construct $a_o(a)$ and $a_i(a)$ in a different way and show that these rules solve the maximization problem for T. Choose some $a \geq 0$ and some $\bar{x} \in \mathbb{R}$ big enough (see below). Consider the set

$$S(a) = \{ (x,\lambda) \mid x \leq MAX \text{ and for all } \alpha \geq 0,$$
$$w^{outs}(\alpha) \leq x + \lambda(\alpha - a) \text{ as well as}$$
$$w^{ins}(\alpha) \leq x + \lambda(\alpha - a) \}.$$

Observe that $S(a) \neq \emptyset$, since $(MAX,0) \in S(a)$. Since $S(a)$ is a compact set, let

$$x(a) = \min \{ x \mid (x,\lambda) \in S(a) \text{ for some } \lambda \}$$

and choose $\lambda(a)$ so that $(x(a),\lambda(a)) \in S(a)$. Observe that we never have $\lambda < 0$, since w^{outs} and w^{ins} are increasing. It is easy to see, that $x(a)$ equals the supremum over all $P_o (T^{outs}w)(a_o) + P_i (T^{ins}w)(a_i)$ in the definition of T. We have to prove that it is actually a maximum for the right choices of a_o and a_i. If $x(a) = w^{outs}(a)$ or $x(a) = w^{ins}(a)$, let $a_o(a) = a_i(a) = a$. It is obvious, that this choice is the solution to the maximization problem in the definition of T.

The case of $w^{outs}(a) \neq x(a) \neq w^{ins}(a)$ remains. Define the line

$$l_a(\alpha) = x(a) + \lambda(a)(\alpha - a).$$

For the proof to proceed, we need to show that the closed sets

$$A\_(a) = \{ \alpha \mid 0 \leq \alpha \leq a,\ w^{outs}(\alpha) = l_a(\alpha) \text{ or } w^{ins}(\alpha) = l_a(\alpha) \}$$

and

$$A_+(a) = \{ \alpha \mid a \leq \alpha \leq \infty,\ w^{outs}(\alpha) = l_a(\alpha) \text{ or } w^{ins}(\alpha) = l_a(\alpha) \}$$

are not empty.

Suppose first to the contrary, that $A_{\_}(a) = \emptyset$. By the compactness of $[0, a]$, we can find some $\lambda > \lambda(a)$, such that we have $\Delta_{\_} > 0$, where

$$\Delta_{\_} = \min_{0 \leq \alpha \leq a} \{ x(a)+\lambda(\alpha-a)-w^{outs}(\alpha), \; x(a)+\lambda(\alpha-a)-w^{ins}(\alpha) \}.$$

Let

$$\Delta_{+} = \min_{a \leq \alpha < \infty} \{ x(a)+\lambda(\alpha-a)-w^{outs}(\alpha), \; x(a)+\lambda(\alpha-a)-w^{ins}(\alpha) \}$$

and observe $\Delta_{+} > 0$, since $w^{outs}(a) \neq x(a) \neq w^{ins}(a)$ and since e.g. $x(a)+\lambda(\alpha-a)-w^{outs}(\alpha) \geq \lambda - \lambda(a)$ for $\alpha \geq a+1$. Let $\Delta = \min\{ \Delta_{+}, \Delta_{\_} \} > 0$. But then

$$(x(a) - \Delta, \lambda) \in S(a),$$

contradicting the definition of $x(a)$.

Suppose secondly to the contrary, that $\Delta_{\_} \neq \emptyset$, but that $\Delta_{+} = \emptyset$. If $\lambda(a) > 0$, then we can find some $\bar{a}$ with

$$\bar{v}(\alpha) \le x + (\lambda(a)/2)(\alpha - a), \text{ all } \alpha \ge \bar{a},$$

since $\lim_{\alpha \to \infty} \frac{\bar{v}(\alpha)}{\alpha} = 0$. The argument is then similar to the argument for Δ_- above, except that we are looking for a λ with $\lambda(a)/2 \le \lambda < \lambda(a)$ and that we replace the compact interval $[0, a]$ by $[a, \bar{a}]$. If $\lambda(a) = 0$, choose some $\alpha \in A_-(a) \neq \emptyset$ such that $w^{outs}(\alpha)=x(a)$ w.l.o.g.. Since w^{outs} is increasing, we have a contradiction.

Thus $A_-(a) \neq \emptyset$ and $A_+(a) \neq \emptyset$. Assume first, that $w^{outs}(\alpha)=l_a(\alpha)$ for some $\alpha \in A_-(a)$. Since w^{outs} is concave and since $w^{outs}(a) \neq x(a) \neq w^{ins}(a)$, we have $w^{outs}(\alpha)<l_a(\alpha)$ for $\alpha \ge a$. Thus $w^{ins}(\alpha)=l_a(\alpha)$ for all $\alpha \in A_+(a)$ and by a similar argument $w^{ins}(\alpha)<l_a(\alpha)$ for all $\alpha \le a$. We then define $a_o = \max A_-(a) < a$ and $a_i = \min A_+(a) > a$. It is obvious that a_i and a_o solve the maximization problem in the definition of T, since e.g. for any other choice of a_i', we have $l_a(a_i') \ge w^{ins}(a_i')$ and thus the same has to hold for the lottery – averages at a. It is also clear, that the minimum distance choice is unique. Assume secondly, that $w^{ins}(\alpha)=l_a(\alpha)$ for some $\alpha \in A_-(a)$. By a similar argument we find $a_i = \max A_-(a) < a$ and $a_o = \min A_+(a) > a$ constitute the unique solution for the maximization problem in the definition of T.

If $w^{outs}(\alpha)=l_a(\alpha)$, then the argument above also shows, that

$$A\_(a) = \{\ \alpha \mid 0 \leq \alpha < \infty,\ w^{outs}(\alpha) = l_a(\alpha)\},$$

etc., that is the sets $A\_(a)$ and $A_+(a)$ and therefore also the choices of a_i and a_o depend on a only via the values of the function l_a.

It remains to prove for part (ii), that the decision rules are increasing. Suppose to the contrary that a < a' and that (w.l.o.g.) $a_o' < a_o$. Consider first the two cases that $a_o = a_i = a$, but $w^{outs}(a_o) < w^{ins}(a_i)$ or $a_o' = a_i' = a'$, but $w^{outs}(a_o') < w^{ins}(a_i')$. Restricting attention to the first of these two cases w.l.o.g., it must be that $a_i' > a'$. By virtue of the choices for a, we have

$$w^{ins}(a) \geq \frac{a - a_o'}{a_i' - a_o'}\, w^{ins}(a_i') + \frac{a_i' - a}{a_i' - a_o'}\, w^{outs}(a_o'),$$

But then, concavity of w^{ins} implies that

$$\begin{aligned} w^{ins}(a') &\geq \frac{a' - a}{a_i' - a}\, w^{ins}(a_i') + \frac{a_i' - a'}{a_i' - a}\, w^{ins}(a) \\ &\geq \frac{a' - a_o'}{a_i' - a_o'}\, w^{ins}(a_i') + \frac{a_i' - a'}{a_i' - a_o'}\, w^{outs}(a_o'), \end{aligned}$$

which for ">" is a contradiction to maximization at a' in the definition of T and which for "=" is a contradiction to the minimal distance choice tie – breaker in the definition of T.

Now consider the remaining case, that $l_a(a_o) = w^{outs}(a_o)$ and $l_{a'}(a'_o) = w^{outs}(a'_o)$. We showed above, that $l_a(\alpha) \equiv l_{a'}(\alpha)$ would imply $a_o = a'_o$, a contradiction. Thus, l_a and $l_{a'}$ are equal to each other for a unique $\bar{\alpha}$. It follows from the concavity of w^{outs} and from $a'_o < a_o$, that $l_a(\alpha) < l_{a'}(\alpha)$ for all $\alpha > \bar{\alpha}$ and thus by definition of x(a), we must have $a \geq \bar{\alpha}$. Likewise we have $a' \leq \bar{\alpha}$, contradicting the assumption that $a < a'$. Thus, a_o (and likewise) a_i must be increasing functions of a.

(iv) Suppose that for some a,

$$a_i(a) < a < a_o(a).$$

Choose some a' with $a_i(a) \leq a' < a$. If we had $a_i(a') < a_i(a)$, then $l_{a'}(\alpha) > l_a(\alpha)$ for all $\alpha \geq a_i(a)$ via concavity of w^{ins} and the dependence of the set $A_-(a')$ on the function $l_{a'}$ only. In particular, we had $x(a') = l_{a'}(a') > l_a(a')$, a contradiction to the choice of x(a'). Thus, $a_i(a') = a_i(a)$. Let in addition $a' \neq a_i(a)$. Then we must have $a_o(a') > a'$. Since $l_a(a_i(a)) = l_{a'}(a_i(a))$, we have $l_a(a') \geq l_{a'}(a')$ if and only if $l_a(a) \geq l_{a'}(a)$. By definition of x(a), we must have $l_a = l_{a'}$ and thus $a_o(a) = a_o(a')$ because $A_+(a)$ depends on a only via l_a. I.e. we have shown that

$$a_i(a')=a_i(a) \text{ and } a_o(a')=a_o(a) \text{ for all } a'\in(a_i(a),a_o(a))$$

as well as

$$a_i(a_i(a))=a_i(a) \text{ and } a_o(a_o(a))=a_o(a).$$

Observe that $a_o(a_o(a))=a_o(a)$ implies that $a_i(a_o(a))\geq a_o(a)$. Hence, if $a' > a$, then we have $a_i(a') \geq a_o(a)$ by the monotonicity of the function a_i.

(v) Continuity of w follows easily now with (iv) by examining the definition for P^{ins} and P^{outs} at jump points for a_i and a_o.

w is strictly increasing: let $a' > a$. Either a_i and a_o are the same for both a' and a, but then the probability on the bigger of the two numbers $w^{ins}(a_i)$ and $w^{outs}(a_o)$ strictly increases or a_i or a_o together with $w^{ins}(a_i)$ or $w^{outs}(a_o)$ strictly increases.

Let $G^{ins} = \{ (a,x) \in \mathbb{R}\times\mathbb{R} \mid x < w^{ins}(a) \}$ and define G^{outs} likewise. Let G^{hull} be the convex hull of $G^{ins} \cup G^{outs}$ and observe that every $(a,x)\in G^{hull}$ can be written as the convex combination of some $(a^{ins},x^{ins})\in G^{ins}$ and $(a^{outs},x^{outs})\in G^{outs}$, since both sets are convex. Since $x^{ins} \leq l_a(a^{ins})$ and $x^{outs} \leq l_a(a^{outs})$, it follows that $G^{hull} \subset G$, where $G = \{ (a,x) \in \mathbb{R}\times\mathbb{R} \mid x < w(a) \}$. On the other hand, we obviously have $G \subset G^{hull}$. This proves that w is concave.

Clearly, $w \leq MAX$, since $w^{outs} \leq MAX$ and $w^{ins} \leq MAX$. $w \geq MIN$ is equally easy to see with the inequality from (ii). •

THEOREM A.V.6:

There is a unique fixed point v of T in W, solving Problem (B) and thus translating into a solution to problem (A). Let $c^{outs}(y)$, $b^{outs}(y)$, $c^{ins}_{i,m}(a)$, $b^{ins}_{i,m}(a)$, $s^{ins}_{i,m}(a)$, $a_o(a)$ and $a_i(a)$ be the decision rules associated with v. These decision rules and v have the following properties:

(i) v is strictly increasing,

(ii) The decision rules $c^{outs}(y)$, $b^{outs}(y)$, $c^{ins}_{i,m}(a)$, $b^{ins}_{i,m}(a)$, $s^{ins}_{i,m}(a)$ as well as the amounts invested into portfolios and the revenue they deliver next periods are unique and Lipschitz continuous,

(iii) The decision rules $c^{outs}(y)$, $b^{outs}(y)$ and $c^{ins}_{i,m}(a)$ as well as the amounts invested into portfolios and the revenues they deliver next periods are increasing,

(iv) $\lim_{y\to\infty} c^{outs}(y) = \lim_{y\to\infty} b^{outs}(y) = \lim_{a\to\infty} c^{ins}_{i,m}(a) = \lim_{a\to\infty} (rev^{ins}_{a,m,g}(a)) = \lim a_i(a) = \lim a_o(a) = \infty$, where $rev^{ins}_{i,m,g}(a) = b^{ins}_{i,m}(a) + (1-g)s^{ins}_{i,m}(a)$,

(v) If $\pi(g|i,m) = \pi(g|i)$ for all i,m, i.e. if messages contain no information, then $v = T^{outs}v$ and no agent will choose to become an insider.

(vi) If $\pi(g|i,m) \neq \pi(g|i)$ for all i, i.e. if messages contain information, then there is some cut – of level $\underline{a}$, such that $a_o(a) = a_i(a) = a$ and $P^{ins}(a) = 1.0$ for all $a \geq \underline{a}$.

PROOF:

Endow W with the usual supremums – metric

$$d_\infty(w,w') = \sup_a |w(a) - w'(a)|$$

and note, that (W,d_∞) is a complete metric space. It is easy to see that T is a contraction mapping of W into itself by checking e.g. Blackwells sufficient conditions. Thus there is a unique fixed point of T in W (see Stokey – Lucas, with Prescott, 1989, Theorems 3.2, p. 50, and 3.3, p. 54). Properties (i), (ii) and (iii) now follow immediately from the previous lemmas.

(iv) We show first, that $\lim_{a\to\infty} c^{outs}(y)$. Suppose not, i.e. $c^{outs}(y) \leq \bar{c}$ for all y and some $\bar{c}$. Let $\delta = \min_{c\leq\bar{c}}(u(c+1)-u(c)) = u(\bar{c}+1) - u(\bar{c})$. Find $\bar{y}$ big enough, so that $v(y-\bar{c}) - v(y-\bar{c}-1) < \delta/\beta$ for all $y \geq \bar{y}$. Choose some $y \geq \bar{y}$. We now have

$$u(c^{outs}(y))+\beta v(b^{outs}(y))$$

$$< u(c^{outs}(y+1))+\beta v(b^{outs}(y)-1),$$

a contradiction to the optimality of the choices c^{outs} and b^{outs}.

The proof for $b^{outs}(y) \to \infty$ is identical, except for reversing the roles of u and v and observing that v is strictly increasing. The proofs for the insider decision rules are similar, except that we may alter the payoff only for one of the two possible future states: to alter the equations above appropriately, the value – function – differences have to be weighed appropriately by the respective probabilities. The details are left away since they are trivial.

The divergence for the rules a_i and a_o follows immediately from part (iv) of the previous lemma and the restriction in the definition of T, that either $a_i(a) \leq a \leq a_o(a)$ or $a_o(a) \leq a \leq a_i(a)$.

(v) is a direct consequence of Jensens lemma applied several times.

(vi) Let $v_e^{outs} = T_e^{outs} v$ and $v^{outs} = T^{outs} v$. Since $v_e^{outs}(y)$ is concave and since $v(a) = \int v_e^{outs}(y+N)dF_N$, it follows from Jensens inequality, that $v^{outs}(a) \leq v_e^{outs}(a+1)$. Fix m and i. If we can show, that $v_{m,i}^{ins}(a) > v_e^{outs}(a+1)$ for all $a \geq \underline{a}$ and some $\underline{a}$, we are done.

To that end, find a difference portfolio $(\Delta b, \Delta s)$, which costs -2, but delivers an expected revenue of 1, conditional on i and m, by solving the two equations

$$-2 = \Delta b + \pi(g{=}0|i)\Delta s,$$

$$1 = R\sum_{g=0}^{1} \pi(g|i,m)\ (\ \Delta b + (1-g)\ \Delta s\ ).$$

Such a portfolio can be found, since $\pi(g|i,m) \neq \pi(g|i)$, $g = 0,1$. Find a level of consumption $\bar{c}$ high enough so that $c + R\Delta b > 0$, $c + R(\Delta b + \Delta s) > 0$ and

$$u(c) \leq \sum_{g=0}^{1} \pi(g|i,m)\ u(c+R(\Delta b+(1-g)\Delta s)).$$

for all $c \geq \bar{c}$. Such a level $\bar{c}$ can be found, since the absolute risk aversion is decreasing with c, i.e. eventually, the risk premium demanded for taking on the fixed gamble $R(\Delta b + (1-g)\Delta s)$ instead of receiving a sure pay of 1 must be smaller than 1. Using (iii), we can now find an initial asset value $\bar{a}$, so that $c_{i,m}^{ins}(a_i(a)) \geq \bar{c}$ and $c^{outs}(a+N) \geq \bar{c}$ for all $i,m,N \geq 0$ and $a \geq \bar{a}$. Finally find $\underline{a}$ so that $Rb^{outs}(a+1) \geq \bar{a}$ for all $a \geq \underline{a}$. Observe that an insider with initial wealth $a \geq \underline{a}$ could follow the following strategy. He consumes one more unit than $c^{outs}(a+1)$ and invests $(b^{outs}(a+1)+\Delta b, \Delta s)$ this period. In the next period then, he saves as much as he would have, had he just saved $b^{outs}(a+1)$ this period, i.e. consumption will completely take care of the random payoffs of the $(\Delta b, \Delta s)$ – part. It is then easy to see that the present value of this strategy his higher than the value $v^{outs}(a+1)$. I.e. formally observe that for all i,m, we have (in the next period) first

$$v_e^{outs}(y') \leq w_e^{outs}(y'),$$
$$v_{i',m'}^{ins}(a') \leq w_{i',m'}^{ins}(a')$$

where

$$w^{outs}(y') = \beta v(Rb^{outs}(y')) + \sum_{g=0}^{1} \pi(g|i,m)u(c^{outs}(y')+R(\Delta b+(1-g)\Delta s))$$

and

$$w_{i',m'}^{ins}(a') = \sum_{g=0}^{1} \pi(g|i,m)u(c_{i',m'}^{ins}(a')+R(\Delta b+(1-g)\Delta s))$$
$$+\beta\sum_{g'=0}^{1} \pi(g'|i',m')v(R(b_{i',m'}^{ins}(a')+(1-g')s_{i',m'}^{ins}(a')).$$

Thus, with

$$w(a) = P_o(a) \int w_e^{outs}(a_o(a)+N)\, dF_N$$

$$+ P_i(a)\sum_{i'=0,m'=0}^{I \quad M} \pi(i')\pi(m'|i')w^{ins}_{i',m'}(a_i(a))$$

we find

$$v^{outs}_e(a) < u(c^{outs}(a+1)+1)+\beta w(Rb^{outs}(a+1))$$
$$\leq v^{ins}_{i,m}(a),$$

which we wanted to prove. •

In the sequel, we need the following notation: let $x \in \mathbb{R}$, $a \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$. Define the line through x at a of slope λ by

$$l_{x,\lambda,a}(\alpha) = x + \lambda(\alpha - a).$$

The following Lemma is rather obvious, but very useful.

LEMMA A.V.8: (A property of continuously differentiable functions)

Let $f:\mathbb{R}_+ \to \mathbb{R}$ be continuously differentiable. Let $\epsilon > 0$ and $0 < a_{min} < a_{max} < \infty$. Then there is $\bar{\nu} > 0$ and for every $v > 0$, $\nu \leq \bar{\nu}$, there is $\delta > 0$ with the following propery.

Let $a_1 \in [\, a_{min}, a_{max} \,]$ and $a_2 \in \mathbb{R}_{++}$ so that $|a_1 - a_1| < \nu$. Let λ be such that $|\lambda - f'(a_1)| \geq \epsilon$. Let $x = l_{f(a_1),f'(a_1),a_1}(a_2)$. Then there is some $a > 0$, $|a-a_2|=\nu$, so that $f(a) \geq l_{x,\lambda,a_2}(a) + \delta$.

PROOF

Find $\bar{\nu}>0$, $\bar{\nu}<a_{min}/3$, so that $|a-a'|$ implies $|f'(a)-f'(a')|\leq\epsilon/4$ for all $a,a' \in [\, a_{min}/3, a_{max} + 2/3\, a_{min}\,]$. Choose $\nu>0$, $\nu\leq\bar{\nu}$ and let $\delta=\nu\epsilon/2$. Fix a_1,a_2,λ and x. We assume w.l.o.g., that $\lambda\geq f'(a_1)+\epsilon$. Choose $a=a_2-\nu$. Observe that by choice of ν and the mean – value theorem,

$$\begin{aligned} f(a) &\geq f(a_1) - (f'(a_1)+\epsilon/4)(a_1-a) \\ &= x + f'(a_1)(a-a_2) - \epsilon/4(a_1-a) \\ &\geq l_{x,\lambda,a_2}(a) + \epsilon(a_2-a) - \epsilon/4(a_1-a) \\ &\geq l_{x,\lambda,a_2}(a) + \nu\epsilon/2. \bullet \end{aligned}$$

LEMMA A.V.9:

v, v^{ins} and v^{outs} are continuously differentiable for $a > 0$, $\lim_{a\to\infty} v'(a) = \lim_{a\to\infty} v'^{outs}(a) = \lim_{a\to\infty} v'^{ins}(a) = 0$ and $\lim_{a\to 0} v'(a) = \infty$.

