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BRIEF REPORT

Prompting Children to Reason Proportionally: Processing Discrete Units as Continuous Amounts

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Recent studies reveal that children can solve proportional reasoning problems presented with continuous amounts that enable intuitive strategies by around 6 years of age but have difficulties with problems presented with discrete units that tend to elicit explicit count-and-match strategies until at least 10 years of age. The current study tests whether performance on discrete unit problems might be improved by prompting intuitive reasoning with continuous-format problems. Participants were kindergarten, second-grade, and fourth-grade students ($N = 194$) assigned to either an experimental condition, where they were given continuous amount proportion problems before discrete unit proportion problems, or a control condition, where they were given all discrete unit problems. Results of a three-way mixed-model analysis of variance examining school grade, experimental condition, and block of trials indicated that fourth-grade students in the experimental condition outperformed those in the control condition on discrete unit problems in the second half of the experiment, but kindergarten and second-grade students did not differ by condition. This suggests that older children can be prompted to use intuitive strategies to reason proportionally.

Keywords: proportional reasoning, mathematical development, spatial cognition, intuitive reasoning

Proportional reasoning involves understanding the multiplicative part-whole relations between rational quantities, and this kind of reasoning is foundational to solving many problems in mathematics and science. For example, proportional information is involved in reasoning about percentages, temperatures, densities, concentrations, velocities, chemical compositions, and economic values (Karplus, Pulos, & Stage, 1983; Moore, Dixon, & Haines, 1991; Sophian & Wood, 1997). Moreover, proportional reasoning and understanding fractions are highly predictive of later mathematics achievement (Bailey, Hoard, Nugent, & Geary, 2012; Siegler et al., 2012; Siegler, Fazio, Bailey, & Zhou, 2013). Despite its ubiquity and importance, elementary school-age children have great difficulty reasoning proportionally, and relatedly, they struggle when dealing with fractions and decimals (Carpenter, Fennema, & Romberg, 1993; Cramer, Behr, Post, & Lesh, 1997/2009; Lesh, Post, & Behr, 1988; Mazzocco & Devlin, 2008; Pitkethly & Hunting, 1996; Smith, Solomon, & Carey, 2005). Thus, it is important to develop and assess promising ways to strengthen this kind of reasoning.

Contrary to earlier research, which suggested a complete absence of proportional reasoning ability until children are capable of formal operations (i.e., around 10–12 years of age; Inhelder & Piaget, 1958; Piaget & Inhelder, 1951/1975), more recent studies report that children as young as 5 to 6 years of age can reason proportionally, as long as spatial-perceptual representational problem formats are used (Ahl, Moore, & Dixon, 1992; Mix, Huttenlocher, & Levine, 2002; Sophian, 2000; Sophian, Garyantes, & Chang, 1997; Sophian & Wood, 1997). Goswami and colleagues, for instance, gave children shape analogy reasoning problems and found that even 6- and 7-year-old children understand spatially represented proportions (e.g., that a 1/2 circle and 1/2 rectangle pair is analogous to a 1/4 circle and 1/4 rectangle pair; Goswami, 1989, 1995; Singer-Freeman & Goswami, 2001). Moreover, a growing body of research indicates that even infants have an intuitive sense of proportional relations (Denison & Xu, 2010; McCrink & Wynn, 2007; Xu & Garcia, 2008). Denison and Xu (2014), for instance, found that 10- to 12-month-old infants make choices between samples based on the relative proportions of preferred and nonpreferred items that make up the population from which they were drawn.

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Another line of research has shown that young children are more successful in solving proportional reasoning problems if the proportions consist of continuous amounts, made up of two undifferentiated spatial extents, rather than two discrete sets, made up of demarcated units (see Figure 1 below for an illustration; Jeong, Levine, & Huttenlocher, 2007; Spinillo & Bryant, 1999). Boyer, Levine, and Huttenlocher (2008) extended this and found that young children are able to solve proportional equivalence problems when they are prevented from counting and matching the absolute number of discrete units presented in two proportions (i.e., when either the target or choice proportional amounts, or both, are presented with continuous amounts rather than discrete units). This pattern of performance was interpreted as reflecting the use of "intuitive strategies," based on an approximate, spatial sense of proportion rather than explicit strategies, based on verbally situated count-and-match operations. Although the former tend to be correct, the latter tend to lead children astray, particularly when a visually salient numerical quantity leads to a different answer than the proportional relation (e.g., 2/8 is judged as equivalent to 2/4 rather than to 1/4 because of an absolute match of their numerators; Mix et al., 2002; Wynn, 1997).

