SAMPLE MPCS Mathematics Placement Test

This is a sample examination of practice problems to help you prepare for the MPCS mathematics placement examination. An answer key is included.

You should expect the problems on the actual mathematics placement examination to consist of mostly proof-based problems, with some computational problems. You will have 2 hours to complete the actual examination. You should answer 70% of the problems correctly in order to pass the exam. Skipping any one of the major topics on the Topics List will likely result in your failing the exam.

Do not consult outside sources when you work on the practice exam.

The actual mathematics placement examination is a **closed-book**, **closed-note**, and **closed-note** exam. You should use this practice exam to simulate the actual math placement exam. Consult the answer key **after** you finish working on the sample exam, or on a problem or a group of problems from the sample exam if you practice at different times.

Problem 1.

1(a) [2 points]
Compute $61^{61} \pmod 9$ by modular arithmetic.

1(b) [1 points]
State Fermat's Little Theorem.

1(c) [2 points]
Use part (b) and modular arithmetic to compute $61^{61} \pmod{11}$.

Problem 2.

Let p, q, and r be distinct prime numbers.

2(a) [3 points]
How many integers n with $1 \leq n \leq pqr$ are relatively prime to pqr?

2(b) [2 points]
Let $p = 2$, $q = 3$, and $r = 5$. Write down all integers n with $1 \leq n \leq pqr$ and relatively prime to pqr, and show that your formula from part a gives the correct number.

Problem 3.

Find the multiplicative inverse of $a \pmod{m}$ for each of the following pairs of integers or prove that it does not exist:

3(a) [1 points]
$a = 15$, $b = 72$.

3(b) [1 points]
$a = 55$, $b = 204$.

3(c) [3 points]
$a = k^2 - 1$, $b = k + 1$.

Problem 4.

Use the Chinese Remainder Theorem to find all solutions of the following system of congruences.

$$
\begin{aligned}
x &\equiv 1 \pmod{2} \\
x &\equiv 2 \pmod{3} \\
x &\equiv 3 \pmod{5} \\
x &\equiv 4 \pmod{11}
\end{aligned}
$$

Problem 5.

5(a) [2 points]
How many ways are there to put n identical objects into m distinct containers so that no container is empty?

5(b) [3 points]
Suppose that S is a set with n elements. How many ordered pairs (A, B) are there such that A and B are subsets of S with $A \subseteq B$?

Problem 6.

You are organizing a dance with students attending from 4 departments: there are 4 CS students, 4 math students, 4 physics students, and 6 statistics students. For one of the dances, you need to arrange 10 students to stand in a circle, all facing the center and holding hands.

6(a) [2 points]
How many different circles of students can you arrange? Two circles are the same if and only if everyone is holding hands with the same neighbors using the same hands. Do not simply your answer (e.g., you can leave in factorials, etc.)

6(b) [3 points]
How many different circles of students can you arrange if there must be exactly 1 math student and 1 physics student, with the math student to the right of the physics student?

Problem 7.

7(a) [2 points]
Prove: if 7 integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11. Is this statement true if 6 integers are selected rather than 7?

7(b) [3 points]
Prove: whenever 25 girls and 25 boys are seated around a circular table, there is always a person both of whose neighbors are boys.

Problem 8.

Let S be the sample space of two-digit **odd** integers (with leading zeros allowed) $n = d_2 d_1$ with $01 \leq n \leq 99$ chosen uniformly at random.

8(a) [1 points]
What is the size of the sample space S?

8(b) [1 points]
Find the probability of the event A that an element of S has distinct digits $d_1 \neq d_2$.

8(c) [1 points]
Define a random variable R to be the sum of the digits of n: thus $R(n) = d_1 + d_2$. Find $E(R)$, the expected value of R.

8(d) [1 points]
Let B be the event that $R(n) = 10$ and event A as in part b. Find the conditional probability $p(B|A)$.

8(e) [1 points]
Let C be the event that $d_2 \geq 5$ and D be the event that $d_1 \leq 5$. Are C and D independent?

Problem 9.

9(a) [1 points]
Let S be a sample space. Complete the definition: "A random variable on S is ...".