(We use e.g. the notation v'^{outs} to denote the derivative of v^{outs})

PROOF:

It is easy to see that $c_e^{outs}(y) > 0$ for $y > 0$ and $c_{i,m}^{ins}(a) > 0$ for $a > 0$, since $\lim_{c\to 0} u'(c) = \infty$. The proof for the differentiability of v^{ins} is now completely analogous to the proof of Theorem 4.11, p.85 in Stokey – Lucas, with Prescott (1989). For the differentiability of v^{outs} for some given

$a_0 > 0$, we similarly have to show that the function

$$w(a) = \int u(c(a_0+N)+a-a_0) + \beta v(b(a_0+N))dF_N$$

is differentiable at a_0. Observe that

$$\frac{w(a)-w(a_0)}{a - a_0} = \int \frac{u(c(a_0+N)+a-a_0)-u(c(a_0+N))}{a - a_0} dF_N$$

and that the function in the integral is dominated for $a > a_1$ (for some $a_1 > 0$, $a_1 < a_0$) by $u'(c(a_1))$. Lebesgues theorem yields the differentiability of $w(a)$ at a_0. Differentiability of v^{outs} now follows in the usual way by applying Theorem 4.10, p.84 in Stokey – Lucas, with Prescott (1989). In particular, it follows that $v'^{outs}(a_0) = \int u'(c^{outs}(a_0+N))\, dF_N$. Since $c^{outs}(y)$ is a continuous, increasing function in y, another application of Lebesgues theorem yields the continuity of v'^{outs}. The continuity of v'^{ins} can be shown with the same argument.

For the continuous differentiability of v, recall the definition of x(a) in the proof for the existence of the decision rules $a_i(a)$ and $a_o(a)$. If $a_i(a) < a < a_o(a)$ (or conversely), it was shown in that proof that the functions $a_i(a)$ and $a_o(a)$ are constant locally around a, i.e. v is just linear locally and thus continuously differentiable with $v'(a) = v'^{outs}(a_o(a)) = v'^{ins}(a_i(a))$. For the remaining possibilities, we can concentrate w.l.o.g. on the case that $v^{ins}(\bar{a}) = v(\bar{a})$ for some $\bar{a}$. Since

$v(a) \geq v^{ins}(a)$, an application of Scheinkmans result (Theorem 4.11, p.85 in Stokey – Lucas, with Prescott (1989)) yields $v'(\bar{a}) = v'^{ins}(\bar{a})$. We thus already established that v is differentiable everywhere. Suppose, v was not continously differentiable at $\bar{a}$, i.e. there was some $\epsilon > 0$ and a sequence $a_n \to \bar{a}$ with $|v'(a_n) - v'(\bar{a})| \geq \epsilon$. Apply the Lemma about a property of continuously differentiable functions to find $\nu = \bar{\nu}$ and δ with the claimed properties. Let a_n be so that $|a_n - \bar{a}| < \nu$ and let $\lambda = v'(a_n)$. Find the point a according to the Lemma. Let $x = l_{v(\bar{a}),v'(\bar{a}),\bar{a}}(a_n)$, $x \geq v(a_n)$. Thus,

$$
\begin{aligned}
v(a) \quad & \geq v^{ins}(a) \geq l_{x,\lambda,a_n}(a) + \delta \geq \\
& l_{v(a_n),v'(a_n),a_n}(a) + \delta \geq v(a) + \delta,
\end{aligned}
$$

a contradiction. Thus, v is continuously differentiable. Finally the results for the limits as $a \to \infty$ follow from the fact, that all three functions are bounded above by MAX. To get the result, that $\lim_{a \to 0} v'(a) = \infty$, recall again that $v'^{outs}(a) = \int u'(c^{outs}(a+N))\, dF_N$ and that c^{outs} is Lipschitzian, i.e. for some constant $L > 0$, we have $c^{outs}(a) \leq La$. Note furthermore, that $a_o(a) \leq a$ for a small enough and that $P_o(a) \to 1$ as $a \to 0$, since $v^{outs}(0) > v^{ins}(0)$. We therefore get, that for a small enough, $v'^{outs}(a) \geq .5\, L^{-\eta} \int (a+N)^{-\eta}\, dF_N \to \infty$ by our assumption about F_N and Levi's theorem on monotone convergence. This concludes the proof. •

<u>LEMMA A.V.10:</u>

Outsiders would not want to hold any stock.

PROOF:

Risk – aversion, tie – breaking rule. •

LEMMA A.V.11:

Agents do not need more "general" lotteries than the choice of a_1 and a_2.

PROOF:

The resulting value function is already concave. •

Finally, we link up the production side and the consumption side of our economy.

A.V.c. Putting it Together.

We now proceed to combine our insights about the production side of the economy with the knowledge about the decision problem of the agents. We would like to show, that for sufficiently uninformative messages and some other, minor restrictions, there is an equilibrium in our economy. The idea is to first analyze the benchmark case of uninformative messages (which amounts to a fairly standard steady – state neoclassical growth economy with random income and a continuum of agents) and then perturb this economy slightly to the case of informative messages, introducing some, but not too many insiders. To make this idea work, we need to show, that everything depends in a continuous fashion on the perturbance parameters. It turns out, that this is prohibitively difficult for the straight version of the model. We will thus proceed by not analyzing exactly the model described in the main body of the paper, but instead a slightly perturbed version of that model. As a rule of thumb, it must be possible to make these perturbances so small that they would not require a change in the computer code for any program computing (approximate) equilibria for our original model numerically on a machine with finite precision. While this pertubation approach is not completely satisfactory from a pure mathematical point of view (one would like to know, wether and when an equilibrium exists for the original economy), this seems a reasonable approach from a numerically oriented point of view, that two models should have similar conclusions, if they result in the same computer code. Another way to motivate our procedure is by noting, that there is always a way to modify the original model in an economic sense in such a way that the perturbances we introduce below become part of the model. While this approach might be more satisfactory in a pure sense, it would make the model a lot more complicated for purely technical reasons. We therefore chose, not to proceed in that fashion.

We first introduce some mathematical notation and show the continuous dependence of the decision problem on the parameters of our model. This includes a proof for the existence of a stationary distribution of assets for the perturbed model and the continuous dependence of that distribution. Next, we show the continuous dependence of the production side under somewhat sharpened assumptions, and finally we combine these for the final existence result.

Let A be a subset of $\mathbb{R}$ and let $f,g:A\to\mathbb{R}$ be two functions defined (at least) on A. We denote by

$$d_\infty(f,g) = \sup \{ |f(a) - g(a)| \mid a\in A \}$$

the metric of uniform convergence on A. If $A = \mathbb{R}_+$, we just write d_∞. We also introduce the metric of uniform convergence on compact sets in $\mathbb{R}_+$ by

$$d_c(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_{\infty,[0,n]}(f,g)}{1 + d_{\infty,[0,n]}(f,g)}.$$

Likewise, we define the metric of uniform convergence on compact sets in $\mathbb{R}_{++}$ by

$$d_{c,\mathbb{R}_{++}}(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_{\infty,[1/n,n]}(f,g)}{1 + d_{\infty,[1/n,n]}(f,g)}.$$

Finally, we use a modified d_∞ – metric, defined by

$$d_\infty^{cut}(f,g)=\sup\{|\max\{f(a),MIN-1\}-nax\{g(a),MIN-1\}| \mid a\in A\}.$$

Observe that d_∞^{cut} is only a quasi – metric in the sense that $d_\infty^{cut}(f,g) = 0$

does not imply $f = g$. This quasi – metric is suitable for our value – functions to avoid problems below with T^{ins} for very low a. These problems then become irrelevant for T, since $T^{outs}w(a) \geq MIN$ for every $w \in W$.

In general we will use the metric d_c for decision rules and the quasi – metric d_{∞}^{cut} for value – functions. Note, that all metrices can take the value $+\infty$.

In the sequel we will keep the probabilites $\pi(g|i)$ and $\pi(i)$ fixed and we may want to choose them so, that $\pi(g|i)$ are sufficiently close to each other in order to satisfy assumption A.V.6 below. However, we will treat

$\theta_{\mathcal{m}} = (P(m|g))_{m,g}$, the message quality,
$\theta_{\mathcal{g}} = (P(g|i))_{g,i}$, the fundamental growth probabilites,
R, the interest rate and
w, the wage

which we compactly write as

$$\theta = (\theta_{\mathcal{m}}, \theta_{\mathcal{g}}, R, w)$$

parametrically. Observe that θ is finite – dimensional. θ is admissable, if the probabilities are non – trivial and if Assumption (A.0) holds (i.e. if $\beta\bar{R} < 1$ and $1 \leq R$). We will subindex decision rules etc. by θ to indicate the dependence on θ. We will implicitely solve for the probabilities $\pi(l)$ and $P(i|l)$. Finally, keeping $\theta_{\mathcal{m}}$ fixed (and thus also $\pi(g|m,i)$ according to formula (3.10)), we will solve for $\theta_{\mathcal{g}}$, R and w in equilibrium. To solve for the probabilities $P(g|i)$ is necessary because of the consistency condition which relates $\pi(g|i)$ and $P(g|i)$. A "forward" approach would be to fix $P(g|i)$ and solve for $\pi(g|i)$, but this would make the analysis more cumbersome, since analysing continuity properties of e.g. the value function would be more complicated. The backward – solving approach of fixing $\pi(g|i)$ and solving for $P(g|i)$ simplifies the analysis.

The wage is only part of the parameter θ, because it will be important for the production side. Due to our normalization to a standard wage of 1, the wage w as part of θ is irrelevant for our subsequent analysis up to Theorem A.V.14 (except for trivial recalculations of the actual aggregate demand for stocks and bonds). We therefore ignore w as part of θ until then.

We will restrict $\theta_{\mathcal{m}}$ to satisfy either $P(m|g=0) = P(m|g=1)$ for all m (the case of uninformative messages) or $P(m|g=0) \neq P(m|g=1)$ for all m (the case of informative messages). Observe that this is equivalent to assuming that

either $\pi(g|i,m) = \pi(g|i)$, all g,i,m (uninformative messages)
or $\pi(g|i,m) \neq \pi(g|i)$, all g,i,m (informative messages).

This was the condition used above to prove that either nobody becomes an insider or everybody becomes an insider beyond a certain $\underline{a}$.

We now modify our operators T, T^{ins} and T^{outs} slightly: this will be the first perturbance of the original model. Let $u_e(a)$, $a \in \mathbb{R}_+$ be bounded from below and above and twice continuously differentiable with $u_e'(a) > 0$, $u_e''(a) < 0$ for all $a > 0$ and where u_e' is convex. If T, T^{ins} and T^{outs} are the operators used above, we now define

$$\tilde{T}^{ins}w = u_e + T^{ins}w,$$
$$\tilde{T}^{outs}w = u_e + T^{outs}w \text{ and}$$
$$\tilde{T}w = u_e + T(\ \tilde{T}^{ins}w,\ \tilde{T}^{outs}w\ ),$$

where $T(\ \tilde{T}^{ins}w,\ \tilde{T}^{outs}w\ )$ is a short – hand notation for the operator T being modified by replacing T^{ins} with $\tilde{T}^{ins}$ and T^{outs} with $\tilde{T}^{outs}$. Note, that all proofs above for the operators T, T^{ins} and T^{outs} also apply to $\tilde{T}$, $\tilde{T}^{ins}$ and $\tilde{T}^{outs}$, except that MIN and MAX have to be modified appropriately.

This function u_e is a purely technical device. What we have in mind is e.g. $u_e(a) = \epsilon\, a\ /\ (1+a)$, where $\epsilon > 0$ is a very small number: this will leave the operators almost unchanged. The mathematical advantage of introducing u_e is, that our resulting functions v^{ins}, v^{outs} and v are strictly concave in a uniform way, enabling us to prove continuity for most of the decision rules in the parameter θ. As for an economic interpretation, one could introduce u_e as modelling envy before and after the lottery in part II of the period, i.e. that an agent cares about his wealth – position relative to some benchmark – wealth $a_t^{bench} = \zeta^t a_0^{bench}$. If ϵ is sufficiently small, this envy would not be detectable observationally, i.e. the economy with envy is in some sense close to the economy without envy which we described in the paper. The factor ϵ could be made small enough to be beyond the precision of a computer calculating an equilibrium for the economy numerically, i.e. introducing u_e would leave the computer code unchanged. We will now proceed with the analysis of this "altered" economy, use the redefined values for MAX and MIN and leave away the tildas. We will first use the new properties resulting from u_e in the proof for the continuity of the decision rules.

LEMMA A.V.12:

T_θ^{outs}, T_θ^{ins} and T_θ are continuous in θ in the sense that for every admissable $\bar{\theta}$, every $w \in W$ and every $\epsilon > 0$, there is a $\delta > 0$ so that for every θ with Euclidean distance $\| \bar{\theta} - \theta \| < \delta$, we have

$$d_\infty^{cut}(T_\theta^{outs} w, T_{\bar{\theta}}^{outs} w) \leq \epsilon,$$
$$d_\infty^{cut}(T_\theta^{ins} w, T_{\bar{\theta}}^{ins} w) \leq \epsilon \quad \text{and}$$
$$d_\infty^{cut}(T_\theta w, T_{\bar{\theta}} w) \leq \epsilon.$$

PROOF:

Fix w and $\bar{\theta} = (\bar{P}(m|g), \bar{P}(g|i), \bar{R})$. Calculate $\bar{\pi}(g|i,m)$ and $\bar{\pi}(i|m)$. Observe that $\pi(g|i,m)$ and $\pi(i|m)$ depend continuously on P. First we show the continuity of T^{ins} in $P(m|g)$ and $P(g|i)$. Let

$\hat{\delta}_P = \min_{i,m} \bar{\pi}(i,m)/2$. Let $c_{min} > 0$ be such that

$$\hat{\delta}_P \left(u(c_{min}) + \beta \,\mathrm{MAX}\right) < \mathrm{MIN} - 1.$$

Now find $\delta_P < \hat{\delta}_P$, $\delta_P > 0$, so that

$$\sum_{i,m} K_1 |\pi(i,m) - \bar{\pi}(i,m)| + \beta \sum_{i,m,g} K_2 |\pi(i,m,g) - \bar{\pi}(i,m,g)| < \epsilon/2$$

for any θ with $\|\theta - \bar{\theta}\| < \delta_P$, where $K_1 = \sup\{|u(c)| \mid c_{min} \leq c < \infty\}$ and $K_2 = \max\{|\mathrm{MIN}|, |\mathrm{MAX}|\}$. Let θ be such that $\|\theta - \bar{\theta}\| < \delta_P$ and $R = \bar{R}$. Note that by construction, we have

$$\Big|\max\{\sum_{i,m} \pi(i,m)(u(c_{i,m}) + \beta \sum_g \pi(g|i,m) w(a'_{i,m,g})), \mathrm{MIN}-1\} - \max\{\sum_{i,m} \bar{\pi}(i,m)(u(c_{i,m}) + \beta \sum_g \bar{\pi}(g|i,m) w(a'_{i,m,g})), \mathrm{MIN}-1\}\Big| < \epsilon/2$$

for any choices of $c_{i,m}$, $a'_{i,m,g}$. Since the price $\pi(g|i)$ for stocks is fixed, we have

$$T_\theta^{ins} w(a) \geq \sum_{i,m} \bar{\pi}(i,m)\Big(u(\bar{c}_{i,m}(a)) + \beta \sum_g \bar{\pi}(g|i,m) w(R(\bar{b}_{i,m}(a) + (1-g)\bar{s}_{i,m}(a))) \Big).$$

It follows that

$$\max\{ T_{\theta}^{ins}w(a), MIN-1\} > \max\{ T_{\bar{\theta}}^{ins}w(a), MIN-1\} - \epsilon/2.$$

By reversing the roles of θ and $\bar{\theta}$, we get

$$d_{\infty}^{cut}(T_{\theta}^{ins}w, T_{\bar{\theta}}^{ins}w)$$

as claimed.

Secondly, we show the continuity of T^{outs} in R. Let δ_R be such that $\|\theta-\bar{\theta}\| < \delta_R$ implies $| \bar{R}/R - 1 | < \epsilon_R$ and $| R/\bar{R} - 1 | < \epsilon_R$, where $\epsilon_R = \epsilon/(2\beta(MAX-MIN+1))$. Let θ be such that $\|\theta-\bar{\theta}\| < \delta_R$ and let $\bar{R} > R$ w.l.o.g.. Define $\hat{w}(a) = w(\bar{R}/R\ a)$ and observe, that $\hat{w} \in W$.

We first claim, that $d_{\infty}^{cut}(\hat{w},w) < \epsilon/(2\beta)$. To show this, consider any $a \geq 0$ and find some $\lambda \geq 0$, so that $w(\alpha) \leq l_{w(a),\lambda,a}(\alpha)$ for all $\alpha \geq 0$. Since $w \in W$, we have

$$\lambda a = w(a) - l_{w(a),\lambda,a}(0) \geq MAX - MIN.$$

Furthermore,

$$w(\bar{R}/R\ a) - w(a) \leq \lambda (\bar{R}/R - 1)\ a < \lambda a \epsilon_R < \epsilon/(2\beta),$$

which establishes our claim.

Observe now that for any $a \geq 0$, $T_{\theta}^{outs} w(a) \leq T_{\theta}^{outs} w(a) = T_{\theta}^{outs} \hat{w}(a)$. On the other hand, we find with our claim above and with the decision rules $\hat{c}_{\theta}^{outs}$ and $\hat{b}_{\theta}^{outs}$ for $T_{\theta,e}^{outs}$ and $\hat{w}$, that

$$
\begin{aligned}
T_{\theta,e}^{outs} \hat{w}(y) &= u(\hat{c}_{\theta}^{outs}(y)) + \beta\, \hat{w}(\hat{b}_{\theta}^{outs}(y)) \\
&\leq u(\hat{c}_{\theta}^{outs}(y)) + \beta\, w(\hat{b}_{\theta}^{outs}(y)) - \beta\epsilon/(2\beta) \\
&\leq T_{\theta,e}^{outs} w(y) - \epsilon/2
\end{aligned}
$$

and thus, after integrating over N, where $y = a + N$:

$$
T_{\theta}^{outs} \hat{w}(a) \leq T_{\theta}^{outs} w(a) - \epsilon/2.
$$

Hence,

$$
d_{\infty}^{cut}(\ T_{\theta}^{outs} w\ ,\ T_{\bar{\theta}}^{outs} w\ ) \leq \epsilon/2,
$$

proving d_{∞}^{cut}–continuity of T_{θ}^{outs} in θ. The same argument goes through for T_{θ}^{ins}, if θ and $\bar{\theta}$ only differ in R. Observe, that the radius δ_R does not change with P for the argument for T_{θ}^{ins}. Thus, with $\delta = \min\{\ \delta_R, \delta_P\ \}$, we find joint continuity of T_{θ}^{ins} (and trivially for T_{θ}^{outs}) with the inequalities in the Lemma being valid. Observe finally for T, that for $\|\theta - \bar{\theta}\| < \delta$, we have

$$
\begin{aligned}
T_{\theta} w(a) \qquad &= P_{o,\theta}(a) T_{\theta}^{outs} w(a_{o,\theta}(a)) \\
&\quad + P_{i,\theta}(a) \max\{\ T_{\theta}^{ins} w(a_{i,\theta}(a))\ ,\ \text{MIN}-1\ \} \\
&\leq P_{o,\theta}(a) T_{\theta}^{outs} w(a_{o,\theta}(a))
\end{aligned}
$$

$$+ P_{i,\theta}(a) \max\{ T_{\theta}^{ins} w(a_{i,\theta}(a)) , MIN-1 \} - \epsilon$$
$$\le P_{o,\bar\theta}(a) T_{\theta}^{outs} w(a_{o,\bar\theta}(a))$$
$$+ P_{i,\bar\theta}(a) \max\{ T_{\theta}^{ins} w(a_{i,\bar\theta}(a)) , MIN-1 \} - \epsilon$$
$$= T_{\bar\theta} w(a) - \epsilon$$

and thus – by reversing the argument – the desired result. •

THEOREM A.V.7:

The unique fixed point v_θ of T_θ is continuous in θ in the metric d_∞ on W.

PROOF:

Fix $\bar\theta$ and let $\bar\epsilon > 0$ be given. Find δ according to the previous lemma to $\epsilon = (1-\beta)\bar\epsilon$. Pick some θ so that $\|\theta-\bar\theta\| < \delta$. We have

$$d_\infty^{cut}(T_{\bar\theta} v_\theta, v_\theta) = d_\infty^{cut}(T_{\bar\theta} v_\theta, T_\theta v_\theta) < (1-\beta)\bar\epsilon$$

by the property of δ and

$$d_\infty^{cut}(v_{\bar\theta}, v_\theta) \le d_\infty^{cut}(T_{\bar\theta} v_\theta, v_\theta) \,/\, (1-\beta) < \bar\epsilon$$

by the contraction mapping theorem, since $T_{\bar\theta}$ is a contraction at the rate β. This proves the Theorem. •

Observe that this theorem and the continuity of the operators T_θ^{ins}

and T_θ^{outs} imply that v_θ^{ins} and v_θ^{outs} are continuous with respect to θ as well.

Theorem A.V.8:

The decision rules c_θ^{outs}, b_θ^{outs}, $c_{i,m,\theta}^{ins}$, $b_{i,m,\theta}^{ins}$ and $s_{i,m,\theta}^{ins}$ are continuous in θ in the metric d_c.

Proof:

Fix $\bar{\theta} = (\bar{P}(m|g) , \bar{P}(g|i) , \bar{R})$, $\bar{\epsilon} > 0$ and calculate $\bar{\pi}(g|i,m)$ and $\bar{\pi}(m|i)$. Fix n. We have to find $\delta>0$ so that $d_{c,[0,n]}(b_\theta^{outs},b_{\bar{\theta}}^{outs}) \leq \bar{\epsilon}$ etc., for all θ with $\|\theta-\bar{\theta}\| < \delta$.