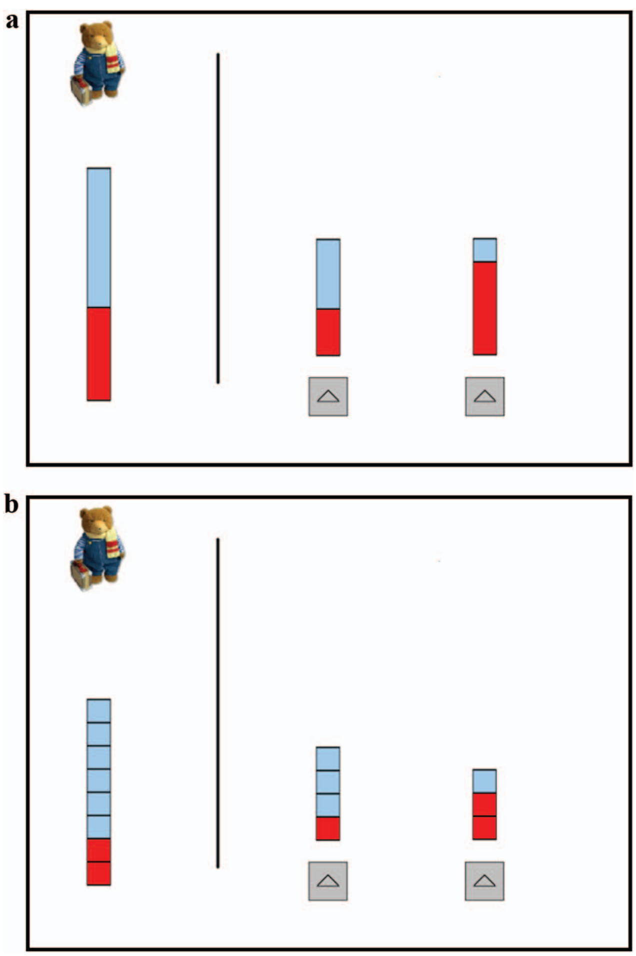

Figure 1. Example screenshots of the proportional equivalence choice task. The target proportion and character photo appear on the left and the two choice alternatives on the right, each above a button that participants mouse-clicked to register their response. (a) A trial presented with a continuous-amount format. The target involves 4 red juice units out of 10 total darker red juice + lighter blue water units, which matches the choice alternative with 2 red juice units of 5 total units. (b) A trial presented with a discrete unit format. The target involves two red juice units out of eight total darker red juice + lighter blue water units, which matches the choice alternative with one red juice unit of four total units. See the online article for the color version of this figure.

Mathematics and science education researchers widely advocate using children's intuitive knowledge to scaffold their learning (e.g., Fischbein, 1982, 1987; Halberda, Mazzocco, & Feigenson, 2008; Lesh et al., 1988), and this approach is espoused by many researchers interested in identifying ways to improve children's understanding of ratios, fractions, and decimals, as well as their proportional reasoning (e.g., Ahl et al., 1992; Falk & Wilkening, 1998; Fujimura, 2001; Pitkethly & Hunting, 1996). The general idea behind this approach is to make use of what children already understand in teaching them challenging concepts. Barth, Baron, Spelke, and Carey (2009), for instance, argued that kindergarten and first-grade students' (i.e., 6- and 7-year-olds') intuitive understanding of doubling and halving operations in a visual-perceptual framework might be used to support understanding of multiplication and division, which are difficult concepts for children to grasp. Consistent with this approach, progressive alignment theory (Kotovsky & Gentner, 1996) suggests that an understanding that comes from solving problems within a child's grasp (e.g., simpler analogies) can facilitate their ability to solve problems that would otherwise be beyond their grasp (e.g., more difficult analogies).

In the current study, our primary aim is to test whether children perform better on proportion problems involving discrete units simply by presenting them with proportion problems involving continuous amounts prior to the more challenging discrete-format problems. We reasoned that presenting continuous-format proportional problems first might prompt children to apply their correct intuitive strategies to the more challenging discrete unit proportional problems. This hypothesis is consistent with the theory of progressive alignment, which posits that easier problems can lead to better performance on more difficult but alignable problems (Gentner & Markman, 1994; Kotovsky & Gentner, 1996). We used a proportional equivalence judgment task that taps proportional reasoning skills in a wide range of age groups—from kindergarten through fifth grade (Boyer & Levine, 2012; Boyer et al., 2008). Half of the participants were randomly assigned to an experimental condition and were given a series of proportional equivalence judgment problems presented with a continuous-amount format followed by a series of problems presented with a discrete unit format. The other half of the participants were randomly assigned to a control condition and were given discrete unit proportional problems throughout. If the intuitive strategies that children have been proposed to use to solve proportion problems involving continuous amounts transfer to proportion problems involving discrete quantities, then performance on the second half of problems, which consisted of the same discrete unit problems for both groups, should be higher for those in the experimental condition than for those in the control condition.