9(b) [1 points]
Prove: If X_i, $i = 1, 2, \ldots, n$, n a positive integer, are random variables on S, then $E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$.

9(c) [2 points]
Suppose that X and Y are random variables on S and that X and Y are nonnegative for all elements s in S. Let U be the random variable defined by $U(s) = \max(X(s), Y(s))$ for all s in S. Prove that $E(U) \leq E(X) + E(Y)$.

9(d) [2 points]
Suppose that $X(s) \leq M$, where M is a positive real number, for all s in S. Let $Z(s) = X(s) \cdot Y(s)$. Prove that $E(Z) \leq M \cdot E(Y)$.

Problem 10.

Let S be the set of permutations of 10 objects.

10(a) [2 points]
How many elements of S leave the third object fixed?

10(b) [3 points]
Consider the experiment of choosing an element of S uniformly at random. Let R be the random variable defined by $R(\sigma) = |\{j : \sigma(j) = j\}|$ where σ is a permutation in S, so $R(\sigma)$ is the number of fixed points of σ. Compute the expected value of R.
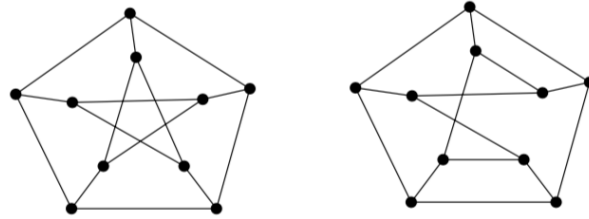
Problem 11.

11(a) [2 points]
Suppose that two fair 8-sided dice are rolled. What is the expected value of the sum of the numbers that come up? What is the variance of the sum of the numbers that come up?

11(b) [3 points]
Consider a fair 6-sided die. Let X denote the number of times that the side 4 shows up over n throws of the die. Compute an upper bound on $P(X \geq n/2)$ using Markov's inequality.

Problem 12.

Is the following pair of graphs isomorphic? If yes, give an explicit isomorphism; if no, explain why they are not isomorphic.

Problem 13.

13(a) [3 points]
Prove that every connected undirected simple graph on n vertices has at least $n - 1$ edges.

13(b) [2 points]
Prove that every undirected simple graph on n vertices with k connected components has at least $n - k$ edges.

Problem 14.

14(a) [1 points]
Complete the following definition: "A tree is ...".

14(b) [4 points]
Prove that a simple undirected graph is a tree if and only if it contains no simple circuits and the addition of an edge connecting two nonadjacent vertices produces a new graph that has exactly one simple circuit (where circuits that contain the same edges are not considered different).

Problem 15.

Let G be a nonempty simple undirected graph.

Prove: if every vertex of G has even degree, then G contains a simple circuit.

Problem 16.

Find a recurrence to describe the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes.

Problem 17.

17(a) [2 points]
Can we draw a closed path made up of 9 line segments, each of which intersects exactly one of the other segments? If yes, give an example of such a path; if no, explain why it is not possible.

17(b) [3 points]
Twenty-five checkers are placed on a 25×25 checkerboard in such a way that their positions are symmetric with respect to one of its diagonals. Prove that at least one of the checkers is positioned on that diagonal.

ANSWERS

Problem 1.

1(a) [2 points]
Compute 61^{61} (mod 9) by modular arithmetic.

Ans. Since $61 \equiv -2$ (mod 9), we have

$$61^{61} \equiv -2^{61} \equiv -2 \cdot 8^{60} \equiv -2 \cdot (-1)^{61} \equiv -2 \equiv 7 \pmod{9}.$$

1(b) [1 points]
State Fermat's Little Theorem.

Ans. If p is prime, then for any integer a, $a^p \equiv a$ (mod p). If $\gcd(a, p) = 1$, then $a^{p-1} \equiv 1$ (mod p).

1(c) [2 points]
Use part (b) and modular arithmetic to compute 61^{61} (mod 11).

Ans. Since $p = 11$ is prime and $\gcd(61, 11) = 1$, Fermat's Little Theorem implies that $61^{10} \equiv 1$ (mod 11). Hence 61^{61} (mod 11) $\equiv 61 \equiv 6$ (mod 11).

Problem 2.