For the outsider – decision rules, define

$$f_\theta(y,b) = u(y-b) + \beta v_\theta(Rb).$$

This function is strictly concave in b, given θ. Since furthermore $b_{\bar{\theta}}^{outs}$ is a continuous function, we can find some $\nu > 0$ so that $f_{\bar{\theta}}(y,b_{\bar{\theta}}^{outs}(y)) - f(y,b) \geq \nu$ for all y, $0 \leq y \leq n$ and all b with $|b_{\bar{\theta}}^{outs}(y)-b| \geq \bar{\epsilon}$, $0 \leq b \leq y$. Find δ to $\epsilon = \nu/(6\beta)$ according to the Theorem about the continuity of v_θ in θ and furthermore in such a way that $d_\infty^{cut}(v_\theta(\bar{R}/R\ a), v_\theta) \leq \nu/(6\beta)$ for $\|\theta-\bar{\theta}\| < \delta$, using the argument given in the proof of the Lemma about the continuity of the T–operators in θ for the continuity of T_θ^{outs} in R. For every y with $0 \leq y \leq n$ and every b with $|b_{\bar{\theta}}^{outs}(y)-b| \geq \bar{\epsilon}$, we thus have

$$
\begin{aligned}
&f_\theta(y, b_{\bar\theta}^{outs}(y)) \\
&\qquad \geq u(y - b_{\bar\theta}^{outs}(y)) + \beta v_\theta(\bar{R} b_{\bar\theta}^{outs}(y)) - \nu/6 \\
&\qquad \geq f_{\bar\theta}(y, b_{\bar\theta}^{outs}(y)) - \nu/3 \\
&\qquad \geq f_{\bar\theta}(y,b) + 2\nu/3 \\
&\qquad \geq f_\theta(y,b) + \nu/3 \\
&\qquad > f_\theta(y,b),
\end{aligned}
$$

hence $b \neq b_\theta^{outs}(y)$ or $d_{c,[0,n]}(b_\theta^{outs}, b_{\bar\theta}^{outs}) \leq \bar\epsilon$. The claim for c_θ^{outs} follows immediately from $c_\theta^{outs}(y) = y - b_\theta^{outs}(y)$.

For the insider decision rules we only show the continuity with respect to variations in the probabilities P. To get continuity in all entries of θ, an argument like the one above for the outsider decision rules has to be applied for R and pieced together with the argument below about the continuity in P (cmp. the proof for the continuity of T_θ^{ins} ins θ).

We concentrate on showing that $s_{i,m,\theta}^{ins}$ varies continuously with P or, equivalently, with $\pi(m|i)$ and $\pi(g|i,m)$. For $\hat\delta_P$ sufficiently small and for a_0 sufficiently small, it is the case that $1/\pi(g{=}0|i) + 1/\pi(g{=}1|i) \leq \bar\epsilon \,/\, a_0$ for all θ with $\|\theta - \bar\theta\| < \hat\delta_P$. Hence, for $a \leq a_0$ and $\|\theta - \bar\theta\| < \hat\delta_P$, we already have $|s_{i,m,\theta}^{ins} - s_{i,m,\bar\theta}^{ins}| \leq \bar\epsilon$. For $a \geq a_0$, observe that $c_{\bar\theta}(a) \geq c_{\bar\theta}(a_0) > 0$. Fix i and m. Let

$$
f_\theta(a,b,s) = u(y - b - \pi(g{=}0|i)s) +
$$

$$\beta \sum_g \pi(g|i,m)\, v_\theta(R(b+(1-g)s)).$$

Observe that f_θ is strictly concave in b and s and continuous in all three arguments, since v_θ is strictly concave by our perturbation of introducing u_e! Thus, there is some $\nu > 0$, so that

$$f_{\bar\theta}(a, b^{in\,s}_{i,m,\bar\theta}(a), s^{in\,s}_{i,m,\bar\theta}(a)) - f_{\bar\theta}(a,b,s) > \nu$$

for all $(b,s) \in S(a) := \{(b,s) \mid 0 \leq s+b,\ 0 \leq b,\ b+\pi(g=0|i)s \leq a\}$ with $|s_{i,m,\bar\theta} - s| \geq \bar\epsilon$, $a_0 \leq a \leq n$. As in the proof for the continuity of T^{ins}_θ, find $\delta_P > 0$, $\delta_P < \hat\delta_P$, so that

$$\sum_{i,m} K_1 |\pi(i,m) - \bar\pi(i,m)| + \beta \sum_{i,m,g} K_2 |\pi(i,m,g) - \bar\pi(i,m,g)| < \epsilon/2$$

for any θ with $\|\theta - \bar\theta\| < \delta_P$, where $K_1 = \sup\{|u(c)| \mid c_{\bar\theta}(a_0) \leq c < \infty\}$ and $K_2 = \max\{|MIN|, |MAX|\}$. Furthermore, let δ_P be such that $d_\infty(v_\theta, v_{\bar\theta}) < \nu/(6\beta)$ for all θ with $\|\theta - \bar\theta\| < \delta_P$. Let θ be such that $\|\theta - \bar\theta\| < \delta_P$ and $R = \bar R$. For $a \geq a_0$, $a \leq n$ and s with $|s - s^{in\,s}_{i,m,\bar\theta}(a)| \geq \bar\epsilon$ and any b such that $(s,b) \in S(a)$ (which is independent of θ!), we have $f_\theta(a, b^{in\,s}_{i,m,\bar\theta}(a), s^{in\,s}_{i,m,\bar\theta}(a)) > f_\theta(a,b,s)$ in an argumentation which is analogous to the argument above for b^{outs}. Since we already argued that $|s^{in\,s}_{i,m,\theta}(a) - s^{in\,s}_{i,m,\bar\theta}(a)| < \bar\epsilon$ for all $a \leq a_0$ and θ with $\|\theta - \bar\theta\| < \delta_P$, we find that $d_{c,[0,n]}(s^{ins}_{i,m,\theta}, s^{ins}_{i,m,\bar{\theta}}(a)) < \bar{\epsilon}$. •

The following general theorem should provide to be useful in other contexts as well: it states that derivatives of concave functions move continuously with some parameter, if the functions itself move continuously with the parameter. This is a somewhat surprising result, since it is obviously false in general for other, non – concave functions functions. The theorem is particularly useful in our context to prove the continuity of the decision rules $a_{i,\theta}(a)$ and $a_{o,\theta}(a)$ which relies on the derivatives of the v^{outs}_{θ} and v^{ins}_{θ}.

<u>THEOREM A.V.9: (The continuity in function space of the derivatives of continuously differentiable, concave functions)</u>

> Let f: $\mathbb{R}_{++} \to \mathbb{R}$ be concave and continuously differentiable. Let $\epsilon > 0$ and $0 < a_{min} < a_{max} < \infty$. Then there is a $\delta > 0$ with the following property. If g: $\mathbb{R}_{++} \to \mathbb{R}$ is a concave, increasing and continuously differentiable function with $d_{\infty,\mathbb{R}_{++}}(f,g) < \delta$, then $d_{\infty,[a_{min},a_{max}]}(f',g') < \epsilon$.
>
> In other words, the derivatives of concave, continuously differentiable functions are continuous in function space (using the metric $d_{c,\mathbb{R}_{++}}$ of uniform convergence on compact subsets of $\mathbb{R}_{++}$) with respect to the d_{∞} – metric on the functions themselves.

PROOF:

Apply the Lemma about a property of continuously differentiable functions and find $\nu = \bar{\nu}$ and δ, which we relabel δ_0. Let $\delta = \delta_0/3$ and suppose to the contrary, that $|f'(\bar{a}) - g'(\bar{a})| \geq \epsilon$ for some $\bar{a} \in [a_{min}, a_{max}]$. Let $\lambda = g'(\bar{a})$, $a_1 = a_2 = \bar{a}$ and apply the Lemma to find a. It then follows that

$$\begin{aligned} g(a) &\geq f(a) - \delta \\ &\geq l_{f(\bar{a}),\lambda,\bar{a}}(a) + 2\delta \\ &\geq l_{g(\bar{a}),g'(\bar{a}),\bar{a}}(a) + \delta \\ &\geq g(a) + \delta, \end{aligned}$$

a contradiction. •

We now redefine the functions $a_i(a)$ and $a_o(a)$ which will leave the properties of the transition measures $P_i(a)\delta_{a_i(a)} + P_o(a)\delta_{a_o(a)}$ unchanged. Note that v, v^{outs} and v^{ins} are continuously differentiable and that e.g. for $a_2 > a_1 \geq 0$, we have

$$\begin{aligned} v_\theta'^{ins}(a_1) - v_\theta'^{ins}(a_2) &\geq u_e'(a_1) - u_e'(a_2) \\ &= -\int_{a_1}^{a_2} u_e''(a)\, da > 0, \end{aligned}$$

i.e. $v_\theta'^{ins}$ is continuously invertible. Observe furthermore that $v'(a) \to 0$ as

$a \to \infty$, etc.. Thus, we define $a_i(a)$ to satisfy

$$v'(a) = v'^{ins}(a_i(a)), \text{ if } \lim_{a \to 0} v'^{ins}(a) > v'(a) \text{ and}$$
$$a_i(a) = 0 \text{ otherwise.}$$

We proceed likewise for $a_o(a)$. Keeping the definitions for $P_i(a)$ and $P_o(a)$, where we use these new $a_i(a)$ and $a_o(a)$ instead of the previously defined functions, we note that these probabilities coincide with the probabilities defined previously and that we changed e.g. $a_i(a)$ only when $P_i(a) = 0$ anyways. Thus, the transition measures do not change.

THEOREM A.V.10:

(i) If messages are informative at $\bar{\theta}$, then there is $\delta > 0$ and $\underline{a} \in \mathbb{R}_+$, so that every agent becomes an insider for θ with $\|\theta - \bar{\theta}\| < \delta$ and $a \geq \underline{a}$.

(ii) If messages are uninformative at $\bar{\theta}$, then for every $\underline{\underline{a}} \in R_+$, there is a $\delta > 0$, so that every agent becomes an outsider for θ with $\|\theta - \bar{\theta}\| < \delta$ and $a < \underline{\underline{a}}$.

(iii) The decision rules $a_{i,\theta}(a)$ and $a_{o,\theta}(a)$ are continuous with respect to θ in the metric $d_{c,\mathbb{R}_{++}}$.

PROOF:

(i) Analyzing the proof of the Theorem about the existence and properties of the unique fixed point v of T, that there is a cut – off level $\underline{a}$, beyond which every agent becomes an insider, and furthermore using the fact, that the decision rules $c^{ins}_{i,m',\theta}$, c^{outs}_{θ} and b^{outs}_{θ} are increasing and continuous in the d_c − metric with respect to θ, it follows that this cut − off level $\underline{a}$ can be chosen independently of θ in some suitable neighbourhood of $\bar{\theta}$, i.e. for all θ with $\| \theta - \bar{\theta} \| < \delta$ for some $\delta > 0$.

(ii) It was shown in the proof for the continuous differentiability of v^{ins} and v^{outs} that

$$v'^{outs}_{\theta}(a) = \int u'(c^{outs}_{\theta}(a+N))dF_N.$$

Since $c^{outs}_{\theta}(y)$ is increasing in y and d_c − continuous in θ, there is some $\delta_0 > 0$ and $\nu > 0$, so that $v'^{outs}_{\theta}(a) \geq \nu$ for all $a \leq \underline{\underline{a}}$ and all θ with $\|\theta - \bar{\theta}\| < \delta_0$. Using a similar argument for v'^{ins} and the fact that $c^{ins}_{i,m}(a) \to \infty$ $(a \to \infty)$, we can find $a_1 > \underline{\underline{a}}$ and $\delta_1 > 0$, $\delta_1 \leq \delta_0$ so that $v'^{ins}_{\theta}(a) < \nu$ for all $a \geq a_1$ and θ with $\|\theta - \bar{\theta}\| < \delta_1$. Since v^{ins}_{θ} and v^{outs}_{θ} are d_∞ − continuous in θ and since $v^{outs}_{\theta}(a) - v^{ins}_{\theta}(a) \geq \epsilon$ for some suitable $\epsilon > 0$ and all a with $0 \leq a \leq a_1$, we can finally find $\delta > 0$, $\delta \leq \delta_1$, so that $v^{ins}_{\theta}(a) < v^{outs}_{\theta}(a)$ for all θ with $\|\theta - \bar{\theta}\| < \delta$ and all a with $0 \leq a \leq a_1$. We claim, that δ delivers the desired result. Suppose, it does not, i.e. suppose that for some $a \leq \underline{\underline{a}}$ and θ with $\|\theta - \bar{\theta}\| < \delta$, $P_{i,\theta}(a) \neq 0$. By construction, $v^{ins}_{\theta}(a) < v^{outs}_{\theta}(a)$ for all a with $0 \leq a \leq a_1$, i.e. we must have $a_{i,\theta}(a) \geq a_1$ and $a_{o,\theta}(a) \leq \underline{\underline{a}}$. However, then we need to have $v'^{outs}_{\theta}(a_{o,\theta}(a)) = v'^{ins}_{\theta}(a_{i,\theta}(a))$, which is not possible by the construction of a_1.

(iii) If messages are uninformative at $\bar{\theta}$, the claim follows directly from (ii). The case of informative messages remains. Let $\epsilon > 0$ and $n > 4/\epsilon$ w.l.o.g.. We need to find a $\delta > 0$, so that

$$d_{\infty,[1/n,n]}(\, a_{i,\theta}, a_{i,\bar{\theta}}\,) \leq \epsilon \text{ and}$$
$$d_{\infty,[1/n,n]}(\, a_{o,\theta}, a_{o,\bar{\theta}}\,) \leq \epsilon$$

for all θ with $\|\theta - \bar{\theta}\| < \delta$.

For convenience of notation, define two continuous, monotonously decreasing functions $g^i_\theta(\lambda)$ and $g^o_\theta(\lambda)$ for $\lambda > 0$ as follows. Let

$$g^i_\theta(\lambda) = (v'^{ins}_\theta)^{-1}(\lambda),$$

if the inverse exists, and $g^i_\theta(\lambda) = 0$ otherwise. Define g^o_θ likewise. I.e. we have $a_{i,\theta}(a) = g^i_\theta(v'_\theta(a))$ and $a_{o,\theta}(a) = g^o_\theta(v'_\theta(a))$. Now let $\nu_0 = v_{\bar{\theta}}'(n)/4 > 0$. Find $a_{max} \geq n$, so that $v_{\bar{\theta}}'^{ins}(a_{max}) \leq \nu_0$ and $v_{\bar{\theta}}'^{outs}(a_{max}) \leq \nu_0$. Let $a_{min} = 1/n$. Since u_e is twice continuously differentiable with $u_e'' > 0$, we can find $\nu_1 > 0$, $\nu_1 < \nu_0$, so that

$$-\int_a^{a+\epsilon/2} u_e''(\alpha)\, d\alpha > 2\, \nu_1$$

for all $a \in [a_{min}, a_{max}]$. Finally, apply the Theorem about the continuity in function space of the derivatives of continuously differentiable, concave functions to find $\delta > 0$, so that

$$d_{\infty,[a_{min},a_{max}]}(v_{\bar\theta}', v_\theta') < \nu_1,$$
$$d_{\infty,[a_{min},a_{max}]}(v_{\bar\theta}'^{,ins}, v_\theta'^{,ins}) < \nu_1, \text{ and}$$
$$d_{\infty,[a_{min},a_{max}]}(v_{\bar\theta}'^{,outs}, v_\theta'^{,outs}) < \nu_1$$

for all θ, $\|\theta - \bar\theta\| < \delta$. We claim, that δ has the desired properties. To that end, let $a \in [1/n, n]$ and choose θ with $\|\theta - \bar\theta\| < \delta$. We will show that $|a_{i,\theta}(a) - a_{i,\bar\theta}| \le \epsilon$. The proof for $a_{o,\theta}$ is analogous.

Let $\bar\lambda = v_{\bar\theta}'(a)$ and $\lambda = v_\theta'(a)$. Let $\bar\lambda_{max} = v_{\bar\theta}'^{,ins}(1/n)$ and $\bar\lambda_1 = \min\{\bar\lambda, \bar\lambda_{max}\}$. Observe, that $\bar\lambda_1 > v_{\bar\theta}'^{,ins}(a_{max})$. Thus, we have $a_1 := g_{\bar\theta}^i(\bar\lambda_1) \in [a_{min}, a_{max}]$. Likewise, let $\lambda_{max} = v_\theta'^{,ins}(1/n)$ and $\lambda_1 = \min\{\lambda, \lambda_{max}\}$. Observe that

$$\lambda_1 \ge v_\theta'(n) \ge v_{\bar\theta}'(n) - \nu_1 \quad \ge 3\nu_0 \quad \ge v_{\bar\theta}'^{,ins}(a_{max}) + 2\nu_0$$
$$\ge v_\theta'^{,ins}(a_{max}) + \nu_0 > v_\theta'^{,ins}(a_{max})$$

and hence $a_2 := g_\theta^i(\bar\lambda_1) \in [a_{min}, a_{max}]$. Let $\bar\lambda_2 = v_{\bar\theta}'^{,ins}(a_2)$. Note that $|\bar\lambda_2 - \lambda_1| \le \nu_1$, $|\lambda - \bar\lambda| \le \nu_1$ and $|\lambda_{max} - \bar\lambda_{max}| \le \nu_1$ by construction of δ. It follows, that $|\bar\lambda_1 - \bar\lambda_2| \le 2\nu_1$.

We claim that $|a_1 - a_1| < \epsilon/2$. Suppose not. Assume w.l.o.g. that $a_2 > a_1$. We noted above, when introducing u_e, that

$$\bar{\lambda}_1 - \bar{\lambda}_2 = v_{\bar{\theta}}^{\prime,ins}(a_1) - v_{\bar{\theta}}^{\prime,ins}(a_2)$$

$$\geq -\int_{a_1}^{a_2} u_e''(\alpha)\, d\alpha > 2\,\nu_1$$

by construction of ν_1 in contradiction to $|\bar{\lambda}_1 - \bar{\lambda}_2| \leq 2\nu_1$. Thus, the claim that $|a_1 \_ a_2| < \epsilon/2$ is established.

Finally note, that $|a_{i,\bar{\theta}}(a) - a_1| \leq 1/n < \epsilon/4$ and likewise $|a_{i,\theta}(a) - a_2| \leq 1/n < \epsilon/4$. It follows, that $|a_{i,\theta}(a) - a_{i,\bar{\theta}}(a)| < \epsilon$ as claimed. This finishes the proof. •

We now need to make the second perturbation of our model. This perturbation forces continuity on the probabilities of becoming an outsider or an insider. To that end, we first have to choose functions $\Delta^v(a;\theta)$ and $\Delta^a(a;\theta)$ with the following properties.

<u>ASSUMPTION A.V.4.</u>

(i) $\Delta^v(a;\theta)$ is continuous in a an θ,

(ii) $\Delta^v(a;\theta) > 0$ for all a, θ,

(iii) $2\Delta^v(a;\theta) < v_\theta^{outs}(a) - v_\theta^{ins}(a)$, if messages are uninformative,

(iv) $2\Delta^v(a;\theta) < v_\theta^{ins}(a) - v_\theta^{outs}(a)$ for $a \geq a_\theta$, some $a_\theta < \infty$, if messages are informative,

(v) $\Delta^a(a;\theta) > 0$, all a, θ,

(vi) $2\Delta^v(a;\theta) < v_\theta^{outs}(a) - v_\theta^{ins}(a)$, all θ and all $a < \Delta^a(a;\theta)$.

It is not difficult to find such a function $\Delta^a(a;\theta)$ (only the last condition puts some constraints on it, which are easily satisfied with the proper choice of Δ^v in light of Lemma A.V.7, part (ii)), but we need to show the existence of a function Δ^v.

LEMMA A.V.14:

Functions Δ^v and Δ^a, which satisfy assumption A.V.4 exist.

PROOF:

We only concern ourselves with the properties (i) through (iv). Fix some $\bar{\Delta} \in (0;1/3)$. With Θ as the space of all possible parameters, let Θ_{inf} be the subset of all θ's with informative messages and Θ_{un} be the subset of all θ's with uninformative messages. Note, that we assumed that Θ_{un} is an open set. It is therefore possible to find a sequence of compact sets Θ_n, $n \in \mathbb{N}$, whose union is Θ_{un} and where Θ_n is contained in the interior of Θ_{n+1}. Consider first any compact subset Θ_c of Θ_{un}. Construct for $\theta \in \Theta_c$ an insider bound $\underline{a}_\theta$ exactly as in the proof of Theorem A.V.6 and note, that then

$$u(c_\theta^{outs}(a+1)+1)+\beta w_\theta(Rb_\theta^{outs}(a+1))-v_{e,\theta}^{outs}(a) > 0$$

for all $a \geq \underline{a}_\theta$ by the last line in the proof to Theorem A.V.6.. Note that the construction of $\underline{a}_\theta$ rests on properties of the decision rules and value functions, and thus, by monotonicity in a and continuity in θ of these rules and functions as well as a finite cover argument for Θ_c, we can choose some $\underline{a}(\Theta_c)$, so that $\underline{a}_\theta < \underline{a}(\Theta_c)$ for all $\theta \in \Theta_c$. Thus construct $\underline{a}_n = \underline{a}(\Theta_n)$ in such a way, that $\underline{a}_n$ is monotonously increasing and $\underline{a}_n \to \infty$ as $n \to \infty$.