Method

Participants

Participants were 194 students (96 girls, 98 boys) recruited from six public elementary schools; 105 were recruited from a larger metropolitan school district in the Midwest area of the United States, and 89 were recruited from a rural school district in the Southeast United States. Sixty-four participants were recruited from kindergarten classrooms (33 girls, 31 boys), 66 from second-grade classrooms (28 girls, 38 boys), and 64 from fourth-grade classrooms (35 girls, 29 boys).[1] Ages of the children at each grade level were as follows: $M_K = 6$ years 0 months, $SD_K = 4$ months; $M_2 = 8$ years 0 months, $SD_2 = 6$ months; and $M_4 = 10$ years 0 months, $SD_4 = 5$ months. All students had written parental consent to participate. Demographic information was not collected from participants, but about half of the students in the participating schools were eligible for free or reduced lunch.

Procedure

We tested participants during regular school hours, in familiar rooms adjacent to their classrooms, on an IBM T20 laptop computer with a 14.1-in. screen. During task instructions, a picture of a teddy bear appeared on the screen, and children were told that his name is "Wally bear." The experimenter explained that Wally bear enjoys drinking all kinds of juice and likes to mix his juice himself. The experimenter then showed participants an example that stressed the importance of maintaining a recipe's proportion when transforming total amount.

During each of the 16 trials, a small photo of the character appeared on the upper left of the screen, and a target proportion column composed of juice and water parts was shown just below the photo. Two potential matches for the target proportion appeared on the right two thirds of the screen (see Figure 1, e.g., screenshots). With the target and choice proportions on the screen, the experimenter asked, "Which of these two (pointing to the two alternatives) is the right mix for the juice Wally bear is trying to make? Which of these two would taste like Wally bear's juice?" Participants registered selections by clicking a button that appeared below either of the choice alternatives with the mouse. After each selection, another target and another two choice alternatives appeared. The "juice" color on each successive trial was randomly chosen from red, green, blue, purple, or yellow with the constraint that it differed on successive trials. Participants completed 16 self-paced trials administered in random order. No performance feedback was provided.

Experimental Design

Participants were randomly assigned to either an experimental or control condition. For participants in the experimental condition, problems in the first half of the task (eight problems) were presented with a continuous-amount format (i.e., the juice and water portions of the target proportion and the choice alternatives formed unitary columns, with visible divisions only occurring at the point where the juice and water parts met), and problems in the second half (eight problems) were presented with a discrete unit format (i.e., lines demarcated each 1-cm^2 unit). The transition from continuous amount to discrete unit problems occurred with no break in the task and a seamless transition from the eighth problem that was presented with a continuous-amount format to the ninth problem that was presented with a discrete unit format. For participants in the control condition, all 16 problems were presented with a discrete unit format.

We also implemented a number of other controls. To be consistent with previous research, we used foil alternatives that matched the target proportion's juice part or whole juice and water quantity, with each kind of foil used on four of the problems in each half of the trials (Boyer et al., 2008). Previous studies have also shown that "half" is an especially salient proportion (Jeong et al., 2007; Spinillo & Bryant, 1991, 1999), so neither the target nor either of the choice alternatives were ever 1/2. However, all foils were on the opposite side of the half boundary from the target (i.e., if the target proportion was less than 1/2, then the foil was greater than 1/2 or vice versa), which we did to make the problems accessible for the younger age groups. All proportions used in the study could be reduced to a multiple of thirds, fourths, or fifths, with numerator values (i.e., juice units) that ranged from 1 to 9 units and denominator values that ranged from 3 to 12 total units (e.g., ranging from 1/5 to 9/12, with all possible numerator and denominator values that reduced to even thirds, fourths, or fifths included). Other factors varied randomly, as determined by the computer program, such as whether the correct choice appeared on the right or left and the "juice" color assigned on a particular trial.