Let p, q, and r be distinct prime numbers.

2(a) [3 points]

How many integers n with $1 \leq n \leq pqr$ are relatively prime to pqr?

Ans. Define

$$P = \{1 \leq n \leq pqr \mid p|n\}$$

and similarly for Q and R. Then $|P| = pqr/p = qr$, while $|Q| = pr$ and $|R| = pq$. Similarly $|P \cap Q| = pqr/pq = r$ since p, q, and r are all distinct primes, and $|P \cap R| = q$, $|Q \cap R| = p$. Finally, $|P \cap Q \cap R| = 1$, so we can apply inclusion-exclusion:

$$
\begin{aligned}
|\{1 \leq n \leq pqr \mid \gcd(n, pqr) = 1\}| &= pqr - |\{1 \leq n \leq pqr \mid p|n \vee q|n \vee r|n\}| \\
&= pqr - |P \cup Q \cup R| \\
&= pqr - qr - pr - pq + r + q + p - 1.
\end{aligned}
$$

2(b) [2 points]

Let $p = 2$, $q = 3$, and $r = 5$. Write down all integers n with $1 \leq n \leq pqr$ and relatively prime to pqr, and show that your formula from part b gives the correct number.

Ans. Putting $p = 2$, $q = 3$, $r = 5$ gives eight numbers 1, 7, 11, 13, 17, 19, 23, 29 between 1 and $pqr = 30$ inclusive and relatively prime to 30. The formula from part b gives

$$pqr - qr - pr - pq + r + q + p - 1 = 30 - 15 - 10 - 6 + 5 + 3 + 2 - 1 = 8.$$

Problem 3.

Find the multiplicative inverse of $a \,(\mathrm{mod}\ m)$ for each of the following pairs of integers or prove that it does not exist:

3(a) [1 points]
$a = 15$, $b = 72$.

Ans. Does not exist because $\gcd(15, 72) = 3 \neq 1$.

3(b) [1 points]
$a = 55$, $b = 204$.

Ans. We use the Euclidean algorithm to find that $\gcd(55, 204) = 1$ and to write as a linear combination of 55 and 204:

$$
\begin{aligned}
204 &= 3 \cdot 55 + 39 & (\Rightarrow 39 = -3 \cdot 55 + 204) \\
55 &= 1 \cdot 39 + 16 & (\Rightarrow 16 = 55 - 39 = 4 \cdot 55 - 204) \\
39 &= 2 \cdot 16 + 7 & (\Rightarrow 7 = 39 - 2 \cdot 16 = -11 \cdot 55 + 3 \cdot 204) \\
16 &= 2 \cdot 7 + 2 & (\Rightarrow 2 = 16 - 2 \cdot 7 = 26 \cdot 55 - 7 \cdot 204) \\
7 &= 3 \cdot 2 + 1 & (\Rightarrow 1 = 7 - 3 \cdot 2 = -89 \cdot 55 + 24 \cdot 204)
\end{aligned}
$$

Now $1 = -89 \cdot 55 + 24 \cdot 204 \equiv -89 \cdot 55 \,(\mathrm{mod}\ 204)$ so the multiplicative inverse of $55 \equiv -89 \equiv 115 \,(\mathrm{mod}\ 204)$.

3(c) [3 points]
$a = k^2 - 1$, $b = k + 1$.

Ans. Does not exist because $\gcd(k^2 - 1, k + 1) = k + 1 \neq 1$.

Problem 4.

Use the Chinese Remainder Theorem to find all solutions of the following system of congruences.

$$
\begin{aligned}
x &\equiv 1 \pmod 2 \\
x &\equiv 2 \pmod 3 \\
x &\equiv 3 \pmod 5 \\
x &\equiv 4 \pmod{11}
\end{aligned}
$$