We now define a_θ in a continuous way recursively in such a way, that $a_\theta > \underline{a}_\theta$ for all $\theta \in \Theta_{un}$. Suppose, we have already defined a_θ on Θ_n in such a way, that $a_\theta \geq \underline{a}_{n+1}$ on the boundary of Θ_n. Define a_θ on $\Theta_{n+2} \backslash \Theta_{n+1}$ to be equal to $\underline{a}_{n+3}$. By Hausdorffs extension theorem, we can thus find a function $\hat{a}_\theta$ on all of Θ_{n+2}, which coincides with a_θ, where it is already defined. Now let a_θ be the maximum of $\hat{a}_\theta$ and $\underline{a}_{n+1}$: it is easy to see that a_θ is continuous, $a_\theta > \underline{a}_\theta$ and $a_\theta \geq \underline{a}_{n+3}$ on the boundary of Θ_{n+2} – the induction can continue.

Define now

$$\Delta^v(a;\theta) = \bar{\Delta}(u(c_\theta^{outs}(a+1)+1)+\beta w_\theta(Rb_\theta^{outs}(a+1))-v_{e,\theta}^{outs}(a))$$

for $\theta \in \Theta_{un}$ and $a \geq a_\theta$: observe that $\Delta^v(a;\theta)$ satisfies parts (ii) and (iv) of our assumption.

Let

$$\Delta^v(a;\theta) = \bar{\Delta}(\ v_\theta^{outs}(a) - v_\theta^{ins}(a)\ )$$

for $\theta \in \Theta_{inf}$. Since $a_\theta \to \infty$ as $\theta \to \Theta_{inf}$, it is easy to see that $\Delta_v(a;\theta)$ can be continuously extended to all of $\mathbb{R}_+ \times \Theta$. •

Note, that by choosing $\bar{\Delta} \in (0,1/3)$ small enough in the proof, we can make Δ^v (and likewise Δ^a) smaller than the numerical accuracy of any computer for the calculations that now follow, which again leaves the computer code to calculate this perturbed model unaltered from a program for the original version.

We now redefine P_i, P_o, a_i and a_o as follows. First define functions

$$\lambda^a(a;\theta) = \min\{\, 1 \,;\, \frac{|a_{i,\theta}(a)-a_{o,\theta}(a)|}{\Delta^a(a;\theta)} \,\} \text{ and}$$

$$\lambda^v(a;\theta) = (1+ \frac{v_\theta^{ins}(a) - v_\theta^{outs}(a)}{\max\{|v_\theta^{ins}(a)-v_\theta^{outs}(a)|\,;\Delta^v(a;\theta)\}})/2.$$

Observe, that $0 \leq \lambda^a \leq 1$ and $0 \leq \lambda^v \leq 1$ and that

$$\lambda^a(a;\theta) = 1 \text{ for } |a_{i,\theta}(a)-a_{o,\theta}(a)| \geq \Delta^a(a;\theta),$$
$$\lambda^v(a;\theta) = 1 \text{ for } v_\theta^{ins}(a) \geq v_\theta^{outs}(a) + \Delta^v(a;\theta),$$
$$\lambda^v(a;\theta) = 0 \text{ for } v_\theta^{ins}(a) \leq v_\theta^{outs}(a) - \Delta^v(a;\theta).$$

Now let

$$\hat{P}_{i,\theta}(a) = \lambda^a(a;\theta)P_{i,\theta}(a) + (1-\lambda^a(a;\theta))\lambda^v(a;\theta),$$

$$\hat{P}_{o,\theta}(a) = \lambda^a(a;\theta)P_{i,\theta}(a) + (1-\lambda^a(a;\theta))(1-\lambda^v(a;\theta))$$
$$= 1-\hat{P}_{i,\theta}(a)\ ,$$
$$\hat{a}_{i,\theta}(a) = a + \hat{P}_{o,\theta}(a)(a_{i,\theta}(a) - a_{o,\theta}(a)) \text{ and}$$
$$\hat{a}_{o,\theta}(a) = a - \hat{P}_{i,\theta}(a)(a_{i,\theta}(a) - a_{o,\theta}(a)).$$

Note that as a consequence,

- if $|a_{i,\theta}(a)-a_{o,\theta}(a)| \geq \Delta^a(a;\theta)$, then $\hat{P}_{i,\theta}(a) = P_{i,\theta}(a)$, $\hat{P}_{o,\theta}(a) = P_{o,\theta}(a)$, $\hat{a}_{i,\theta}(a) = a_{i,\theta}(a)$ and $\hat{a}_{o,\theta}(a) = a_{o,\theta}(a)$,
- $a_{i,\theta}(a) - a_{o,\theta}(a) = \hat{a}_{i,\theta}(a) - \hat{a}_{o,\theta}(a)$,
- $\hat{P}_{i,\theta}\hat{a}_{i,\theta}(a) + \hat{P}_{o,\theta}\hat{a}_{o,\theta}(a) = a$,
- if $a_{i,\theta}(a) = a = a_{o,\theta}(a)$, then $\hat{a}_{i,\theta}(a) = a = \hat{a}_{o,\theta}(a)$. If furthermore $|v_\theta^{ins}(a)-v_\theta^{outs}(a)| \geq \Delta^v(a;\theta)$, then $\hat{P}_{i,\theta}(a) = P_{i,\theta}(a)$, $\hat{P}_{o,\theta}(a) = P_{o,\theta}(a)$,
- all four functions $\hat{P}_{i,\theta}$, $\hat{P}_{o,\theta}$, $\hat{a}_{i,\theta}$ and $\hat{a}_{o,\theta}$ are continuous in a: this follows from the continuity of $P_{i,\theta}$, $P_{o,\theta}$ on $\{a|\ |a_{i,\theta}(a)-a_{o,\theta}(a)| \geq 0\ \}$ and the fact that we have

$$\lambda^a(a_n;\theta) \to \lambda^a(\bar{a};\theta) = 0$$

for any $\bar{a}$ and $a_n \to \bar{a}$ with $|a_{i,\theta}(a_n)-a_{o,\theta}(a_n)| \to 0$: exactly, when $P_{i,\theta}$ or $P_{o,\theta}$ might have a discontinuity, this discontinuity is smoothed away with λ^a.

<u>**Lemma A.V.14:**</u>

(i) The new decision rules $\hat{P}_{i,\theta}$, $\hat{P}_{o,\theta}$, $\hat{a}_{i,\theta}$ and $\hat{a}_{o,\theta}$ are continuous with respect to θ in the metric $d_{c,\mathbb{R}_{++}}$,

(ii) $\hat{P}_{o,\theta}(a) \to 1$ as $a \to 0$ and $\theta \to \bar{\theta}$ for some $\bar{\theta}$,

(iii) Theorem A.V.10, parts (i) and (ii) remain valid for the new decision rules $\hat{a}_{i,\theta}$ and $\hat{a}_{o,\theta}$.

PROOF:

(i) follows from tracing through the construction of the altered decision rules above. Note that v_θ^{ins} and v_θ^{outs} are d_∞ – continuous in θ by the remark after Theorem A.V.7 and that the unaltered decision rules $a_{i,\theta}$ and $a_{o,\theta}$ are $d_{c,\mathbb{R}_{++}}$–continuous with respect to θ by Theorem A.V.10 (iii). The only difficulty could arise at some $\bar{\theta}$, $\bar{a}$, where $a_{i,\bar{\theta}}(\bar{a}) = a_{o,\bar{\theta}}(\bar{a})$ and $v_{\bar{\theta}}^{ins}(\bar{a}) = v_{\bar{\theta}}^{outs}(\bar{a})$, since then the unaltered decision rules $P_{i,\theta}(a)$ and $P_{o,\theta}(a)$ may not be continuous at $\bar{\theta}$, $\bar{a}$. But since they are bounded above and below by 1 and 0 respectively and since $\lambda^a(a;\theta) \to 0$ as $a \to \bar{a}$ and $\theta \to \bar{\theta}$, if $\bar{a} > 0$, we get continuity there as well.

(ii) Observe that $\lambda^v(a;\theta) \to 1$ as $a \to 0$ and $\theta \to \bar{\theta}$. Furthermore $P_{o,\theta}(a) \to 1$ as $a \to 0$ and $\theta \to \bar{\theta}$: this can be seen from the definition of $P_{o,\theta}(a)$ and from the continuity of v_θ^{ins} and v_θ^{outs} together with $v_\theta^{outs}(0) > v_\theta^{ins}(0)$, delivering the fact, that locally $a_{i,\theta}(a) \geq \bar{a} > 0$ for some suitable $\bar{a}$, if $P_{o,\theta}(a) < 1$.

(iii) is a direct consequence of Lemma A.V.13. •

It is clear, what this perturbation does: it avoids sharp transitions in the choices of whether to become an insider or an outsider, if the lotteries become small in distance (for example, if the agent chooses to just retain the original asset holdings with certainty), and if the value – functions $v_\theta^{ins}(a)$ and $v_\theta^{outs}(a)$ get very close to each other. So instead of switching 100 % from outsider to insider e.g. for no – gamble – cases, if a small change in the assets a (or in the parameter θ) brings $v_\theta^{ins}(a)$ just above $v_\theta^{outs}(a)$, the agents make that transition more "carefully". The model could be altered to make a feature of that type happen as the solution to the optimal program of different agents, if we generalize the concept of the utility functions u_e introduced above by making these extra utilities different and independently random across agents and across the outsider – insider – branches. That would have the effect, that some agents with the same assets end up choosing to become an outsider and others choose to become an insider, where the relative fraction of these two populations changes, as the underlying "common" value functions v^{outs} and v^{ins} move. These fractions are then, what utimately matters for the population asset distribution via a law of large numbers. Again, writing these changes out in detail would make the model unnecessarily complicated just for purely technical reasons.

We now write $a_{i,\theta}$ for $\hat{a}_{i,\theta}$, $P_{i,\theta}$ for $\hat{P}_{i,\theta}$, etc., and proceed with these perturbed decision rules.

With these new rules and given some θ, define the transition probabilities

$$Q^{outs}(a;A') = \int 1_{A'}(Rb^{outs}(a+N)\, dF_N,$$

$$Q^{ins}(a;A') =$$

$$\sum_{i=0}^{I} \sum_{m=0}^{M} \sum_{g=0}^{1} \pi(i)\pi(m|i)\pi(g|m,i) 1_{A'}(R(b_{i,m}^{ins}(a)+(1-g)s_{i,m}^{ins}$$

$$(a)),$$

$$Q(a;A') = P_o(a)\, Q^{outs}(a_o(a);A') + P_i(a)\, Q^{ins}(a_i(a);A'),$$

where $1_A(a) = 1$, if $a \in A$ and $1_A(a) = 0$, if $a \notin A$ (we left away the subindex θ for clearer notation). Note, that only the definition for Q, but not the definitions for Q^{outs} or Q^{ins} are affected by our perturbing the decision rules $a_{i,\theta}$, $P_{i,\theta}$, etc..

We finally introduce our third and last perturbation of the model, which allows us to apply Doeblins condition or a strengthening thereof to prove existence and in particular uniqueness of a stationary asset distribution (see Stokey – Lucas, with Prescott (1989), p. 345 and p. 348): we introduce a small probability for bankrupcy, where the assets of the bankrupt agents are redistributed to all other agents in proportion of their asset holdings.

That is, we fix some $\bar{\epsilon} > 0$ and alter our transition probability Q to

$$\hat{Q}(a:A') = (1-\bar{\epsilon})\, Q(a;A'/(1-\bar{\epsilon})) + \bar{\epsilon}\, Q^{outs}(0;A'),$$

i.e. with probability $\bar{\epsilon}$, the agent goes bankrupt and starts anew as outsider. Again, choosing $\bar{\epsilon}$ small enough should allow us to stay with the same numerical computations as for the original version of the model before. Again, we proceed writing Q instead of $\hat{Q}$ and use it in place of the original transition probabilities Q.

It should be mentioned, that this bankrupcy probability is not necessary in order to prove existence of an invariant distribution: the Feller property (see Stokey – Lucas, with Prescott (1989), p.220 and p.376), appropriately modified to work on all of $\mathbb{R}$, will suffice. However, we need more here: we also need that the invariant distribution moves continuously with the parameter vector θ: to this end, we want to apply a version of Theorem 12.13 in Stokey – Lucas, with Prescott(1989) and for that, we need uniqueness of the invariant distribution. This is proved in the following Theorem. For that Theorem, let

$$C_0(\mathbb{R}_+) = \{f:\mathbb{R}_+ \to \mathbb{R} \mid f \text{ is continuous, bounded, } \lim_{a\to\infty} f(a) \text{ exists }\}$$

with the d_∞ – metric and

$$\Lambda(\mathbb{R}_+) = \{\mu \mid \mu \text{ is a probability measure on } (\mathbb{R}_+, \mathscr{B})\},$$

with the weak topology $\sigma(\Lambda(\mathbb{R}_+), C_0(\mathbb{R}_+))$ induced by the space $C_0(\mathbb{R}_+)$ and where $\mathscr{B}$ is the set of Borel sets of $\mathbb{R}_+$.

<u>**Theorem A.V.11:**</u>

(i) There is a unique, invariant probability measure μ_θ to Q_θ on $\mathbb{R}_+$,

(ii) the map $\theta \mapsto \mu_\theta$ is continuous (in the weak topology $\sigma(\Lambda(\mathbb{R}_+),C_o(\mathbb{R}_+))$ on $\Lambda(\mathbb{R}_+)$),

(iii) the maps $\theta \mapsto \mu_\theta^{ins}$ and $\theta \mapsto \mu_\theta^{outs}$ are continuous in the weak topology $\sigma(\Lambda(\mathbb{R}_+),C_o(\mathbb{R}_+)$ on $\Lambda(\mathbb{R}_+)$, where

$$\mu_\theta^{ins}(A) = \int P_{i,\theta}(a)\, 1_A(a_{i,\theta}(a))\, d\mu_\theta \text{ and}$$
$$\mu_\theta^{outs}(A) = \int P_{o,\theta}(a)\, 1_A(a_{o,\theta}(a))\, d\mu_\theta.$$

Note, that these invariant distribution μ_θ, μ_θ^{ins} and μ_θ^{outs} were called F_a, F_a^{ins} and F_a^{outs} in the definition of an equilibrium in part IV of the main text.

<u>**PROOF:**</u>

(i) It is clear, that Q_θ is a transition function on $(\mathbb{R}_+,\mathscr{B})$ in the sense of Stokey – Lucas, with Prescott (1989), p. 212. Observe, that condition M on p. 348 in Lucas – Stokey, with Prescott(1989) is satisfied for $\epsilon = \bar{\epsilon}$ and $N = 1$: if $A' \in \mathscr{B}$, then $Q(a,A') \geq \bar{\epsilon}$, if $0 \in A'$ and $Q(a,A'^c) \geq \bar{\epsilon}$, if $0 \notin A'$. The existence and uniqueness of an invariant probability distribution μ_θ on $\mathbb{R}_+$ now follows immediately from Theorem 11.12 in Stokey – Lucas, with Prescott(1989).

(ii) We want to apply Theorem 12.13 in Stokey – Lucas, with Prescott(1989). To that end, choose some strictly monotone, continuous map $\psi:\mathbb{R}_+ \cup \{+\infty\} \to [0;1]$, which is onto (take e.g. $\psi(x) = x/(1+x)$). Now define transition probabilities $P_\theta(s,A)$ on

$[0;1]$ by transforming $\mathbb{R}_+$ onto $[0;1)$, i.e. for $s \in [0;1)$, $A' \in \mathscr{B}([0;1])$, let

$$P_\theta(s,A') = Q_\theta(\psi^{-1}(s),\mathbb{R}_+ \cap \psi^{-1}(A'))$$

and for $s = 1$, let

$$P_\theta(1,A') = \bar{\epsilon}\, 1_{A'}(0) + (1-\bar{\epsilon})\, 1_{A'}(1)$$

(please note, that for this proof, we use P to denote some transition function on $[0;1]$ and <u>not</u> e.g. the signalling probabilities $P(m|g)$, etc.). Observe, that by repeating the argument under (i), we have

$$\lambda_\theta = \mu_\theta \circ \psi^{-1}$$

as the unique invariant distribution for P_θ. Observe furthermore that $C_0(\mathbb{R}_+)$ corresponds to $C([0;1])$, the space of continuous functions on $[0;1]$ via the transformation ψ and that the weak topology $\sigma(\Lambda(\mathbb{R}_+),C_0(\mathbb{R}_+))$ corresponds therefore to the usual weak topology (in the language of the probability theorists; in the language of the functional analysts: weak–star) on the space of probability measures on $[0;1]$ with the usual Borel sets.

Thus, all that remains to check to prove (ii) by applying the above – mentioned Theorem 12.13 is that

$$(s_n,\theta_n) \to (\bar{s},\bar{\theta})$$

implies that

$$P_{\theta_n}(s_n;\cdot) \to P_{\bar{\theta}}(\bar{s};\cdot) \text{ weakly.}$$

For $\bar{s}$, this can easily be checked: observe that at $\bar{\theta}$, we have

$$a_i(a) \to \infty \ (a\to\infty),$$
$$a_o(a) \to \infty \ (a\to\infty),$$
$$b^{outs}(y) \to \infty \ (y\to\infty) \text{ and}$$
$$rev^{in\,s}_{i,m,g}(a) \to \infty \ (a\to\infty)$$

and that furthermore these functions are increasing for any θ and $d_{c,\mathbb{R}_{++}}$ – or d_c – continuous respectively in θ.

For $\bar{s}\neq 1$, we rewrite the conditions above in $\mathbb{R}_+$ – space in terms of the transition function Q: we need to show for $\bar{a} \in \mathbb{R}$ and $\bar{\theta}$, that

$$(a_n,\theta_n) \to (\bar{a},\bar{\theta})$$

implies that

$$Q_{\theta_n}(a_n;\cdot) \to Q_{\bar\theta}(\bar a;\cdot) \text{ in the topology } \sigma(\Lambda(\mathbb{R}_+),C_o(\mathbb{R}_+)).$$

Using Theorem A.V.8 and the continuity of the outsider– and insider – decision rules by Theorem A.V.6, we can see that

$$Q^{outs}_{\theta_n}(a_{o,n};\cdot) \to Q^{outs}_{\bar\theta}(\bar a_o;\cdot) \text{ and}$$
$$Q^{ins}_{\theta_n}(a_{o,n};\cdot) \to Q^{ins}_{\bar\theta}(\bar a_o;\cdot)$$

for any sequences $a_{o,n}\to\bar a_o$ and $a_{i,n}\to\bar a_i$: to prove that, all that is needed to show is the convergence of $S^{outs}_{\theta_n}(f)(a_{o,n})$ to $S^{outs}_{\bar\theta}(f)(\bar a_o)$ and $S^{ins}_{\theta_n}(f)(a_{o,n})$ to $S^{ins}_{\bar\theta}(f)(\bar a_o)$ for any $f \in C_o(\mathbb{R}_+)$, where

$$S^{outs}_{\theta}(f)(a) = \int f(\, Rb^{outs}_{\theta}(a + N))\, dF_N \text{ and}$$
$$S^{ins}_{\theta}(f)(a) = \sum_{i=0,m=0,g=0}^{I \quad M \quad 1} f(R(\, b^{in\,s}_{i,m,\theta}(a)+ (1-g)s^{in\,s}_{i,m,\theta}(a)).$$

For S^{ins}_{θ}, this is immediately clear with the continuity of the decision rules in θ and a; for S^{outs}_{θ}, this follows additionally from Lebesgues theorem on dominated convergence. Thus, it remains to be checked that for $f \in C_o(\mathbb{R}_+)$, we have

$$S_{\theta_n}(f)(a_n) \to S_{\bar\theta}(f)(\bar a),$$

where

$$S_\theta(f)(a) = (1-\bar{\epsilon})P_{o,\theta}(a)S_\theta^{outs}(f)(a_{o,\theta}(a)) + (1-\bar{\epsilon})P_{i,\theta}(a)S_\theta^{ins}(f)(a_{i,\theta}(a)) + \bar{\epsilon}S_\theta^{outs}(f)(0).$$

But this follows immediately from Lemma A.V.13.