Results

Overall, participants selected the correct proportion on 62.0% of the trials. Preliminary analyses indicated no gender difference in performance, $t(192) = -.49$, $p = .63$, and no significant difference as a function of the geographical region from which the data were collected, $t(192) = 1.06$, $p = .29$. A 3 × 2 × 2 mixed-model analysis of variance examined number of correct responses as the dependent variable with condition (experimental, control) and school grade (kindergarten, second-grade, fourth-grade) as between-subjects factors, experiment half (first half, second half) as a within-subjects factor, and with the school that children were recruited from included as a covariate. This revealed a significant main effect of school grade, $F(1, 187) = 18.94$, $p < .001$, $\eta_p^2 = .168$, with higher scores in higher grades (M = 53.0%, 58.0%, and 74.9% correct in kindergarten, second grade, and fourth grade, respectively, although only the difference between fourth-grade students and the other two grade groups was significant; both $t \geq 4.11$, both $p < .001$). There was also a main effect of condition, $F(1, 187) = 7.87$, $p = .006$, $\eta_p^2 = .040$, with children in the experimental condition outperforming those in the control condition (M = 66.0% and 57.7% correct, respectively), which generally replicates the previous finding of superior performance on problems illustrated with continuous than discrete quantities (e.g., Boyer et al., 2008). Most relevant to the primary question of our study, there was a significant school grade × condition × experiment half interaction, $F(2, 187) = 3.97$, $p = .020$, $\eta_p^2 = .041$. All

[1] Due to variability in their availability, the number of children drawn from each of the six schools varied across the age groups. Importantly, however, the available children in each class were randomly assigned to the experimental and control conditions.

remaining main effects and interactions were nonsignificant (all $F \leq 1.99$, all $p \geq .16$).

We next unpacked this three-way interaction to compare performance of children in the experimental and control conditions in the first and second halves of problems. Consistent with previous research, which has found better performance on continuous- than discrete-format proportional equivalence problems, during the first half of the experiment, participants in the experimental condition outperformed those in the control condition, $t(192) = 2.97$, $p = .003$, Cohen's $d = .42$ (M = 64.8% vs. 50.0% in kindergarten, 64.3% vs. 53.1% in second grade, and 80.9% vs. 73.8% in fourth grade). The more important question for the current study, however, is whether the children in the experimental condition performed better than the children in the control condition on the set of discrete problems given to both groups in the second half of the trials. As illustrated in Figure 2, planned pairwise comparisons reveal a significant difference in performance on the second half of problems in the experimental versus control conditions in fourth grade, $t(62) = 2.21$, $p = .03$, Cohen's $d = 0.55$, but not in kindergarten or second grade, $t(62) = 0.075$, $p = .94$, Cohen's $d = 0.018$ and $t(64) = 0.183$, $p = .86$, Cohen's $d = 0.047$, respectively.

Discussion

Consistent with previous studies, we find that kindergarten through fourth-grade students perform better on continuous-format proportional equivalence problems than on discrete-format proportional equivalence problems (Boyer & Levine, 2012; Boyer et al., 2008; Jeong et al., 2007; Spinillo & Bryant, 1999). We believe that this better performance reflects the use of intuitive perceptual strategies on the continuous-format problems and the use of erroneous count-and-match strategies on the discrete-format problems (Boyer et al., 2008; Boyer & Levine, 2012). Furthermore, as expected, we found a significant effect of school grade from kindergarten to fourth grade, which likely, at least in part, reflects the increasing focus on fractions and proportions beginning in third grade in the mathematics curriculum.

Turning to our main finding, fourth-grade students' performance on proportional equivalence problems involving discrete quantities was better in the experimental condition, which first experienced continuous-quantity proportional problems, than in the control condition, which first experienced a set of discrete-quantity proportional problems. The continuous-format problems precluded counting and, therefore, were likely to be solved using an intuitive, spatial sense of proportion. We believe that solving these problems prior to solving the discrete problems led the fourth-grade children to relinquish the count-and-match strategies they typically apply to these problems. In contrast, kindergarten and second-grade students showed no such benefit, performing no better on discrete-format problems regardless of whether they were preceded by a set of continuous- or discrete-format problems. This pattern suggests that count-and-match strategies for solving proportional equivalence problems may be more robust in the earlier grades. These sorts of strategies, as Siegler and colleagues (2013) have argued, are maladaptive, because proportions and fractions do not abide by the same operational rules as whole-number mathematics. As such, these strategies can interfere with proportional reasoning processes that children might actually understand (Mix & Paik, 2008; Paik & Mix, 2003).