Ans. The Chinese remainder theorem needs to be used here. Following the notation on page 278 of Rosen, $a_1 = 1$, $m_1 = 2$, $a_2 = 2$, $m_2 = 3$, $a_3 = 3$, $m_3 = 5$, $a_4 = 4$, $m_4 = 11$, $m = 330$, $M_1 = 330/2 = 165$, $M_2 = 330/3 = 110$, $M_3 = 330/5 = 66$, $M_4 = 330/11 = 30$. We find inverses y_i of M_i modulo m_i for $i = 1, 2, 3, 4$: $y_1 = 1$, $y_2 = 2$, $y_3 = 1$, and $y_4 = 7$, respectively (for the last inverse, $30 \equiv 8 \pmod{11}$, so we want to solve $8y_4 = 1 \pmod{11}$, i.e., $y_4 = 7$). The simultaneous solution is $x = 1 \cdot 165 \cdot 1 + 2 \cdot 110 \cdot 2 + 3 \cdot 66 \cdot 1 + 4 \cdot 30 \cdot 7 = 1643 \equiv 323 \pmod{330}$. Thus the solutions are all integers of the form $323 + 330k$, where k is an integer.

Problem 5.

5(a) [2 points]
How many ways are there to put n identical objects into m distinct containers so that no container is empty?

Ans. Since the objects are identical, all that matters is the number of objects put into each container. If we let x_i be the number of objects put into the ith container, then we are are asking for the number of solutions to the equation $x_1 + x_2 + \cdots + x_m = n$ with the restriction that each $x \geq 1$. This is $\binom{m + (m-n) - 1}{m-n} = \binom{n-1}{n-m}$.

5(b) [3 points]
Suppose that S is a set with n elements. How many ordered pairs (A, B) are there such that A and B are subsets of S with $A \subseteq B$?

Ans. Each element x of S falls into exactly one of three categories: either it is an element of A, i.e., $x \in A$; or it is not an element of A but is an element of B, i.e., $x \in B - A$; or it is not an element of B, i.e., $x \in S - B$. So the number of ways to choose sets A and B to satisfy these conditions is the same as the number of ways to place each element x of S into one of these three categories. Therefore the answer is 3^n.

Problem 6.

You are organizing a dance with students attending from 4 departments: there are 4 CS students, 4 math students, 4 physics students, and 6 statistics students. For one of the dances, you need to arrange 10 students to stand in a circle, all facing the center and holding hands.

6(a) [2 points]
How many different circles of students can you arrange? Two circles are the same if and only if everyone is holding hands with the same neighbors using the same hands. Do not simply your answer (e.g., you can leave in factorials, etc.)

Ans. There are 18 students altogether. There are $P(18, 10) = 18!/8!$ ways to arrange 10 of the students in different orders standing in a row (not a circle). These orders may be grouped into groups of 10, corresponding to the same circle, depending of where the first person in the circle is, and proceeding clockwise. So the answer is $P(18, 10)/10$.

6(b) [3 points]
How many different circles of students can you arrange if there must be exactly 1 math student and 1 physics student, with the math student to the right of the physics student?

Ans. Proceeding clockwise around the circle, we must encounter the math student and physics student in the order M-P. The number of ways to assign the M and the P is $4 \times 4 = 16$. The number of spaces continuing clockwise from P around to M is 8. There are $P(10, 8)$ ways to order 8 of the remaining 4 CS students and 6 statistics students to fit into these spaces. Thus there are $16 \times P(10, 8) = 16 \times 10!/2!$ circles.

Problem 7.

7(a) [2 points]
Prove: if 7 integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11. Is this statement true if 6 integers are selected rather than 7?

Ans. Group the first 10 positive integers into five subsets of two integers each, each subset adding up to 11: $\{1, 10\}$, $\{2, 9\}$, $\{3, 8\}$, $\{4, 7\}$, $\{5, 6\}$. If we select 7 integers from this set, then by the pigeonhole principle (at least) two of them must come from the same subset. These two integers have a sum of 11. Now if we ignore the two in the same subset, there are 5 more integers and 4 more subsets; again by the by the pigeonhole principle (at least) two of them must come from the same subset. This gives us two pairs of integers, as desired. The statement is not true if 6 integers are selected: the set $\{1, 2, 3, 4, 5, 6\}$ has only 5, 6 from the same subset, so there is only one pair with a sum of 11.

7(b) [3 points]
Prove: whenever 25 girls and 25 boys are seated around a circular table, there is always a person both of whose neighbors are boys.