(iii) Observe that for $f \in C_o(\mathbb{R}_+)$, $|f|\leq 1$, we have

$$\int f(a)\, d\mu_\theta^{outs} = \int P_{o,\theta}(a)\, f(a_{o,\theta}(a))\, d\mu_\theta,$$

and we have to show that this expression is continuous in θ. Thus, given $\bar{\theta}$ and $\epsilon>0$, find first a neighbourhood and $\underline{a} > 0$, $\underline{\underline{a}} > 0$, so that $\mu_\theta([0;\underline{a}]) < \epsilon/8$ and $\mu_\theta([\underline{\underline{a}},\infty)) < \epsilon/8$ for all θ within that neighbourhood and where $\underline{a}$, $\underline{\underline{a}}$ are continuity points of $\mu_{\bar{\theta}}$. Apply Lemma A.V.13 to shrink that neighbourhood even further, so that for all θ within,

$$|P_{o,\theta}(a)f(a_{o,\theta}(a)) - P_{o,\bar{\theta}}(a)f(a_{o,\bar{\theta}}(a))| < \epsilon/4, \text{ all } a\in[\underline{a},\underline{\underline{a}}].$$

Finally shrink that neighbourhood to those θ, such that

$$\left|\int_{\underline{a}}^{\underline{\underline{a}}} P_{o,\theta}(a)f(a_{o,\theta}(a))\, d\mu_\theta - \int_{\underline{a}}^{\underline{\underline{a}}} P_{o,\bar{\theta}}(a)f(a_{o,\bar{\theta}}(a))\, d\mu_\theta\right| < \epsilon/4$$

by part (ii) and the fact that $\underline{a}$ and $\underline{\underline{a}}$ are continuity points of $\mu_{\bar{\theta}}$. It now follows that

$$
\begin{aligned}
&\left|\int P_{0,\theta}(a)f(a_{0,\theta}(a))\,d\mu_\theta - \int P_{0,\bar{\theta}}(a)f(a_{0,\bar{\theta}}(a))\,d\mu_\theta\right| \\
&\leq \mu_\theta([0;\underline{a}]) + \mu_\theta([\underline{\underline{a}},\infty)) \\
&+ \mu_{\bar{\theta}}([0;\underline{a}]) + \mu_{\bar{\theta}}([\underline{\underline{a}},\infty)) \\
&+ \int_{\underline{a}}^{\underline{\underline{a}}} |P_{0,\theta}(a)f(a_{0,\theta}(a)) - P_{0,\bar{\theta}}(a)f(a_{0,\bar{\theta}}(a))|\,d\mu_\theta \\
&+ \left|\int_{\underline{a}}^{\underline{\underline{a}}} P_{0,\theta}(a)f(a_{0,\theta}(a))\,d\mu_\theta - \int_{\underline{a}}^{\underline{\underline{a}}} P_{0,\bar{\theta}}(a)f(a_{0,\bar{\theta}}(a))\,d\mu_\theta\right| \\
&\leq \epsilon.
\end{aligned}
$$

The proof for μ_θ^{ins} is analogous. •

We now aim at showing that certain aggregates like

$$
\begin{aligned}
D_{std,\theta}(i,g) &= \pi(i,g)\sum_{m=0}^{M} P(m|g)\int s_{i,m,\theta}^{ins}(a)\,d\mu_\theta^{ins} \text{ and} \\
B_{std,\theta}(i,g) &= \pi(i,g)\Big(\int_a\int_N b_\theta^{outs}(a+N)dF_N\,d\mu_\theta^{outs} \\
&+ \sum_{m=0}^{M} P(m|g)\int b_{i,m,\theta}^{ins}(a)\,d\mu_\theta^{ins}\Big)
\end{aligned}
$$

are continuous functions of θ. The key to proving this assertion is to prove that

$$\int_{\underline{\underline{a}}}^{\infty} a \, d\mu_{\theta}^{ins} \to 0 \; (\underline{\underline{a}} \to \infty)$$

locally uniformly in θ: note that the integrand function is unbounded! This and another important continuity property are established in the next Lemma.

However, in order to prove this Lemma, we need a property of the value – functions involved, which seems intuitively clear, but can only be formulated here as a conjecture: suppose, that instead of giving agents choices between becoming an outsider or an insider, we give them zero income, but a sure rate of return of $\bar{R}$ on their assets. We want, that agents would then receive more as a result of their savings next period, given a certain amount of disposable income this period, than in the model.

<u>CONJECTURE A.V.1:</u>

Consider the solution to the following dynamic programming problem

$$v(a) = \max_{c} \left\{ \frac{c^{1-\eta}-1}{1-\eta} + \beta \, v\left(\bar{R}(a-c)\right) \right\}$$

and let $\hat{a}'(a) = \bar{R}(a - \hat{c}(a))$, where $\hat{c}(a)$ is the decision rule to this problem (Note, that $\bar{R}$ is the maximal return, an insider can earn on his portfolio). Then

$$\hat{a}'(a) \geq \sum_{g=0}^{1} \pi(g|i,m)(R(b_{i,m}^{ins}(a)+(1-g)s_{i,m}^{ins}(a))$$

for all i,m, all a and

$$\hat{a}'(y) \geq Rb^{outs}(y)$$

for all y.

<u>**LEMMA A.V.15:**</u>

(i) The fraction of outsiders $\bar{n}_\theta = \int P_{o,\theta}(a)\, d\mu_\theta$ is a continuuous function of θ.

(ii) If conjecture A.V.1 is true, then $\int_{\underline{\underline{a}}}^{\infty} a\, d\mu_\theta^{ins}$ exists and converges locally uniformly in θ to 0 as $\underline{\underline{a}} \to \infty$.

<u>**PROOF:**</u>

(i) Follows from the fact that $P_{o,\theta} \to P_{o,\bar{\theta}}$ in the $d_{c,\mathbb{R}_{++}}$ – metric and from $\mu_\theta \to \mu_{\bar{\theta}}$: separate the integrals into three parts, where we observe that $\int_0^{\bar{a}} d\mu_\theta < \epsilon$ and $\int_{\underline{\underline{a}}}^{\infty} d\mu_\theta < \epsilon$ in a neighbourhood of θ for suitably small $\underline{a}$ and suitably big $\underline{\underline{a}}$.

(ii) Note first, that since $\bar{R} < 1/\beta$, we have $\hat{a}'(a) \leq \gamma a$ with $\gamma < 1$ in Conjecture A.V.1 ($\hat{a}$ can be calculated directly: $\hat{a}'(a) = (R\beta)^{1/\eta}a$).

Consider first the case, that $\bar{\theta}_{m}$ is informative. Denote with a'(a) the expected assets of agents next period, given asset holdings a now. I.e.,

$$a'(a) = \bar{\epsilon}\, 0 + (1-\bar{\epsilon})P_o(a) \int Rb^{outs}(a_o(a)+N)\, dF_N +$$
$$(1-\bar{\epsilon})P_i(a) \sum_{i,m} \pi(i,m) \sum_g \pi(g\,|\,i,m)$$
$$R(b^{in\,s}_{i,m}(a_i(a)) + (1-g)s^{in\,s}_{i,m}(a_i(a))).$$

Note, that by our conjecture,

$$a'(a) \le (1-\bar{\epsilon})\, P_o(a) \int \gamma(a_o(a)+N)\, dF_N +$$
$$(1-\bar{\epsilon})P_i(a) \sum_{i,m} \pi(i,m)\ \gamma a_i(a)$$
$$\le \gamma\, a + \gamma.$$

Choose $\bar{a}$ high enough, so that everybody with $a \ge \bar{a}$ becomes an insider for every θ in a neighbourhood of $\bar{\theta}$ and so that

$$\gamma a + \gamma \le \nu a$$

for some $\nu < 1$, all $a \ge \bar{a}$. Given $\epsilon > 0$, find $\underline{\underline{a}}$, so that

$$Q_\theta(a,[\underline{\underline{a}};\infty)) < (1-\nu)\epsilon/(\gamma\bar{a} + \gamma)$$

and so that $a_i(a) < \underline{\underline{a}}$ for all $a \le \bar{a}$ in the same neighbourhood w.l.o.g.. Note then that we have

$$\int_{\underline{\underline{a}}} a \, d\mu_\theta^{ins} = \int_{\underline{\underline{a}}} a \, d\mu_\theta.$$

It is therefore enough to show, that the latter integral is smaller than ϵ in the neighbourhood of $\bar{\theta}$. For that, note that Theorem 11.12 in Stokey – Lucas, with Prescott (1989), which we used to prove existence of a stationary distribution for Theorem A.V.11, also implies, that the iterations μ_θ^n of any initial measure μ^0 with the transition probabilities Q_θ converge to μ_θ in the norm – topology on $\Lambda(\mathbb{R}_+)$, induced by the supremums – norm on $C_0(\mathbb{R}_+)$ (again, using the a 1–1 transformation of $\bar{\mathbb{R}}_+$ onto [0;1]). Thus, choose some initial measure μ_θ^0, which places zero mass on $a \geq \underline{\underline{a}}$. It follows from the calculations below and Levis theorem, that $\int a \, d\mu_\theta^n < \infty$: we therefore proceed, using this fact. We find by definition of μ_θ^n and by the properties of $\underline{\underline{a}}$, $\bar{a}$

$$\begin{aligned}
\int_{\underline{\underline{a}}} a \, d\mu_\theta^n &= \int_{a'(a) \geq \underline{\underline{a}}} a'(a) \, d\mu_\theta^{n-1} \\
&= \int_{\substack{a'(a) \geq \underline{\underline{a}} \\ a \geq \bar{a}}} a'(a) \, d\mu_\theta^{n-1} + \int_{\substack{a'(a) \geq \underline{\underline{a}} \\ a < \bar{a}}} a'(a) \, d\mu_\theta^{n-1} \\
&\leq \int_{\underline{\underline{a}}} \nu a \, d\mu_\theta^{n-1} + (\gamma\bar{a}+\gamma)\,(1-\nu)\epsilon/(\gamma\bar{a}+\gamma).
\end{aligned}$$

Thus, we can establish via induction that

$$\int_{\underline{\underline{a}}} a \, d\mu_\theta^n \leq (1-\nu)\epsilon \sum_{k=0}^{n-1} \nu^k \leq \epsilon.$$

It now follows quickly from Levis theorem that

$$\int_{\underline{\underline{a}}} a \, d\mu_\theta \leq \epsilon,$$

since convergence of the iterations imply that

$$\int_{\underline{\underline{a}}} \min(a;C) \, d\mu_\theta < \epsilon$$

for all constants C. •

THEOREM A.V.12:

If conjecture A.V.1 is true, then $D_{std,\theta}(i,g)$ and $B_{std,\theta}(i,g)$ are continuous functions of θ.

PROOF:

We only show the result for $D_{std,\theta}(i,g)$, the result for $B_{std,\theta}(i,g)$ follows similarly. Note, that by Theorem A.V.6, the functions $s_{i,m,\theta}^{in\,s}$ are Lipschitz continuous. An analysis of the proof to Lemma A.V.6 reveals, that the Lipschitz constant can be chosen to be

$$L = \frac{1}{\min\{\pi(g=0|i), \pi(g=1|i)\}}.$$

Note, that $s^{ins}_{i,m,\theta}(0) = 0$ and hence $|s^{ins}_{i,m,\theta}(a)| \leq La$. Thus, given $e > 0$ and $\bar{\theta}$, choose a neighbourhood of $\bar{\theta}$ and $\underline{\underline{a}}$ high enough, so that

$$\int_{\underline{\underline{a}}}^{\infty} a \, d\mu^{ins}_{\theta} \leq \epsilon / 6L$$

in that neighbourhood, where $\underline{\underline{a}}$ is a continuity point of $\mu^{ins}_{\bar{\theta}}$. Furthermore, apply Theorem A.V.8 to shrink that neighbourhood, so that for all θ within, we have

$$|s^{ins}_{i,m,\theta}(a) - s^{ins}_{i,m,\bar{\theta}}(a)| < \epsilon/3,$$

all $a \leq \underline{\underline{a}}$ and so that

$$|\int_0^{\underline{\underline{a}}} s^{ins}_{i,m,\bar{\theta}}(a) \, d\mu^{ins}_{\theta} - \int_0^{\underline{\underline{a}}} s^{ins}_{i,m,\bar{\theta}}(a) \, d\mu^{ins}_{\bar{\theta}}| < \epsilon/3.$$

It follows, that for θ in that neighbourhood,

$$|\int s^{ins}_{i,m,\theta}(a) \, d\mu^{ins}_{\theta} - \int s^{ins}_{i,m,\bar{\theta}}(a) \, d\mu^{ins}_{\bar{\theta}}| < \epsilon,$$

which we had to show. •

Recall now, that, given a wage w and prices $q_1(l)$, the actual insider demand for stocks can be calculated according to the proof of Theorem A.V.5

and the wage – standardization at the beginning of the analysis of the decision problem of the agent as

$$D_\theta(l,i,g) = w\ R\ D^{(i,g)}_{std,\theta} \,/\, (q_1(l) - q_1(l+1)),$$

where $D_{std,\theta}$ does not depend on the wage – part of θ because of our wage normalization for the agents decision problem.

We now find as a consequence of the stock market clearing condition the announced consistency condition,

THEOREM A.V.13: (The Consistency Condition)

At θ, a necessary condition on the probabilities $\theta_{\mathcal{g}}$ for the equilibrium to exist is that for all categories (l,i,g), we have

$$P(g|l,i) = (1 + \chi(l,i)\pi(1-g|i)(\ D(0,i,g) - D(0,i,1-g)\ ))\ \pi(g|i),$$

where

$$\chi(l,i) = \frac{\pi(l)\ \pi(i)}{\bar{k}\ F_k(l)\ P(i|l)\ f(1,x(i,l))}\,\frac{q_1(0) - q_1(l)}{q_1(l) - q_1(l+1)}.$$

PROOF

Stock market clearing and Theorem A.V.6 imply

$$F_k(l)\ P(i,g|l)\ f(1,x(l,i))\ \bar{k}$$

$$=$$

$$\varphi(l,i)\ \pi(l)\ \pi(i)\ \pi(g|i)$$

$$+$$

$$\pi(l)\ \pi(i)\ \pi(g|i)\ \frac{q_1(0)\ -\ q_1(l)}{q_1(l)\ -\ q_1(l+1)}\ D(0,i,g).$$

Observing, that $\varphi(l,i)$ does not depend on g yields the conclusion. •

The key for proving the existence of an equilibrium is now to solve for consistent fundamental probabilities P and $\pi(l)$ "backwards" in the spirit of Sims (1984,1990) in order to satisfy the consistency condition. We aim at probabilities $\pi(l)$, $P(i|l)$ and $P(g|l,i) = P(g|i)$, which satisfy

$$\pi(l) = \bar{\chi}\,\frac{q_1(l)-q_1(l+1)}{q_1(0)-q_1(l)}\,\frac{F_k(l)\ \bar{k}}{\sum_{\iota=0}^{I}\ \pi(\iota)\ /\ f(1,x(l,\iota))},$$

$$P(i|l) = (\ \pi(i)\ /\ f(1,x(l,i))\ )\ /\ (\ \sum_{\iota=0}^{I}\ \pi(\iota)\ /\ f(1,x(l,\iota))\ )$$

and

$$P(g|l,i) = (1 + \bar{\chi}\ \pi(1-g|i)(\ D(0,i,g)-D(0,i,1-g)\ ))\ \pi(g|i),$$

where $\overline{\chi}$ is chosen such that $\sum_{l=0}^{\infty} \pi(l) = 1$. There is quite a bit of freedom in choosing equations for the backsolving of the probabilities. The particular choices made here are justified in the numerical appendix.

To satisfy these equations in equilibrium, we will solve directly for probabilities $\pi(l)$ and $P(i|l)$ via a variation of Theorem A.V.4 under somewhat altered assumptions. All that then remains are the probabilities $P(g|i)$ and the wage w, which are part of our parameter vector θ: we will solve for these probabilities and the wage jointly with the interest rate R in the final theorem, which employs a fixed point argument.

We now adapt Theorem A.V.2 by simultaneously solving for

$$P(i|l) = \frac{\pi(i)}{f(1,x(l,i))} \Big/ \sum_{\iota=0}^{I} \frac{\pi(\iota)}{f(1,x(l,\iota))},$$

which we simply substitute out in the formulas and recheck the conditions for contraction. Note, that starting with the following theorem, we need the wage – part of our parameter θ.

<u>**Theorem A.V.14:**</u>

Given $\theta_{\mathcal{m}}, \theta_{\mathcal{g}}$ and R, there exists a wage $\bar{w}(\theta_{\mathcal{m}}, \theta_{\mathcal{g}}, R)$, so that for every wage $w > \bar{w}(\theta_{\mathcal{m}}, \theta_{\mathcal{g}}, R)$ and $\theta = (\theta_{\mathcal{m}}, \theta_{\mathcal{g}}, R, w)$, there are strictly positive prices $q_{1,\theta}$, $q_{2,\theta}$ and $q_{3,\theta}$ as well as an investment rule x_θ

and probabilities $P_\theta(i,l)$, solving the maximization problem of the investment firm and the mutual fund as well as the equation for $P(i|l)$ above.

PROOF:

The proof is similar to the proof of Theorem A.V.2 except that we proceed directly from q_3 to q_1 and substitute out $P(i|l)$ in the formulas. To that end, fix θ_m, θ_g and R. Keep the definition of $\tilde{q}(w)$, but define $\bar{\bar{q}}$ via

$$\bar{\bar{q}} = \sup\{ \tilde{q} \mid |\sum \delta_i(q^0,..,q^I)| < R \text{ for all } q^0,..,q^I \in [0;\tilde{q}] \},$$

where

$$\delta_i(q^0,..,q^I) = P_i(q^0,..,q^I)\,(\,h'(q^i) + \frac{\frac{d}{dq}f(1,X(q))\,|_{q=q^i}}{f(1,X(q^i))}\,(\sigma(q^0,..,q^I)(\sum_{\iota=0}^{I} h(q^\iota)) - h(q^i)))$$

and where

$$P_i(q^0,..,q^I) = \sigma(q^0,..,q^I)\,\frac{\pi(i)}{f(1,X(q^i))} \text{ and}$$

$$\sigma(q^0,..,q^I) = \sum_{\iota=0}^{I} \frac{\pi(\iota)}{f(1,X(q^\iota))},$$

that is, $P_i(q^0,..,q^I) = P(i|l)$, if $q^i = q_3(l,i)$. Observe that $\bar{\bar{q}} > 0$, since by assumption A.V.0 (or by the assumption, that $\kappa_1 > 1/2$ in our CES capital production function), we have $\frac{\frac{d}{dq}f(1,X(q))}{f(1,X(q))} \to 0$ as $q \to 0$ (as well as $h(q) \to 0$, $f(1,X(0)) > 0$ and $\sigma(q^0,..,q^I) \to 1/(f(1,X(0)))$ as $q^i \to 0$, $i = 1,..,I$). Find $\bar{w}$ and q^* as before. Skip the operator Q_2 to define directly

$$Q_1 : D_1 \to \ell^\infty,$$

$$(Q_1(q))(l) = d(l) + \sum_{i=0}^{I} P_i(q(l,0),..,q(l,I))\, h(q(l,i)),$$

$$D_1 = \{q \in \ell^\infty_{I+1} \mid 0 \le q(l,i) \le q^* \}.$$

Define Q_3 as before and $Q : D_3 \to D_3$ via $Q = Q_1 \circ Q_3$. Some messy algebra reveals, that

$$\frac{\partial}{\partial q(l,i)}(Qq)(l) = \delta_i(q(l,0),..,q(l,I))$$

and as before, we conclude, that Q is a contraction mapping. The rest of the arguments go through as well. •

<u>**Theorem A.V.15:**</u>

Let $\bar{\theta}$ be such that $\bar{w} > \bar{w}(\bar{\theta}_{\mathcal{m}}, \bar{\theta}_{\mathcal{J}}, \bar{R})$. Then the rules $q_{1,\theta}$, $q_{2,\theta}$, $q_{3,\theta}$ and P_θ are continuous amd $w > \bar{w}(\theta_{\mathcal{m}}, \theta_{\mathcal{J}}, R)$ in an $\mathbb{R}^{2M + 2I + 2}$ – open set of θ's around $\bar{\theta}$.

<u>**Proof:**</u>

This follows from the usual continuity argument for contraction mappings: given $\epsilon > 0$, given $\bar{\theta}$, we can choose a $\mathbb{R}^{2M + 2I + 2}$ – open neighbourhood of $\bar{\theta}$, where we still have $w > \bar{w}(\theta_{\mathcal{m}}, \theta_{\mathcal{g}}, R)$ and where we can choose the same q^* for all θ's in the proof above. Since Q_1 and Q_3 only use continuous functions of θ, we can shrink that neighbourhood even further, so that within it,

$$\| Q_\theta(q) - Q_{\bar{\theta}}(q) \| < \epsilon(1-\nu),$$

where ν is the contraction factor to $\bar{\theta}$. But then observe that with q_θ denoting the fixed point to Q_θ, we get

$$\| Q^n_{\bar{\theta}}(q_\theta) - q_\theta \| \leq \| Q_\theta(q_\theta) - q_\theta \| \sum_{k=0}^{n-1} \nu^k$$
$$= \frac{1-\nu^n}{1-\nu} \| Q_\theta(q_\theta) - \theta_\theta(q_\theta) \| \leq \epsilon$$

and thus $\| q_{\bar{\theta}} - q_\theta \| \leq \epsilon$. This delivers the continuity for q_1, the continuity for the other pieces follows from the continuity of the operators which calculate them from q_1. •

Contrary to Lemma A.V.2, we may no longer have monotonicity in the wage w, since the relevant derivatives δ_i may or may not be negative. Note, however, that part (ii) and part (iii) of that Theorem are still valid for fixed $(\theta_{\mathcal{m}}, \theta_{\mathcal{g}}, R)$: the proof for part (ii) still works verbatim and for (iii), we can just take the solution to $P(i|l)$ we get and apply Lemma A.V.2, using it. In particular, note that for fixed θ,

$$1 / \sum_{i=0}^{I} \frac{\pi(i)}{f(l,x_\theta(l,i))}$$

is strictly monotonously decreasing, since $x_\theta(l+1,i) < x(l,\iota)$ for all l, all i, all ι. Thus, in order to prove an analogue of theorem A.V.4, let

$\Theta \subset \Theta_{max}$, where

$$\Theta_{max} = \{ \theta \mid \sum_{i=0}^{I} P(g=0|i)\, \pi(i) \Big/ \sum_{i=0}^{I} \frac{\pi(i)}{f(l,x_\theta(l,i))} < \zeta \text{ for } l\geq 1,\ w > \bar{w}(\theta_{\mathscr{m}}, \theta_{\mathscr{g}}, R),\ 1 < R < \zeta^{\eta}/\beta,\ \theta_{\mathscr{m}} \text{ is either informative or uninformative, } \theta_{\mathscr{m}} \text{ and } \theta_{\mathscr{g}} \text{ are are probabilities and not trivial}\}.$$

In light of Theorem A.V.15, we assume

ASSUMPTION A.V.5:

- Θ is an open, nonempty set in $\mathbb{R}^{2M+2I+2}$,
- for any $\bar{\theta}_{\mathscr{m}}$, $\Theta_{\mathscr{m}} = \{\theta\in\Theta \mid \theta_{\mathscr{m}} = \bar{\theta}_{\mathscr{m}}\}$ is a convex set,
- if $\bar{\theta}_{\mathscr{m}}$ is uninformative, then $\Theta_{\mathscr{m}} = \{\theta\in\Theta_{max} \mid \theta_{\mathscr{m}} = \bar{\theta}_{\mathscr{m}}\}$.