It is important, however, to note that our manipulation was subtle—we did not directly instruct students about proportions, call their attention to the analogy between the continuous and discrete problems, or even provide performance feedback. Children were simply given several proportional judgment problems presented with continuous amounts, followed by several identically structured proportion problems involving discrete units. It is certainly possible that if children were explicitly instructed to apply their intuitions about continuous-format proportions to discrete-format problems (e.g., by comparing parallel continuous- and discrete-format problems; Gentner & Markman, 1994), then younger children might show even greater performance gains on discrete-format proportion problems compared to a control group.

In terms of theoretical implications, we take our findings to suggest that children's intuitive proportional reasoning strate-

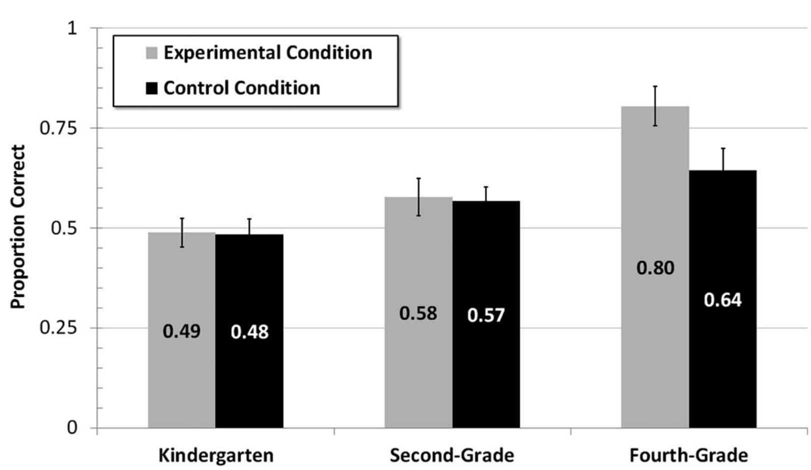

Figure 2. Mean proportion correct for the problems for the second half of trials, all of which were presented with a discrete unit format, for the experimental and control conditions in each grade level. Error bars represent ± SEM.

gies can be primed, at least in the latter elementary school years, by presenting problems that preclude the use of count-and-match strategies (i.e., problems that involve proportions consisting of continuous quantities). This hypothesis could be put to a further test by inverting the order of the trial formats used in the experimental condition (i.e., by presenting discrete-format problems prior to continuous-format problems). That is, if we are correct about the role that the continuous-format problems play in improving children's performance on the discrete trials, a block of discrete trials prior to a block of continuous trials should have no effect, or perhaps even a deleterious effect, on performance on subsequent continuous-format problems. That is, after solving discrete-format proportional judgment problems, children might approach the continuous problems with more of an absolute matching strategy and possibly even superimpose mental units onto the continuous quantities and then apply the sorts of count-and-match strategies they tend to use on discrete unit problems. Such a finding would bolster our view that continuous-format proportional problems can be valuable in scaffolding children's understanding of discrete proportions. Although the current study did not include enough items to identify the limits of intuitive, approximate proportional reasoning, there clearly are limits to the accuracy of this system. In a previous study, we found that the scale relationship between the target proportion and the match affected performance, with larger scale factors leading to more difficulty (Boyer & Levine, 2012). We also know that the approximate magnitude system has a ratio limit, so it would be difficult for the spatial sense of proportion to discriminate proportions that do not differ from each other by at least a threshold ratio, which gets smaller with increasing age (e.g., Cantrell & Smith, 2013; Cordes & Brannon, 2009). Although the current study was not designed to identify these limits, it is clearly important to do so for different age groups.

In terms of practical implications, several caveats should be noted. First, in terms of the generalizability of our findings, the school demographics we gathered indicate that our sample included many children from underserved backgrounds. Our findings for children in this demographic group may be of particular importance in view of data showing that mathematics achievement is strongly associated with socioeconomic status (e.g., Jordan & Levine, 2009; National Mathematics Advisory Panel, 2008). Second, it is important to note that our manipulation, in and of itself, is too subtle to be an effective educational intervention, although our findings do hint that instruction that highlights the relation between proportions involving continuous and discrete quantities could enhance student learning by encouraging students to make use of their intuitions about proportional relations. Perhaps it might be even more effective to align equivalent proportions involving discrete and continuous quantities, as well as encourage children to more explicitly compare these proportions (Christie & Gentner, 2010; Gentner & Markman, 1994; Namy & Gentner, 2002). In view of findings showing that these concepts are challenging yet powerful predictors of later mathematics achievement (e.g., Bailey et al., 2012; Siegler et al., 2012), such efforts to identify effective instructional strategies to increase children's understanding of proportions and fractions are important.

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