Ans. Number the seats around the table from 1 to 50, with seat 50 adjacent to seat 1. There are 25 seats with odd numbers and 25 seats with even numbers. If no more than 12 boys occupied the odd-numbered seats, then at least 13 boys would occupy the even-numbered seats, and vice versa. Assume that at least 13 boys occupy the 25 odd-numbered seats. Then at least two of those boys must be in consecutive odd-numbered seats. The person sitting between those two boys will have boys as both of his or her neighbors.

Problem 8.

Let S be the sample space of two-digit **odd** integers (with leading zeros allowed) $n = d_2 d_1$ with $01 \le n \le 99$ chosen uniformly at random.

8(a) [1 points]
What is the size of the sample space S?

Ans. 50.

8(b) [1 points]
Find the probability of the event A that an element of S has distinct digits $d_1 \ne d_2$.

Ans. Since A is all of S except for 11, 33, 55, 77, and 99, A has cardinality 45 and $p(A) = 45/50 = 9/10$.

8(c) [2 points]
Define a random variable R to be the sum of the digits of n: thus $R(n) = d_1 + d_2$. Find the expected value $E(R)$.

Ans. Since expectation is linear, $E(R) = E(d_2) + E(d_1)$. Since the digits are equally probable, $E(d_2) = (0+1+2+\cdots+9)/10$ and $E(d_1) = (1+3+5+7+9)/5$. Summing, $E(R) = 9.5$.

8(d) [2 points]
Let B be the event that $R(n) = 10$ and A be as in part b. Find the conditional probability $p(B|A)$.

Ans. If $R(n) = 10$, then n is one of the 5 numbers 19, 37, 55, 73, and 91. Only 55 is not in A, so

$$ p(B|A) = \frac{p(B \cap A)}{p(A)} = \frac{4/50}{45/50} = \frac{4}{45}. $$

8(e) [1 points]
Let C be the event that $d_2 \ge 5$ and D be the event that $d_1 \le 5$. Are C and D independent?

Ans. Yes: $p(C \cap D) = p(C) \cdot p(D)$. Proof: $p(C \cap D) = 15/50$, $p(C) = 25/50$, $p(D) = 30/50$, so

$$ p(C) \cdot p(D) = \frac{30}{50} \cdot \frac{25}{50} = \frac{15}{50} = p(C \cap D). $$

26

Problem 9.

9(a) [1 points]
Let S be a sample space. Complete the definition: "A random variable on S is
$\ldots$".
Ans. A random variable on S is a function that assigns a real number to each
element $s \in S$.

9(b) [1 points]
Prove: If X_i, $i = 1, 2, \ldots, n$, n a positive integer, are random variables on S,
then $E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$.

Ans.

$$
\begin{aligned}
E(X_1 + X_2) &= \sum_{s \in S} p(s)(X_1(s) + X_2(s)) \\
&= \sum_{s \in S} p(s)X_1(s) + \sum_{s \in S} p(s)X_2(s) \\
&= E(X_1) + E(X_2)
\end{aligned}
$$

Prove true for $n > 2$ by induction.

9(c) [2 points]
Suppose that X and Y are random variables on S and that X and Y are non-
negative for all elements s in S. Let U be the random variable defined by
$U(s) = \max(X(s), Y(s))$ for all s in S. Prove that $E(U) \leq E(X) + E(Y)$.

Ans. Since X and Y are nonnegative on S, $X(s) \leq X(s) + Y(s)$ and $Y(s) \leq$
$X(s) + Y(s)$ for all s in S. Since $U(s)$ is either $X(s)$ or $Y(s)$ for all s in S,
$U(s) \leq X(s) + Y(s)$ for all s in S. Therefore $E(U) \leq E(X+Y) = E(X) + E(Y)$.

9(d) [2 points]
Suppose that $X(s) \leq M$, where M is a positive real number, for all s in S. Let
$Z(s) = X(s) \cdot Y(s)$. Prove that $E(Z) \leq M \cdot E(Y)$.