For the following version of Theorem A.V.4, in which we backsolve for

$P(i|l)$, we leave out the level $-0-$ stationarity condition

$$\sum_{i=0}^{I} P(g=0|i)\,\pi(i) \Big/ \sum_{i=0}^{I} \frac{\pi(i)}{f(1,x_\theta(0,i))} = \zeta,$$

which we will enforce instead in the final fixed point theorem.

<u>**Theorem A.V.16:**</u>

(i) For any $\theta \in \Theta$ and with the rules given by Theorem A.V.15, there is a capital distribution $F_{k,\theta}$ and aggregate capital $\bar{k}_\theta$, so that $F_{k,\theta}$ satisfies the stationary condition for all levels $l \geq 1$ (i.e. except possibly for level $l = 0$) and so that the labor market clears.

(ii) $F_{k,\theta}(l)$, $l=0,1,..$, $\bar{k}_\theta$ and χ_θ are continuous in θ, where

$$\chi_\theta = 1/\sum_{l=0}^{\infty} \pi_\theta(l) \text{ and}$$

$$\pi_\theta(l) = \frac{q_{1,\theta}(l)-q_{1,\theta}(l+1)}{q_{1,\theta}(0)-q_{1,\theta}(1)} \frac{F_{k,\theta}(l)\bar{k}_\theta}{\sum_{i=0}^{I} \frac{\pi(i)}{f(1,x_\theta(l,i))}}.$$

<u>**Proof:**</u>

As in the proof to theorem A.V.4, let

$$\psi_\theta(l) = \frac{\sum_{i=0}^{I} P(g=1|i)\pi(i) \Big/ \sum_{i=0}^{I} \frac{\pi(i)}{f(1,x_\theta(l-1,i)}}{\zeta - \sum_{i=0}^{I} P(g=0|i)\pi(i) \Big/ \sum_{i=0}^{I} \frac{\pi(i)}{f(1,x_\theta(l,i)}},$$

where we substituted out (or "backsolved for")

$$P_\theta(i|l) = \frac{\pi(i)}{f(1,x_\theta(l,i))} \Big/ \sum_{\iota=0}^{I} \frac{\pi(\iota)}{f(1,x_\theta(l,\iota))}.$$

Note that by our definition of Θ, $\psi(l)$ is well defined and positive. Note furthermore, that x_θ is ℓ_∞ – continuous in θ and that $1 \Big/ \sum_{i=0}^{I} \frac{\pi(i)}{f(1,x_\theta(l,i))}$ is monotonously decreasing in l with $x_\theta(l,i) \to 0$ ($l \to \infty$). We can thus find $l_{\bar\theta}$, so that in a neighbourhood of $\bar\theta$ in Θ, we have $\psi_\theta(l) \leq \nu$, where $\nu = (1+\zeta)/(2\zeta) < 1$. As in the proof to Theorem A.V.4, define $\tilde F_{k,\theta}(0) = 1$ and $\tilde F_{k,\theta}(l) = \psi_\theta(l)\, \tilde F_{k,\theta}(l-1)$ for $l \geq 1$. Since

$$\tilde F_{k,\theta}(l+l_{\bar\theta}) \leq \nu^l\, \tilde F_{k,\theta}(l_{\bar\theta}),\ l = 0,1,..,$$

it is now easy to see that $\tilde F_{k,\theta}(l)$ as well as

$$\bar{\tilde F}_{k,\theta} = \sum_{l=0}^{\infty} \tilde F_{k,\theta}(l)$$

are continuous functions in $\mathbb{R}_+$ of θ around $\bar\theta$, since $\tilde F_{k,\theta}(l)$ converges to zero as $l \to \infty$ at the locally uniform geometric rate ν. Thus,

$$F_{k,\theta}(l) = \tilde{F}_{k,\theta}(l) \ / \ \bar{\tilde{F}}_{k,\theta}$$

has to be continuous in θ as well and it is easily checked, that the same is true for

$$\bar{k}_\theta = \bar{n}_\theta \ / \sum_{l=0}^{\infty} F_{k,\theta}(l) \ n_\theta(l)$$

(recall, that $\bar{n}_\theta$ is continuous in θ by Lemma A.V.15, part (ii)) and for χ_θ, because of the locally uniform geometric convergence of $F_{k,\theta}(l)$ to zero as $l \to \infty$. •

Fixing $\theta_{\mathcal{M}}$, we have to solve for the following remaining I + 2 unknowns of θ in equilibrium:

- $P(g=0|i)$, $i \in \{0,..,I\}$,
- R, the equilibrium interest rate
- w, the wage.

To solve for them, we need to satisfy the following I+2 equations

(E.1) The Consistency Conditions for i=1,..,I:

$$P(g=0|i) = (1+\chi_\theta \pi(g=1|i)(D_\theta(0,i,0)-D_\theta(0,i,1))\ \pi(g=0|i),$$

(E.2) The Market Clearing Condition

$$A_\theta^{sup} = A_\theta^{dem},$$

where the supply of assets A_θ^{sup} is given by

$$A_\theta^{sup} = \bar{k}_\theta \sum_{l=0,i=0,g=0}^{\infty\quad I\quad 1} (\ q_{3,\theta}(l,i)P(g|i)\pi(i)$$
$$F_{k,\theta}(l)\ /\ \sum_{\iota=0}^{I} \frac{\pi(\iota)}{f(1,x_\theta(l,\iota))}\ )$$

and the demand for assets A_θ^{dem} is given by

$$A_\theta^{dem} = w \sum_{i=0,g=0}^{I\quad 1} \pi(i,g)(B_{std,\theta}(i,g)+\pi(g=0|i)D_{std,\theta}(i,g))$$

(E.3) The stationarity condition for level–0 capital

$$\sum_{i=0}^{I} P(g=0|i)\ \pi(i)\ /\ \sum_{i=0}^{I} \frac{\pi(i)}{f(1,x_\theta(0,i))} = \zeta$$

It is important to note, that we have an equilibrium, once these I+2 equations (E.1), (E.2) and (E.3) are satisfied. The consistency condition

(E.1) enables us to solve for mutual fund share holdings $\varphi(l,i)$ and at the same time guarantees the stock market clearing conditions by construction. The asset market clearing condition is equivalent to the mutual fund market clearing condition, conce the stock market clearing condition holds: this follows from substituting out $\varphi(l,i)$ in the mutual fund market clearing condition of the equilibrium definition in part IV of the main text via the stock market clearing condition. All other conditions are already satisfied by construction except for the consumption goods market clearing condition. However, this market clearing condition can be shown to hold as well (after some messy algebra) from the other equations (including (E.1), (E.2) and (E.3)): this is just Walras' law.

Thus, all that remains to be shown is that the I+2 equations (E.1) through (E.3) have a solution in the I+2 unknowns listed above. We will show this in the last theorem under certain assumptions, using a fixed point argument.

To that end, we first consider the case of uninformative θ's. In that case, (E.1) collapses to $P(i|l) = \pi(i)$, i.e. we only have to solve for the interest rate R and the wage w, using the market clearing condition in the assets and the stationarity condition on capital: everything can be parameterized just by R and w.

We have to impose the following variation of Assumption A.V.2.

ASSUMPTION A.V.6:

(i) For some interest rate R_{low}, $1 \leq R_{low} < \zeta^{\eta}/\beta$, and all $R \in [R_{low}, \zeta^{\eta}/\beta]$, there is a wage w(R), which solves (E.3) for $P(g|i) = \pi(g|i)$. Furthermore, this function w(R) can be chosen to be continuous on $[R_{low}, \zeta^{\eta}/\beta]$,

(ii) There is some $\Delta^{w} > 0$, so that for fixed $P(g|i) = \pi(g|i)$, fixed $\theta_{\mathcal{m}}$ and any fixed $R \in [R_{low}, \zeta^{\eta}/\beta]$, the function

$$g(w) = \zeta \sum_{i=0}^{I} \frac{\pi(i)}{f(1,x_{\theta}(0,i))}$$

is strictly increasing in the wage $w \in [w(R)-\Delta^{w}, w(R)+\Delta^{w}]$, where $\theta = (\theta_{\mathcal{m}}, \theta_{\mathcal{g}}, R, w)$, $\theta_{\mathcal{m}}$ uninformative and $\theta_{\mathcal{g}}$ defined via $P(g|i) = \pi(g|i)$.

Again, the restrictiveness of this assumption is similar to the restrictiveness of assumption A.V.2. Note, that $\theta_{\mathcal{m}}$ is irrelevant on the production side, once P(g|i) as well as the interest rate R and the wage are given. Also, the continuity of w(R) is not a severe restriction: consider the benchmark case, that all $\pi(g|i)$'s are the same. In that case, $x_{\theta}(i,1)$ does not depend on i (or, equivalently $P_{\theta}(i|1) = \pi(i)$) and Theorem A.V.2, Lemma A.V.2 and Theorem A.V.4 apply directly. With a generalization of Lemma A.V.2, monotonicity in the interest rate R can be established. Finally, analyzing the stationarity condition (E.3), we can see, that we can choose w(R) monotonously. Continuity then follows, because all functions involved are continuous. If we move away from the benchmark assumption, that all $\pi(g|i)$s are the same, continuity of w(R) should continue to hold at least as long as the $\pi(g|i)$'s stay sufficiently close to each other and at least for some interval $[R_{low}, \zeta^{\eta}/\beta]$, as considered in the assumption above, by checking the way w(R) changes around $R = \zeta^{\eta}/\beta$ as we change the $\pi(g|i)$'s. The same logic applies to (ii) and (iii): for the case, where all $\pi(g|i)$'s are the same, the function g is already a strictly increasing function of w and A_{θ}^{sup} is a decreasing function of R.

LEMMA A.V.16: (Properties of specific value function operators)

(i) Let $u: \mathbb{R}_{+} \to \mathbb{R}$ be differentiable and finite on $\mathbb{R}_{++}$ with u' continuous, strictly convex, strictly positive, strictly decreasing and $\lim_{y \to 0} u'(y) = \infty$. Let $w: \mathbb{R}_{+} \to \mathbb{R}$ be differentiable, where w' is continuous, convex, strictly positive, strictly decreasing. Let $R > 0$, $\beta > 0$. Then

$$Tw(y) = \max \{ u(c) + \beta\, w(R(y-c)) \mid 0 \le c \le y \}$$

is differentiable and (Tw)' is continuous, convex, strictly positive, strictly decreasing. If additionally $\lim_{y \to 0} w'(y) = \infty$, then (Tw)' is strictly convex with $\lim_{y \to 0} (Tw)'(y) = \infty$.

(ii) If additionally $u(c) = \dfrac{c^{1-\eta} - 1}{1 - \eta}$ in (i) and $\eta > 0$, then the decision rule c(y) is strictly concave.

PROOF:

(i) We first proceed with the proof for the second case, that $\lim_{y \to 0} w'(y) = \infty$. The proof the for differentiability of Tw, the applicability of first order conditions and the proof for the fact, that $(Tw)'$ is strictly increasing with $\lim_{y \to 0} (Tw)'(y) = \infty$, are standard and can be omitted. It remains to show strict convexity of $(Tw)'$. We have for the decision rule $c(y)$, that

$$(Tw)'(y) = u'(c(y)) = \beta R\, w'(R(y-c(y))).$$

Now, let $0 < y_0 < y_1$ and $\lambda \in (0;1)$. Let $y_\lambda = \lambda y_0 + (1-\lambda) y_1$ and $c_\lambda = \lambda c(y_0) + (1-\lambda) c(y_1)$. Let $\tau_\lambda = \lambda (Tw)'(y_0) + (1-\lambda)(Tw)'(y_1)$, we need to show that $\tau_\lambda > (Tw)'(y_\lambda)$. By our assumption about concavity, we get

$$\tau_\lambda > u'(c_\lambda) \text{ and}$$
$$\tau_\lambda \geq \beta R\, w'(R(y-c_\lambda)).$$

There are now two cases: if $c(y_\lambda) \geq c_\lambda$, then

$$(Tw)'(y_\lambda) = u'(c(y_\lambda)) \leq u'(c_\lambda) < \tau_\lambda$$

and if $c(y_\lambda) < c_\lambda$, then

$$(Tw)'(y_\lambda) = \beta Rw'(R(y-c(y_\lambda))) < \beta Rw'(R(y-c_\lambda)) < \tau_\lambda.$$

The general case can now be derived by properly taking care of the boundary cases, where $c(y) = y$.

(ii) Note, that $c(y) = ((Tw)')^{-1/\eta}(y)$, where (Tw)' is strictly convex according to (i). The claim now follows easily from the general observation, that for any strictly convex, strictly positive function f and any $\alpha < 0$, the function f^α is strictly concave: if $x < y$, $\lambda \in (0;1)$, $z = \lambda x+(1-\lambda)y$, then

$$\lambda f(x) + (1-\lambda)f(y) > f(z), \text{ hence}$$
$$(\lambda f(x) + (1-\lambda)f(y))^\alpha < f^\alpha(z) \text{ and}$$
$$\lambda f^\alpha(x) + (1-\lambda)f^\alpha(y)) \leq (\lambda f(x) + (1-\lambda)f(y))^\alpha$$

by concavity of $x \mapsto x^\alpha$. •

Lemma A.V.17:

Let messages θ_m be uninformative and $R \in [R_{low}, \zeta^\eta/\beta]$.

(i) Consider the benchmark case, that agents receive a sure wage of 1 (or, equivalently, that $F_{N,bench}$ is the distribution, which assigns unit mass at $N = 1$) and that there is no perturbation u_e. Let μ be some asset distribution on $\mathbb{R}_+$ with $0 < \int a \, d\mu < \infty$. Then there is some scaling factor σ so that the economy is in equilibrium at $R = \zeta^\eta/\beta$, $P(g|i) = \pi(g|i)$ and with asset distribution $\mu_\sigma(A) = \mu(\sigma A)$.

(ii) Let $c_{bench}(y;R)$, $b_{bench}(y;R)$ be the decision rules for part (i), if $y = a + 1$ is total disposable wealth, and $C(a;R)$, $B(a;R)$ the integrals over F_N of the decision rules $b_\theta^{outs}(a+N)$ for our problem (at the assumed asset distribution F_N, satisfying assumption A.V.3, and with the perturbation u_e). Then $B(a;R) > b_{bench}(a+1;R)$ and $B(\cdot;R)$ is strictly convex.

(iii) there is a solution $\theta = (\theta_{\mathscr{m}}, \theta_{\mathscr{g}}, R, w)$ with $R < \zeta^\eta/\beta$, which solves (E.1), (E.2) and (E.3), i.e. delivers an equilibrium for the case of uninformative messages.

PROOF:

Note first, that since messages are uninformative, there will not be any insiders and both parts $\theta_{\mathscr{m}}, \theta_{\mathscr{g}}$ are not needed any further in the analysis of the decision problem of agents. Fix $\theta_{\mathscr{g}}$ at $P(g|i) = \pi(g|i)$ as implied by (E.1) throughout.

(i) Transform the decision problem back to $\tilde{\zeta} = 1$, $\tilde{R} = R/\zeta$, $\tilde{\beta} = \beta\zeta^{1-\eta}$ and $\tilde{w} = 1$ as outlined at the beginning of the analysis of the decision problem of the agent. The solution for the value function is now a standard problem (see e.g. Stokey – Lucas, with Prescott (1989), chapter 5.17 for $\tilde{R} < 1/\tilde{\beta}$) and can also be obtained from a straight – forward generalization of Theorem A.V.6 above for any $\tilde{R} > 0$ (note, that the assumption on the interest rate, that $\tilde{R} < 1/\tilde{\beta}$ was not needed there). Let $c(a;R)$ denote the decision rule for this problem and note, that we get from the first – order conditions and the envelope theorem at $\tilde{R} = 1/\tilde{\beta}$ (which corresponds to $R = \zeta^{\eta}/\beta$ in the untransformed problem)

$$c(a)^{-\eta} = \tilde{R}\tilde{\beta}\, c(\ \tilde{R}(a-c(a)+1)\ )^{-\eta},$$

which is satisfied, if

$$a = \tilde{R}(a - c(a) + 1)$$

or

$$c(a) = \frac{\tilde{R} - 1}{\tilde{R}} a + 1 > 0.$$

I.e. the agent exactly eats the interest payments to his assets as well as his wage and saves $a/\tilde{R}$. His savings become $a = \tilde{R}(a/\tilde{R})$ next period, i.e. his asset holdings remain constant. Thus, any distribution over assets is stationary. Let $R = \tilde{R}\zeta = \zeta^{\eta}/\beta$ and solve for the equilibrium wage w(R) according to assumption A.V.6. Calculate aggregate asset demands (from the retransformed decision rules, which are integrated with μ). Note, that (E.1) and (E.3) are satisfied by construction. Choosing the scale factor σ right, it is now easy to see, that (E.2) can be satisfied as well, establishing the equilibrium.

(ii) Again, transform the decision problem back to $\tilde{\zeta} = 1$, $\tilde{R} = R/\zeta$, $\tilde{\beta} = \beta\zeta^{1-\eta}$ and $\tilde{w} = 1$. Fix the interest rate R. To make the notation clearer, we leave away the superindex "outs", we subindex solutions to the benchmark problem of (i) with bench, and subindex solutions to the model problem with θ, which is the parameter vector containing R, among other things. First note for the value function v_{bench} of the benchmark problem under (i), that starting from a bounded value function v_0, which is differentiable and where v_0' is strictly decreasing and convex, and iterating on $v_n(a) = (Tv_{n-1})(a+1)$, where T is the operator from the previous lemma, we find from the usual contraction mapping argument and with Theorem A.V.9, that v_{bench} is differentiable, concave and that v_{bench}' is convex. We now use $v_{\theta,0} = v_{bench}$ as a starting point to find v_θ, the value function for our problem via

$$v_{\theta,n}(a) = S(Tv_{\theta,n-1})(a),$$

where T is the operator from the previous Lemma and where

$$S(w)(a) = 2u_e(a) + w(0) + \int w(a+N)\, dF_N.$$

(note, that we included our perturbation u_e).

Note that since $v_{\theta,0}$ is differentiable, strictly concave, strictly increasing, where $v_{\theta,0}'$ is convex, the same now holds for

$v_{\theta,1}$, except that we also get $\lim_{a\to\infty} v'_{\theta,1}(a) = \infty$ through our operator S. Reapplying the previous lemma and proceeding through the standard contraction mapping argument delivers that $v_{\theta,n}$ for $n \geq 1$ as well as v_θ itself are differentiable, strictly concave, strictly increasing and in particular, that $v'_{\theta,n}$ and v'_θ as well as $(Tv_{\theta,n})'$ and $(Tv_\theta)'$ are strictly convex.

We now show by induction, that

$$v'_{\theta,n}(a) > v'_{bench}(a)$$

for all $n > 1$. Note, that we get by Jensens inequality and because F_N has a density

$$\begin{aligned} v'_{\theta,n}(a) &= \int (Tv_{\theta,n-1})'(a+N)\, dF_N \\ &> (Tv_{\theta,n-1})'(a+1). \end{aligned}$$

Since $v'_{\theta,n-1}(a) \geq v'_{bench}(a)$ (either via induction hypothesis or trivially for $n = 1$), we must have $c_{\theta,n-1}(y) \leq c_{bench}(y)$, where these are the decision rules derived from the operator T, when applied to $v_{\theta,n-1}$ or v_{bench} respectively. Since $(Tw)'(y) = c(y)^{-\eta}$, it follows that

$$(Tv_{\theta,n-1})'(a+1) \geq (Tv_{bench})'(a+1) = v'_{bench}(a),$$

establishing our induction hypothesis.

Via a limit argument (use e.g. Theorem A.V.9), it now follows that

$$v'_\theta(a) > v'_{bench}(a).$$

Applying the previous Lemma to v_θ, we find that the decision rule $b_\theta(y) = y - c_y(y)$, delivered by the operator T applied to v_θ, is strictly convex and that furthermore

$$b_\theta(y) > b_{bench}(y) = y - c_{bench}(y).$$

It follows, that

$$B(a;R) = \int b_\theta(a+N)\, dF_N > b_\theta(a+1) = b_{bench}(a+1;R).$$

This finishes the proof. •

In light of the previous lemma, it is clear, that $R = \zeta^\eta/\beta$ cannot be an equilibrium interest rate in our model. For if agents own assets a at the beginning of the period, they will save more than in the benchmark model and therefore have more than a at the beginning of the next period. In other words, the only stationary asset distribution would have $\int a\, d\mu = \infty$ and demands for assets must exceed supply.

On the other hand, observe that for the benchmark model with any interest rate $R < \zeta^{\eta}/\beta$, agents will eventually eat their "cake" completely (see e.g. Stokey – Lucas, with Prescott (1989), chapter 5.17) and the only stationary asset distribution would have $\int a \, d\mu = 0$. Through the distribution F_N, this is avoided in the model, since heuristically, agents would never risk eating all they have. Thus, the following assumption, which is essentially an assumption on F_N is reasonable, although it is admittedly a complex assumption and not an assumption on the fundamentals underlying the economy.

<u>ASSUMPTION A.V.7:</u>

F_N is such that for the case of uninformative messages, there are interest rates $\bar{R}_{min}, \bar{R}_{max} \in [R_{low}, \zeta^{\eta}/\beta)$, $\bar{R}_{min} < \bar{R}_{max}$, so that

$$A^{dem}_{\bar{\theta}_{min}}(\bar{R}_{min}) < A^{sup}_{\bar{\theta}_{min}}(\bar{R}_{min})$$

and

$$A^{dem}_{\bar{\theta}_{max}}(\bar{R}_{max}) > A^{sup}_{\bar{\theta}_{max}}(\bar{R}_{max})$$

where $\bar{\theta}_{min} = (\theta_{\mathcal{m}}, \theta_{\mathcal{g}}, \bar{R}_{min}, w(\bar{R}_{min}))$, $\bar{\theta}_{max} = (\theta_{\mathcal{m}}, \theta_{\mathcal{g}}, \bar{R}_{max}, w(\bar{R}_{max}))$ with $\theta_{\mathcal{m}}$ uninformative, $\theta_{\mathcal{g}}$ defined via $P(g|i) = \pi(g|i)$.

Judging from the numerical experiments, a standard exponential distribution for F_N seemed to work.

We now prove our last and final theorem. It is important to note, that we made eight assumptions (A.V.1 through A.V.8), three perturbations to the model and one conjecture to prove this result.

THEOREM A.V.17:

If Conjecture A.V.1 is true and all assumptions hold, then there is a parameter vector $\theta \in \boldsymbol{\theta}$, so that (E.1), (E.2) and (E.3) are satisfied, i.e. we have an equilibrium.

PROOF:

It follows Assumption A.V.6 and A.V.7 and the reasoning below, that we can choose interest rates R_{min}, R_{max} with $\bar{R}_{min} < R_{min} < R_{max} < \bar{R}_{max}$, a wage distance $0 < \delta^W \leq \Delta^W$, probabilities $P_{min}(i) < \pi(g=0|i) < P_{max}(i)$, distances $\epsilon_{\mathcal{m}} > 0$, $\epsilon_{\mathcal{g}} > 0$ and a factor $\nu > 0$ so that we have

– $\boldsymbol{\theta}_{sub} = \{ \theta \mid \theta = (\theta_{\mathcal{m}}, \theta_{\mathcal{g}}, R, w)$ with

$R \in [R_{min}, R_{max}]$,

$w \in [w(R)-\delta^W, w(R)+\delta^W]$,

$|P(m|g=0) - P(m|g=1)| < \epsilon_{\mathcal{m}}$, all m,

$|P(g|i) - \pi(g|i)| < \epsilon_{\mathcal{g}}$, all i,g $\}$

is a subset of $\boldsymbol{\theta}$,

– $f_{1,i}(\theta) \in [\, P_{min}(i), P_{max}(i)\,]$, where $f_{1,i}(\theta) = (1 + \chi_\theta \pi(g{=}1|i)(D_\theta(0,i,0) - D_\theta(0,i,1))\, \pi(g{=}0|i)$,

– $f_2(\theta) \in [\, R_{min}, R_{max}]$ for all θ_{sub}, where $f_2(\theta) = R + \nu(A_\theta^{sup} - A_\theta^{dem})$,

– $f_3(\theta) \in [w(R) - \delta^W, w(R) + \delta^W]$ for all $\theta \in \theta_{sub}$, where

$$f_3(\theta) = w + \nu\Big(\sum_{i=0}^{I} P(g{=}0|i)\pi(i) - \zeta \sum_{i=0}^{I} \frac{\pi(i)}{f(1, x_\theta(1,i))}\Big).$$

Note, that for uninformative θ's with $P(g{=}0|i) \in [P_{min}(i), P_{max}(i)]$, $R \in [R_{min}, R_{max}]$ and $w \in [w(R) - \delta^W, w(R) + \delta^W]$, we have $f_{1,i}(\theta) = \pi(g{=}0|i)$, which is strictly in the interior of $[P_{min}(i), P_{max}(i)]$, $R_{min} < f_2(\theta) < R_{max}$ and $w(R) - \delta^W < f_3(\theta) < w(R) + \delta^W$ by proper choice of $\nu > 0$ and the assumption about the monotonicity of the functions involved.

Thus, if we fix informative $P(m|g)$'s, but where $P(m|g{=}0)$ is sufficiently close to $P(m|g{=}1)$, these properties continue to hold for these θ's as well by the continuity in θ of all functions involved.

Now rewrite (E.1), (E.2) and (E.3) as fixed point equations according to

(F.1) $P(g{=}0|i) = f_{1,i}(\theta)$,

(F.2) $R = f_2(\theta)$,

(F.3) $w = f_3(\theta)$,

Again, by continuity in θ of the functions $f_{1,i}$, f_2 and f_3, it follows directly from the Brouwer fixed point theorem (see e.g. Stokey – Lucas, with Prescott (1989)), that our equations (E.1), (E.2) and (E.3) have a solution. This finishes the proof and the analysis of the model. •

Appendix VI. The Numerical Analysis of the Model.

1. Overview.

The computations were done on a SPARC–station, using the programming language C and a source code of roughly 4000 lines. The computation of one experiment took usually between 40 and 90 minutes of CPU–time for the economy with the insiders and 6 to 8 minutes for the economy without the insiders.

Structurally, the program consists of several nested loops and models, the hierarchy of which can be seen in the following overview.

Program Structure

1. Input and Initializations,
2. R–Loop,
   - 2.1 Decision Problem Module,
     - 2.1.1 Value Function Loop,
     - 2.1.2 Asset Distribution Loop,
     - 2.1.3 Calculation of Aggregate Demands at the Normalized Wage = 1,
   - 2.2. Probability Adjustment Loop,
     - 2.2.1 Wage Loop,
       - 2.2.1.1 Price Loop,
     - 2.2.2 Calculation of Capital Distribution, Aggregate Capital and Aggregate Demands,
     - 2.2.3 Update Probabilities "Backwards"
   - 2.3. Calculation of Remaining Aggregates
3. Output

Many of the strategies and structures for the computations are in common with the theoretical analysis of the model in appendix V. The key difficulty (here as well as in the theoretical analysis) is to disentangle the maze of interdependencies between the different modules of the program into a tree – type hierarchy. There are two key insights, which make the hierarchy described above possible. The first insight is to use the backsolving idea in the spirit of Sims (1984,1990) and to fix the pricing – probabilities π for the entire program rather than solving for it. Since these pricing – probabilities are important parameters for the decision problem, computation time is cut down considerably and the decision problem can be solved, once given only one additional parameter: the interest rate R. The wage is not needed as a parameter, since the solution for any wage can be obtained by direct calculation from the solution for the standard wage = 1.0, see appendix V. Then, given R, and the solution to the decision problem, the production side of the economy can be solved separately. This identification of the interest rate R as the only common link between the two sides of the economy – the decision problem side and the production side – is the second key insight.

We now introduce briefly each individual loop and their interdependencies. The value function loop uses a value–function iteration approach for the value function, defined by interpolation over a logarithmic grid on the assets, to compute the value functions, the decision rules and the transition probabilities. It assumes a normalized wage of 1, fixed stock prices $\pi(g|i,m)$ and the given interest rate R from the R–loop. The asset distribution loop iterates on an asset–distribution, defined by interpolation over a logarithmic grid on the assets, using the probability transitions on assets as calculated in the value function loop. The probability adjustment loop adjusts iteratively in 2.4 the fundamental transition probabilities $P(g|l,i)$ and $P(i|l)$ via the consistency condition (relying on a contraction mapping property of that condition), using a fixed interest rate R as given by the R–loop, aggregate demands at the normalized wage = 1 as calculated by module 1.3 and the wage from the last pass through (for the first pass, we use $P(g|1,i) = \pi(g|i)$ and $P(i|1) = \pi(i)$). The wage loop adjust the wage, until the capital stock on level 0 replicates itself exactly, using the transition probabilities P as given by the probability adjustment loop and the investment rules calculated within by the price–loop. The price–loop calculates prices q_1, q_2 and q_3 as well as the investment rule x via iteration, taking as given the interest rate R, the transition probabilities P and the dividends calculated from the given wage.

Only the wage loop and the R–loop iterate until a criterion reaches a desired degree of precision. All other loops iterate for a fixed number of iterations. There are several reasons for that approach. First, fixing the number of iterations allows us to control the computation time of the program, which becomes important when running many experiments. Secondly, since the R–loop uses an interval–dissection method to find the next R, it is important that the calculation inside the R–loop yield similar results if given similar values for R: this is automatically ensured with iterations of fixed length (for the wage – loop, the precision can be made high enough so that no problem is created there), which always start from the same initial conditions. Finally, and most importantly, it is easier to compare results of different experiments with a fixed number of iterations. For suppose, that instead of terminating the value function loop after a fixed number of iterations, we chose to terminate it after the change of the value function after an additional period is not too big. Suppose, we want to compare the results of two experiments. Effectively then, the program would sometimes solve a finite – horizon problem for e.g. T = 60 periods in one experiment and for T = 200 periods in another experiment. To compare the average welfare levels, the degree of precision needs to be so sharp, that the difference between termination upon reaching the desired precision and the theoretical infinite – horizon limit of the computation is small compared to the differences in the average welfare levels (computed as the infinite – horizon limit) between the two experiments. In other words, we need to ensure that the value function iteration terminates only after a very high number of periods so that the (excluded) future is discounted sufficiently much. By contrast, fixing the number of iterations will always result in comparable results, since every experiment effectively calculates a finite–horizon decision problem version of the model, where the horizon T does not vary across the experiment. By choosing that horizon far enough, one can ensure that e.g. the decision rules come reasonably close to the theoretical infinite – horizon limit of the computation, but even if the mistake is noticable, it will be of the same systematic kind and therefore roughly of the same size across different experiments, making the results comparable. Evidence for this claim is delivered e.g. by the smoothness of the curves in figures 2.1 through 2.7. We provide further insights for judging the precision of the computation below by looking at intermediate results of the experiment with spread = .4 and signal quality = .7 in figures 6.1.1 through 6.7.2.

We proceed to describe the parts of the program in greater detail.

2. The R–Loop.

The R–loop is the global loop of the program, all other parts (except input and output) are internal to it. It computes the equilibrium by searching for the equilibrium interest rate R. The criterion used is the excess supply ΔB for the mutual fund, defined by the difference of the supply and the demand for the mutual fund. The other parts of the program take care of the other equilibrium conditions except for market clearing on the goods market. By Walras' law, the goods market has to clear simultaneously with the market for mutual fund shares: this is checked.

The loop starts from the benchmark neoclassical growth model interest rate $R_0 = \zeta^{\eta}/\beta$. It proceeds with

$$R_t = R_{min} + (R_{t-1} - R_{min}) \,/\, 1.3,$$

where we used $R_{min} = \beta$, until the relative excess supply becomes positive (it was negative in all calculations for R_0). This procedure delivers two interest rates $R_l < R_u$, where the excess supply $\Delta B_l > 0$ is bigger than zero for R_l and smaller than zero ($\Delta B_u < 0$) for R_u. The program then calculates λ so that

$$\lambda \, \Delta B_l + (1-\lambda)\Delta B_u = 0$$

and readjusts λ to be in the interval [.05, .95] (otherwise there is some danger, that the iterations get stuck for a long time at one side of the interval, if the ΔB – function is very steep at the other end). It then uses

$$R_t = \lambda R_l + (1 - \lambda)R_u$$

for the next round of computation. Depending on whether the resulting ΔB is negative or positive, R_u or R_l is replaced with R_t and the computation enters the next cycle. The procedure terminates, when the absolute excess supply $|\Delta B|$ relative to the total demand for the mutual fund is less than 1.0 %.

2.1 The Decision Problem Module.

The decision problem module consists of the three parts listed in the overview above. It takes as given the interest rate R given by the R–loop as well as the fixed parameters $\pi(g|i)$, $\pi(g|i,m)$, $\pi(m|i)$, β and η. Note that the parameter $\pi(g|i)$ is the information revealed by prices in equilibrium: to fix π parametrically rather than solving for it is a backsolving approach following in spirit the technique developed by Sims (1984,1990). Since the value function loop below is the most computation intensive part of the program and since it needs π to compute its results, this backsolving approach saves time in a crucial way: a forward approach would have to iterate over various choices of π to get convergence to an equilibrium, effectively multiplying the computation time of the entire program by the number of these iterations. Even if these iterations are combined with a search for the equilibrium interest rate, the number of iterations necessary is likely to be much higher than just from the search for the interest rate alone as done in this program.

2.1.1 The Value Function Loop.

In the value function loop, the value function and the decision rules are computed. This is the most computation intensive part of the program.

We used a value function iteration approach, where the value functions and the decision rules are defined by interpolation through the computed values on a grid in the underlying state space, starting from $v_0 \equiv 0$. For functions defined on the asset holdings a (such as the value functions v, v^{outs}, v^{ins}, but also the insider – decision rules $s_{i,m}^{ins}(a)$, etc., with the additional dimensions of the state space i = 0,1 and m = 0,1), the grid used was a logarithmic grid with 26 grid points, where the first grid point corresponds to a = .01 and the last grid point corresponds to a = 999.0 (these values are to be understood as multiples of the average wage). The outsider – decision rules are defined on a logarithmic grid of total disposable income y with 51 grid points, where the first grid point corresponds to y = .02 and the last grid point corresponds to y = 1998.0. We then use a logarithmic grid on N with 21 grid points ranging from .01 to 999.0 to integrate out the random wage for the outsider – part, using an exponential distribution for N with mean equal to 1: the formula is given below.

Given initial asset holdings $\bar{a}$ (i.e. some number between .01 and 999.0 from the logarithmic grid) and values $i \in \{0,1\}$, $m \in \{0,1\}$ for the other dimensions of the state – space for an insider, the decisions for $\bar{c}$,$\bar{s}$ and $\bar{b}$ are computed by a speed – enhanced grid – maximization method as follows. First, a coarse grid for possible values for c is defined by partitioning the entire possible range [0, $\bar{a}$] by selecting 15 equally spaced grid points. For each grid point c_j, the possible range

$$[-\frac{\bar{a}-c_j}{\pi(g=1|i)}, \frac{\bar{a}-c_j}{\pi(g=0|i)}]$$

of stock – positions (resulting from the constraint, that the total value of the agents assets have to positive in the next period, regardless of whether g = 0 or g = 1 is drawn) is partitioned into a coarse grid of 13 equally spaced grid points s_k. The purchases of the mutual fund shares b and the next – period asset holdings a'_0 (if g = 0) as well as a'_1 (if g = 1) is then calculated from the budget constraint

$$c + \pi(g=0|i)s + b = \bar{a}$$

and the interest rate R for each pair (c_j,s_k). We then determine the value of the objective function

$$f(c,s) = u(c) + \beta(\pi(g=0|i,m)v(a'_0)+\pi(g=1|i,m)v(a'_1))$$

for each grid point and select the maximizing pair (j^*,k^*), where the value function v(a) is evaluated by interpolating the calculated values for v on the grid points from the previous round of the value function loop by linear interpolation of v over log(a), i.e. by interpolating linearly between grid points (and thereby domain – logarithmically over the corresponding grid on asset values).

Now, a second round with a finer grid on c and s is started by applying the same algorithm again, but using the range $[c_{j^*-1}, c_{j^*+1}]$ for consumption and, given a consumption grid value c, the range

$$[s_{k^*-1}, s_{k^*+1}] * \frac{a - c}{a - c_{j^*}}$$

for the stock position s. Special care of course is taken for the cases, where (j,k) is on the boundary of the coarser grid by not subtracting or adding in the boundary – crossing direction from the index j^* or k^* when calculating the new intervals. This finally delivers optimizing values c^* and s^* in this finer sub – grid together with step sizes Δc and Δs of this finer grid. It need not be the case that c^* and s^* would have been the optimizing values, if we used the finer partitioning on the entire initial range for c and s. Because of concavity of the objective function, this would typically result in c^* and s^* being on the boundary of the finer grid: a warning message is printed in that case (and was printed at most for some of the initial iterations on the value function in the experiments performed). If c^* and s^* are not on the boundary, we have

$$f(c^*, s^* - \Delta s) < f(c^*, s^*) > f(c^*, s^* + \Delta s) \text{ and}$$
$$f(c^* - \Delta c, s^*) < f(c^*, s^*) > f(c^* + \Delta c, s^*).$$

For the final round of maximization, a quadratic function is therefore fitted through the three points $(c^*, s^* - \Delta s)$, (c^*, s^*) and $(c^*, s^* + \Delta s)$ with the values given by f. The maximizing $\bar{s}$ of that quadratic function is calculated and satisfies

$$s^* - \Delta s < \bar{s} < s^* + \Delta s$$

because of the inequalities above. We proceed likewise to find the decision for $\bar{c}$ and finally calculate $\bar{b}$ from the budget constraint. This also delivers the value – function $v_{i,m}^{ins}(a)$ for the insider, given i and m. $v^{ins}(a)$ itself is now found by calculating

$$v^{ins}(a) = \sum_{i=0}^{1} \pi(i) \sum_{m=0}^{1} \pi(m \mid i)\, v_{i,m}^{ins}(a).$$

To calculate $v^{outs}(a)$, we proceed similarly: first we find $v_e^{outs}(y)$, $c^{outs}(y)$ and $b^{outs}(y)$ in very much the same manner as the decision rules for the insider described above, except that we only need a one – dimensional grid (for consumption), since the outsider only faces a consumptions/savings – tradeoff. Then we calculate the integral

$$\int v_e^{outs}(a + N)\, dF_N$$

by evaluating instead the sum

$$v^{outs}(a) =$$

$$F_N[0]v_e^{outs}(a+.01)+(1-F_N[20])v_e^{outs}(a+999.0) +$$
$$\sum_{n=1}^{20}(v_e^{outs}(a+N(n))+v_e^{outs}(a+N(n)-1))(F_N[n]-F_N[n-1])/2$$

,

where N(n) delivers the value of N to the index n and where we used the square brackets [] rather than the round brackets () for F_N to indicate, that the values inside the brackets are to be interpreted as indices rather than values for the exponential distribution function, which has been calculated once at the initializations of the program. This numerical calculation of the integral amounts to the integration of a linearly interpolated function with a step–function type density function.

Finally, the value function v itself needs to be calculated. The model itself requires, that the agent chooses some lottery over his initial asset holdings and then finds himself to either be an insider or an outsider, depending on the outcome of the lottery. Furthermore, these lotteries might vary as the initial asset holdings a vary, with the agent for some assets a choosing to become an insider when he ends up with assets after the lottery below a and for other assets, when he ends up above. In fact, while a great chunk of the difficulties of the theoretical analysis in appendix V stem from the complications arising from these lotteries (which were only introduced in the first place to make the resulting value function concave), we cut through this knot in a very pragmatic fashion and demonstrate below, that this approach is indeed legitimate in our calculations. First, we only search for some cut – off level $\underline{a}$, so that

$$v^{outs}(a) > v^{ins}(a) \text{ for all } a < \underline{a} \text{ and}$$
$$v^{outs}(a) < v^{ins}(a) \text{ for all } a > \underline{a}.$$

Whether such a cut – off level $\underline{a}$ exists, is an open theoretical question. For our program, we simply assume, that it does and calculate it as the first value of a, where v^{ins} crosses v^{outs}. For values a on the logarithmic grid, we then completely ignore the possibilities of lotteries and set

$$v(a) = v^{outs}(a), \text{ if } a < \underline{a} \text{ and}$$
$$v(a) = v^{ins}(a), \text{ if } a > \underline{a}.$$

There is no reason yet from a theoretical perspective, that this technique works. Since lotteries were introduced only to generate concave value functions, we regard this technique as numerically justified, if for our calculations, we have

a) the value functions v^{ins}, v^{outs} and v are concave and
b) the value function v^{ins} crosses v^{outs} only once.

That this is indeed the case can be seen in figures 6.1.1, 6.1.2 and 6.1.3.

Figure 6.1.1 demonstrates the concavity for the value – functions v^{ins} and v^{outs} by plotting the slopes

$$Dv^{ins}(a_j) = (v^{ins}(a_j) - v^{ins}(a_{j-1}))/(a_j - a_{j-1}),$$

and likewise $Dv^{outs}(a_j)$, where j counts through the grid points for the assets holdings a. The value functions are (numerically) concave if these functions Dv^{ins} and Dv^{outs} are monotonously decreasing: they are. Figure 6.1.2 does the same for the value function v. The concavity of this function is the critical, desired property. Figure 6.1.3 finally plots $v^{ins}(a)-v^{outs}(a)$ and demonstrates that this difference crosses the value 0 only once. After crossing 0, it bends back to 0 somewhat of course, since $v^{ins}(a)-v^{outs}(a) \to 0$ as $a \to \infty$.