Ans.

$$
\begin{aligned}
E(Z) &= \sum_{s \in S} p(s) \cdot Z(s) \quad \text{(by definition)} \\
&= \sum_{s \in S} p(s)(X(s) \cdot Y(s)) \\
&\leq \sum_{s \in S} p(s)(M \cdot Y(s)) \quad \text{(since } X(s) \leq M \ \forall s \in S) \\
&= M \sum_{s \in S} p(s) \cdot Y(s) \\
&= M \cdot E(Y) \quad \text{(by definition)}
\end{aligned}
$$

Problem 10.

Let S be the set of permutations of 10 objects.

10(a) [2 points]
How many elements of S leave the third object fixed?

Ans. Permutations that fix the third object correspond precisely to permutations of the $n - 1 = 9$ objects numbered $1, 2, 4, 5, \ldots, 10$. Thus there are 9! of them.

10(b) [3 points]
Consider the experiment of choosing an element of S uniformly at random. Let R be the random variable defined by $R(\sigma) = |\{j : \sigma(j) = j\}|$ where σ is a permutation in S, so $R(\sigma)$ is the number of fixed points of σ. Compute the expected value of R.

Ans. Let $n = 10$. For $j = 1$ to n define an indicator random variable R_j by $R_j(\sigma) = 1$ if σ fixes j and $R_j(\sigma) = 0$ otherwise. Then the number of fixed points is $R = R_1 + R_2 + \cdots + R_n$. Since expectation is linear,

$$E(R) = E(R_1) + E(R_2) + \cdots + E(R_n).$$

But precisely $(n - 1)!$ of the permutations of n objects fix the first object, as in part (a), so

$$E(R_1) = \frac{(n - 1)!}{n!} = \frac{1}{n}.$$

Since there was nothing special about the first object, $E(R_j) = 1/n$ for each j. Thus $E(R) = n \cdot 1/n = 1$.

Problem 11.

Suppose that two fair 8-sided dice are rolled.

11(a) [2 points]
Suppose that two fair 8-sided dice are rolled. What is the expected value of the sum of the numbers that come up?

Ans. Since expected value is linear, the expected value of the sum is equal to the sum of the expected values. For each die, each of the outcomes 1 through 8 occurs with probability $1/8$, so the expected value is $(1/8)(1+2+3+\cdots+8) = 9/2$. Therefore the answer is 9.

What is the variance of the sum of the numbers that come up?

Ans. Since variance is linear for independent random variables, and these random variables are independent, the variance of the sum is equal to the sum of the variances. For each die, $V(X) = E(X^2) - E(X)^2 = (1/8)(1^2 + 2^2 + 3^2 + \cdots + 8^2) - (9/2)^2 = (51/2) - (81/4) = 21/4$. Therefore the answer is $21/2$.

11(b) [3 points]
Consider a fair 6-sided die. Let X denote the number of times that the side 4 shows up over n throws of the die. Compute an upper bound on $P(X \geq n/2)$ using Markov's inequality.

Ans. Let $X_i = 1$ if the ith throw shows 4 and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{n} X_i$ and $E(X) = nE(X_i) = n/6$. So Markov's inequality implies

$$P\left(X \geq \frac{n}{2}\right) \leq \frac{n/6}{n/2} = \frac{1}{3}.$$

Problem 12.

Is the following pair of graphs isomorphic? If yes, give an explicit isomorphism; if no, explain why they are not isomorphic.
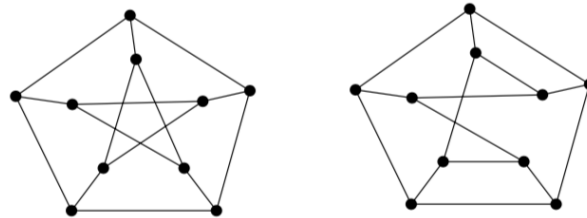
Isomorphic: Yes/No

Ans. No, the two graphs are not isomorphic.

Justification: For instance, the first graph has no simple circuit of length 4, the second graph has two simple circuits of length 4.

Problem 13.

13(a) [3 points]
Prove that every connected undirected simple graph on n vertices has at least $n-1$ edges.