Thus, while this approach to calculate the value function v without lotteries and with just one crossing point lacks a theoretical justification, it is numerically justified. It is numerically sensible, since numerically calculating the lotteries would rely on properties of the slope of the value – functions and therefore introduce numerical instabilities, the mistakes of which might be greater in the end than leaving the lotteries away altogether. Finally, the approach also makes intuitive sense: since agents have decreasing absolute risk – aversion, and since richer agents have more assets to "speculate" with, there should be increasing returns to information, i.e. the richer an agent, the greater should be the advantage of becoming an insider rather than an outsider. Unfortunately, this argument cannot be made firm theoretically, since the value – function can have linear parts by the introduction of the lotteries. It is then conceivable, that a relatively poor agent may want to become an insider (since he faces the linear, i.e. risk – neutral parts of the value function with his asset – speculation for the next period) whereas a somewhat richer agent decides to become an outsider again (since now, he has to face a strictly concave, i.e. risk – averse part of the value function next period). Figure 6.1.3 shows, that effects of this type do not matter in our calculations. Lotteries, however, would only become trivial everywhere (as we assumed in our calculations), if the derivative of $v^{outs}(a)$ coincides with the derivative of $v^{ins}(a)$, whenever the functions themselves coincide: we could not find a proof for that conjecture. The numerical calculations and the figure 6.1.1 and 6.1.2 suggest, that these derivatives coincide well enough with each other at the crossing point(s) to make lotteries superfluous in the context of our calculations.

Our approach of using iterations on interpolated functions on a grid is similar to Colemans (1990) technique except that we iterate on the value functions instead of on the Euler equations. Iterating on the value function directly is more suitable for our purposes, since we want to compute average welfare and since agents have to make a decision, whether to become an insider or an outsider in part II of the period, i.e. agents have to compare absolut values and not just margins. A combined approach is conceivable, but was not tried. We used a grid – method, since the state space is fairly small, so that grid – methods are still feasible, and since we are interested in the behaviour of the population over the entire range of the state space and not just around some steady state.

The number of iterations was fixed at 40, which seemed to be a reasonable compromise between computation time and precision. The last ten of these 40 value function iterations (for the last pass through the R–loop) are plotted in figure 6.2.1 (Note, that the value functions do not appear concave, since they are plotted over the index a, which provides a logarithmic scale for the asset holdings). While differences between the iterations are still quite visible, these functions might not be far from the theoretical infinite – horizon limit. To analyze this, we fitted an exponential trend (see explanation below) for each index $a_{ind} = 1,..,26$ through all ten iterations and plotted the extrapolated next 190 iterations in figure 6.2.2. In that figure, convergence is clearly visible and the distance between the limit and iteration 40 is roughly of the same size as the distance between iteration 40 and iteration 30. Figure 6.2.2 demonstrates that for a comparison across experiments without a fixed number of iterations, we would need a higher degree of precision than available after 40 iterations. When fixing the number of iterations, however, 40 iterations deliver comparable results, since the mistake made in different experiments is roughly of the same size and not too big.

To fit an exponential trend through points x_n, $n = N_1,..,N_2$, as done for figure 6.1.2, we need to estimate x_∞, α and λ in the equation $x_n = x_\infty - \alpha\, e^{\lambda n}$. We used the estimators

$$\hat{\lambda} = \frac{1}{N_2-N_1-1} \sum_{n=N_1+1}^{N_2-1} \log \frac{x_{n+1}-x_n}{x_n-x_{n-1}},$$

$$\hat{\alpha} = \frac{1}{N_2-N_1} \sum_{n=N_1}^{N_2-1} \frac{x_{n+1}-x_n}{(1-e^{\hat{\lambda} n})e^{\hat{\lambda} n}} \quad \text{and}$$

$$\hat{x}_\infty = \frac{1}{N_2-N_1+1} \sum_{n=N_1}^{N_2} x_n + \hat{\alpha} e^{\hat{\lambda} n}.$$

In all cases, where we checked the quality of the interpolation by plotting the actual and the fitted data as functions of n, the two curves where virtually indistinguishable. The suggestion of this experience to use exponential extrapolation to find an estimate for the theoretical infinite – horizon calculations has not been tried.

In figure 6.3.1, we plotted the stock–investment decision rules $s^n_{i,m}(a)$ for the normalized stocks as described in appendix 5, problem B, for $i = m = 0$ and the last ten iterations $n = 31, .., 40$. Note, that there is virtually no more change in the decision rule! To make the differences better visible, we plotted and exponentially extrapolated the percent differences

$$100*(s^{40}_{i,m}(a) - s^n_{i,m}(a)) \;/\; s^{40}_{i,m}(a)$$

of the decision rules, relative to the last decision rule $s^{40}_{i,m}$ in figure 6.3.2. The curves with positive percent values are the (fitted) calculated iterations $n = 31,..,40$, whereas the negative values are the next 190 exponentially extrapolated iterations. Figure 6.3.2 demonstrates, that the deviation of the last decision rule $s^{40}_{i,m}(a)$ from the theoretical infinite – horizon limit of the calculation is unlikely to exceed 5 %. Again, since the number of 40 iterations is fixed across the experiments, this error is systematic, making the results comparable.

2.1.2 The Asset Distribution Loop.

The asset distribution loop calculates the distribution F_A of assets across the population. Since we ignored lotteries in the value function loop, there is no distinction between the pre – lottery and the post – lottery asset distribution. The loop starts from a distribution with point – mass of 1 at the lowest asset value .01 and iterates 200 times on the distribution, using the transition probabilities calculated in the value function loop.

One iteration consists of two parts: one to calculate the transition for the outsiders and one to calculate the transition for the insiders. Suppose, we have calculated F^n_A and want to calculate F^{n+1}_A of the (n+1)st iteration at some target asset level $\bar{a}$. We need to calculate

$$F^{n+1}_A(\bar{a}) = P^{outs}_{\bar{a}} + P^{ins}_{\bar{a}},$$

where

$$P^{outs}_{\bar{a}} = \int_{a \leq \underline{a}} \int_N 1_{\{a^{outs}(a+N) \leq \bar{a}\}} \, dF_N \, dF^n_A \text{ and}$$

$$P_{\bar{a}}^{ins} = \sum_{i,m} \pi(i,m) \int_{a \geq \underline{a}} 1_{\{a_{i,m}^{ins}(a) \leq \bar{a}\}} \, dF_A^n$$

and where $\underline{a}$ is the cut – off level determined in the value function loop.

The probability for the outsider is calculated in two steps. First, the transition probability, conditional on a starting level of assets a, is calculated as

$$P_{\bar{a}}^{outs}(a) = F_N(a_{inv}^{outs}(\bar{a}) - a),$$

where $a_{inv}^{outs}(\bar{a})$ denotes the level of income y so that $a^{outs}(y) = \bar{a}$. It is calculated by first determining the grid points y_j and y_{j+1}, so that

$$a^{outs}(y_j) \leq \bar{a} < a^{outs}(y_{j+1})$$

and then using linear interpolation between y_j and y_{j+1} to solve for $a_{inv}^{outs}(\bar{a})$. The integration up to the cut – off level $\underline{a}$ of these probabilities then follows according to the formula

$$\begin{aligned} P_{\bar{a}}^{outs} = {} & F_A^n[0]*P_{\bar{a}}^{outs}[0] + \\ & \sum_{k=1}^{\underline{k}} (P_{\bar{a}}^{outs}[k]+P_{\bar{a}}^{outs}[k-1])*(F_A^n[k]-F_A^n[k-1])/2 + \\ & (\underline{\lambda}/2*P_{\bar{a}}^{outs}[\underline{k}+1]+(1-\underline{\lambda}/2)P_{\bar{a}}^{outs}[\underline{k}])(F_A^n(\underline{a})-F_A^n[\underline{k}]), \end{aligned}$$

where brackets [] contain indices and brackets () contain values, where $\underline{k}$ is the index of the last asset grid point below the cut – off level $\underline{a}$, where $\underline{\lambda} \in [0,1]$ is defined to be the difference between the (real) index – value of $\underline{a}$ (i.e. after the logarithmic transformation) and $\underline{k}$ and where $F_A^n(\underline{a})$ is found from linear interpolation of $F_A[\underline{k}+1]$ and $F_A[\underline{k}]$. This summation procedure essentially amounts to integrating a linearly interpolated function with a step–function type density function.

To calculate the probability for the insider, find inversion functions $a^{ins}_{i,m,g,inv}$ in the same way as a^{outs}_{inv}–functions above. P^{ins} is then simply given by

$$P^{ins}_{\bar{a}} = \sum \pi(g,i,m) \max\{F^n_A(a^{ins}_{i,m,g,inv}(\bar{a})) - F^n_A(\underline{a});\ 0\}.$$

Special care for the various boundary cases has to be taken in all of the above calculations.

To judge the precision of the computations, the differences $F_A^{200}[j] - F_A^n[j]$ have been plotted for the last ten iterations n = 191,..,200 in figure 6.4.1 and exponentially extrapolated for 190 more iterations in figure 6.4.2. It is obvious from these figures, that the convergence is extremely good.

<u>2.1.3. The Calculation of Aggregate Demands at the Normalized Wage = 1.</u>

The last part 2.1.3 calculates aggregate demand for bonds and stocks, using the decision rules computed in the value function loop and the asset distribution from the asset distribution loop. The integration routines used are similar to the integration routines described in the asset distribution loop to calculate $P^{outs}_{\bar{a}}$, only that a similarly complicated formula has to used as well to calculate integrals of the type

$$\sum_{i,m} \pi(i,m) \cdot \int_{a \geq \underline{a}} b^{ins}_{i,m}(a) \, dF_A .$$

Also, we calculate the aggregate labor supply at this point as

$$\bar{n} = F_A(\underline{a}),$$

where $\underline{a}$ is the cut – off level, where agents switch to become insiders.

2.2. The Probability Adjustment Loop.

Recall that the probabilities $\pi(g \mid i)$, $\pi(i)$ as well as $P(m \mid g)$ are fixed parameters for the entire calculations. We need to calculate probabilities $P(g \mid l,i)$, $P(i \mid l)$ and $\pi(l)$ (which denotes the distribution of agents across the different levels), making use of the consistency condition as stated in appendix V. As starting point, we chose $P_0(g \mid l,i) = \pi(g \mid i)$ and $P_0(i \mid l) = \pi(i)$. After each pass through the loop we finally update these probabilities backwards in the module 2.2.3. The number of iterations of the price – loop are fixed at five. To judge the quality of the convergence of these five iterations, the differences

$$P_n(i=0 \mid 1) - P_5(i=0 \mid 1), n = 1,..,5$$

(for the last pass through the R–loop) are plotted for l = 1,..,6 in figure 6.5: the rapid convergence should be obvious. Likewise, we plotted the five values of $P_n(g=0 \mid l,i=0)$ (which is independent of l, see the comments below in the description of module 2.2.3) for n = 1,..,5 in figure 6.6: again, the convergence is very rapid. Therefore, it did not seem necessary to use more than 5 iterations in this loop.

2.2.1. The Wage Loop.

The wage loop calculates the equilibrium wage, given the interest rate R and the fundamental growth probabilities P. The wage loop is the only other loop in the program besides the R–loop, which iterates until a degree of precision is reached, since convergence is computationally is very fast. The algorithm for the wage loop is almost identical to the algorithm of the R–loop described above. The criterion used is

$$\text{stat} = \frac{1}{\zeta}\sum_{i=0}^{1} P(g=0 \mid i,l=0)\ P(i \mid l=0)\ f(1,x(l=0,i)),$$

i.e. the fraction of capital on level 0 this period, which arrives on level 0 next period, divided by the steady state growth rate. ζ has to be equal to 1.0 in equilibrium to get a steady – state distribution for capital: if stat equals 1.0, the distribution will replicate itself. To find the equilibrium wage, we use as starting point the wage calculated from the benchmark neoclassical growth model (with one consumer, no randomness, but otherwise the same parameters), multiplied with the factor 100.0. The minimal wage is 0.0. The adjustment speed is 2.0 instead of 1.3 until a right and a left wage are found, noting that stat is a decreasing function of the wage (appendix V contains a proof of that assertion). The minimal and maximal λ allowed are .2 and .8. The calculation terminates when stat differs from 1.0 by less than .00001.

2.2.1.1. The Price Loop.

The price loop performs a fixed number of 100 iterations on the prices for l = 0,..,40 via the formulas

$$q_1^n(l) = d(l,wage) = \sum_{i=0}^{1} P(\, i \mid l \,)\, q_2^{n-1}(l,i),$$
$$q_3^n(l,i) = (\ \pi(g=0 \mid i) * q_1^n(l) + \pi(g=1 \mid i) * q_1^n(l+1)\ )/R,$$
$$x^n(l,i) = f_x(\ 1\ ,\ 1/(\ .0000001 + q_3^n(l,i)\ )\ ) \text{ and}$$
$$q_2^n(l,i) = f_k(\ 1\ ,\ x^n(l,i)\ ) * q_3^n(l,i),$$

for iteration n, level l = 0,..,40 and i =0,1, where dividends are calculated as

$$d(l,wage) = \rho\ (\ \xi_l\ \gamma_0\ )^{1/\rho}\ (1-\rho)^{(1-\rho)/\rho}\ wage^{1\,-\,1/\rho}$$

and where f_x and f_k are the partial derivatives of the CES – function

$$f(k,x) = (\ .94\ k^{.7} + 1.0\ x^{.7}\ )^{1/.7},$$

see appendix V. As starting point, we have $q_2^0(l,i) \equiv 0$. Note that we need to declare $q_1(l)$ for l ranging from 0 to 41 instead of just 0 to 40, we set $q_1(41) = 0$ throughout.

To judge the precision of these calculations, the last ten relative differences

$$100 * \frac{q_1^{100}(0) - q_1^n(0)}{q_1^{100}(0)}$$

for the last ten iterations n = 91,..,100 (for last pass through the R–loop and the last pass through the wage – loop) have been plotted in figure 6.7.1 and exponentially extrapolated in figure 6.7.2, demonstrating the high precision of the convergence. The convergence for higher levels of l was even better. Likewise, we plotted and extrapolated the last ten x(0,0)–iterations in figures 6.8.1 and 6.8.2, demonstrating the high degree of precision here as well.

2.2.2. The Calculation of Capital Distribution, Aggregate Capital and Aggregate Demands.

The capital distribution can now be calculated directly, using the formulas given in appendix V. That is, we set

$\tilde{F}_k(0) = 1.0$ and

$\tilde{F}_k(l) = \psi(l)\, \tilde{F}_k(l-1),\ l = 1,..,40,$

where

$$\psi(l) = \frac{\sum_{i=0}^{1} P(g=1,i \mid l-1)\, f(\,l\,,x(l-1,i))}{\zeta - \sum_{i=0}^{1} P(g=0,i \mid l\,)\, f(\,l\,,\, x(l,i)\,)}.$$

Find the sum

$$\text{sum} = \sum_{l=0}^{40} \tilde{F}_k(l)$$

and calculate finally

$$F_k(l) = \tilde{F}_k(l) \;/\; \text{sum}.$$

Aggregate capital is calculated via

$$\bar{k} = \bar{n} \;/ \sum_{l=0}^{40} F_k(l)\, n(l),$$

where $\bar{n}$ is the aggregate labor supply calculated in module 2.1.3 and where the labor – demand n(l) for one unit of capital is given by

$$n(l) = (\,\xi_l\, \gamma_0\, (1-\rho)\; /\; \text{wage}\,)^{1/\rho},$$

see appendix V. Aggregate insider demands D(0,i,g) for stocks of level 0 at the current wage, which appears in the updating rules in module 2.2.3, is found from the aggregate demands $D_{std}(i,g)$ calculated in modul 2.1.3 for the standardized wage = 1.0 and the standardized stocks (paying R, if g=0 and nothing if g=1) via the formula

$$D(0,i,g) = wage*R*D_{std}(i,g)/(\ q_1(0) - q_1(1)\ ).$$

Aggregate mutual fund shares demand for all agents on level l=0, and aggregate consumption is similarly found by applying the formulas

$$B(0,i,g) = wage*(B_{std}(i,g) - D_{std}(i,g)*q_1(1)/(q_1(0)-q_1(1))\ \text{and}$$
$$C(l,i,g) = wage*C_{std}(i,g).$$

2.2.3. Update Probabilities "Backwards"

The remaining probabilities are updated according to

$$\pi_{n+1}(l) = \bar{\chi}\,\frac{q_1(l)-q_1(l+1)}{q_1(0)-q_1(1)}\;\frac{F_k(l)\,\bar{F}}{\sum_{\iota=0}^{I} \pi(\iota)\ /\ f(l,x(l,\iota))},$$

for l = 0,..,30, where $\bar{\chi}$ is chosen such that $\sum_{l=0}^{\infty} \pi_{n+1}(\,l\,) = 1,$

$$P_{n+1}(i|l) = (\pi(i)/f(1,x(l,i)))/(\sum_{\iota=0}^{I} \pi(\iota)/f(1,x(l,\iota)))$$

and

$$P_{n+1}(g|l,i) = (1+\overline{\chi}\pi(1-g|i)(D(0,i,g)-D(0,i,1-g)))\ \pi(g|i).$$

Here, D is the actual aggregate insider demand for stocks, calculated for the prevalent wage in part 2.2.2.

Since there is quite a bit of freedom a priori in choosing the updating algorithm for these probabilities, the appeal of the formulas above should be explained. Firstly, note that

$$P(\ i\ |\ l\ ) \rightarrow \pi(\ i\ ) \text{ as } l \rightarrow \infty,$$

since $f(\ 1\ ,\ x(l,i)\ ) \rightarrow f(\ 1\ ,\ 0\ )$, as $l \rightarrow \infty$. This is an important property, since we should not make big errors for the levels l which are not included in the calculations. Secondly, the value of $P(\ g\ |\ l,i\ )$ does not depend on l, a property shared with $\pi(g|i)$ (which had to be independent of l to make insiders indifferent between different levels of the stock). The advantage of this is that it indicates that the error made by the mutual fund is of the same size, regardless of the level l, i.e. the bias of the wheel of choosing bad stocks over good stocks is the same, given the price, regardless of the level l.

Finally, a consequence of the formulas above and some algebra is that the conditional growth probabilities

$$P(g|l) = \sum_i P(g \mid i,l) \frac{P(i \mid l)\, f(1,x(l,i))}{\sum_{i'} P(i'|l)\, f(1,x(l,i'))}$$

(where we integrate over the new capital on each level after prices are observed, i.e. take account of the different levels of production of new capital for different indices i's) and the unconditional growth probability P(g) (that is, P(g|l), integrated over the capital distribution $F_k(l)$) compute to

$$P(g|l) = P(g) = \pi(g) + \bar{\chi} \sum_i \pi(i,g)(\, D(0,i,g) - D(0,i,1-g)\,)).$$

That is, the growth probability P(g|l) is also independent of the level l, which is again a property shared with $\pi(g)$ and is therefore appealing for the same reason as the independence of P(g |l,i) outlined above.

2.3. The Calculation of Remaining Aggregates

The remaining aggregate variables (such as aggregate output, aggregate investment, etc.) can now be computed either in a straight – forward manner or following techniques similar to the ones above. We only give the formula to calculate actual average welfare $\bar{v}$ from the average welfare $\bar{v}_{std}$ of the standardized, wage = 1 – problem in module 2.1.3. The formula is:

$$\bar{v} = wage^{1-\eta} * \left(\bar{v}_{std} + \frac{1}{(1-\beta)(1-\eta)} \right) - \frac{1}{(1-\beta)(1-\eta)}.$$

3. Concluding Remarks for the Numerical Analysis

While there is room for improvement (most notably, a method which keeps track of the derivative of the value functions as well as the level to improve on the value function loop, possibly introducing lotteries in the way the model demands), we have demonstrated above and with the smoothness of the relevant graphs in figures 2.1 through 2.7, that the program delivers reasonably sharp answers to the questions asked of it, i.e. to questions of the type

- "How does aggregate output and aggregate and individual welfare change between steady states when moving from the no – insider economy economy to the with – insider economy?",
- "How does the answer vary with variations in the parameters, most notably the assumed probabilities π and $P(m \mid g)$?"
- "How do other interesting aggregates, such as the equilibrium interest rate, the average return on the insider portfolios, the fraction of assets hold by the insiders and the fraction of stocks demanded by the insiders, change with these parameters?"

More tests of the accuracy of these answers (e.g. along the lines of the analysis in Taylor – Uhlig (1990)) are desirable.

Several techniques used here are useful for other types of calculations as well. The three – step procedure to solve for the decision problem of an agent is a good speed – improving technique for grid – methods and deserves further evaluation in other models. Also, this is the first model to our knowledge, which does not use a linear reproduction function for capital and which uses different types of capital with random growth rates of the productivity. For further models, which use one or both of these elements, the techniques found here to compute prices as well as the formula developed above to compute the capital distribution is a useful computational tool. Finally, the overall approach of untangling the mesh of interdependencies between the various elements of the economy into a computable hierarchy by concentrating on and backsolve – reducing the number of parameters relevant for the decision problem is a powerful vehicle for thinking about how to compute equilibria in reasonably complicated models.

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