Ans. Proof by induction on n. Basis: $n = 1$. A simple graph with 1 vertex has $1-1 = 0$ edges. Inductive step: Assume the inductive hypothesis: every connected simple graph on k vertices has at least $k-1$ edges, for some $k \geq 2$. Let G be a connected simple graph with $k+1$ vertices: we must show that G has at least k edges.
Since G is connected, every vertex of G must have degree 1 or greater.
Case 1: if every vertex of G has degree 2 or greater, then the sum of the degrees is at least $2(k+1)$. Therefore, by the handshake theorem, the number of edges is at least $k+1$, which is greater than k.
Case 2: suppose G has at least one vertex of degree 1. Call it v. Remove v and its adjacent edge from G. Since $\deg(v) = 1$, this does not disconnect G. Thus $G - v$ is still connected, and it has k vertices, so by the inductive hypothesis, $G - v$ has at least $k-1$ edges. Since G has one more edge than $G - v$, G has at least k edges, and the proof is complete.

13(b) [2 points]
Prove that every undirected simple graph on n vertices with k connected components has at least $n-k$ edges.

Ans. Generalize your proof for part a!

Problem 14.

14(a) [1 points]
Complete the following definition: "A tree is …".

Ans. A tree is a connected undirected graph with no simple circuits.

14(b) [4 points]
Prove that a simple undirected graph is a tree if and only if it contains no simple circuits and the addition of an edge connecting two nonadjacent vertices produces a new graph that has exactly one simple circuit (where circuits that contain the same edges are not considered different).

Ans. First assume that G is a tree. We must show that G contains no simple circuits (which is immediate by definition) and that the addition of an edge connecting two nonadjacent vertices produces a graph that has exactly one simple circuit. Clearly the addition of such an edge $e = (u, v)$ results in a graph with a simple circuit, namely u, e, v, P, u, where P is the unique simple path joining v to u in G. Since P is unique, this is the only simple circuit that can be formed. (You may need to prove property: An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.)
To prove the converse, suppose that G satisfies the given conditions; we want to prove that G is a tree; i.e., that G is connected (since one of the conditions is already that G has no simple circuits). Suppose, to the contrary, that G is not connected. Then there exists two vertices u and v that lie in different connected components of G. Then edge (u, v) can be added to G without the formation of any simple circuits, contradicting the assumed condition. Therefore G is indeed a tree.

Problem 15.

Let G be a nonempty finite simple undirected graph.

Prove: if every vertex of G has even degree, then G contains a simple circuit.

Ans. Let $P = u_1, u_2, \ldots, u_k$, where u_i are vertices in G, be a simple path of maximal length in G. Since $\deg(u_1) \geq 2$, there is at least one other edge (v, u_1) in G having u_1 as a vertex. If v is one of the vertices of the path P, then the subpath from u_1 to v, concatenated with the edge (v, u_1), forms a simple circuit in G. If v is not a vertex of path P, then the path $v, u_1, u_2, \ldots, u_k$ is a simple path whose length is greater than the length of P, contradicting the assumption that the length of P is maximal.

Problem 16.

Find a recurrence to describe the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes.

Ans. Let a_n be the number of coverings. Consider separately the covering where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.

If the rightmost domino is positioned vertically, then we have a covering of the leftmost $n - 1$ columns, and this can be done in a_{n-1} ways. If the rightmost domino is positioned horizontally, then there must be another domino directly beneath it, and these together cover the last two columns. The first $n - 2$ columns therefore will need to contain a covering by dominoes, and this can be done in a_{n-2} ways. Thus we obtain the recurrence $a_n = a_{n-1} + a_{n-2}$, the Fibonacci recurrence.

Problem 17.

17(a) [2 points]
Can we draw a closed path made up of 9 line segments, each of which intersects exactly one of the other segments? If yes, give an example of such a path; if no, explain why it is not possible.

Ans. Such a closed path is not possible. If such a closed path were possible, then all the line segments could be partitioned into pairs of intersecting segments. But then the number of segments would have to be even.

17(b) [3 points]
Twenty-five checkers are placed on a 25×25 checkerboard in such a way that their positions are symmetric with respect to one of its diagonals. Prove that at least one of the checkers is positioned on that diagonal.

Ans. If no checker occurred on the diagonal, then the checkers could be partitioned into pairs, placed symmetrically with respect to the diagonal. Therefore, there must be one (and in fact an odd number) of checkers on the diagonal.