Journal of Experimental Psychology: General
1994, Vol. 123, No. 3, 284–296

0096-3445/94/$3.00

A Mental Model for Early Arithmetic

Janellen Huttenlocher, Nancy C. Jordan, and Susan Cohen Levine

The authors examined young children's ability to solve nonverbal calculation problems in which they must determine how many items are in a hidden array after items have been added into or taken away from it. Earlier work showed that an ability to reliably solve such problems emerges earlier than verbal calculation ability but did not examine when it first appears. The authors propose that the ability to solve such problems involves domain-general symbolic processes similar to those involved in symbolic play and the use of physical models. Hence the ability to calculate should appear at about 2 years and should be related to overall level of intellectual competence. The authors show that the ability to reliably solve nonverbal calculation tasks emerges only after 2 years of age and that performance on nonverbal calculation problems is highly related to overall level of intellectual competence in children between 3 and 4 years of age.

It is well known that children can carry out numerical transformations involving addition and subtraction by 5 or 6 years of age (e.g., Siegler & Shrager, 1984; Siegler & Jenkins, 1989; and our own earlier papers, Levine, Jordan, & Huttenlocher, 1992; Jordan, Huttenlocher, & Levine, 1992). They can answer questions such as "What is 3 + 2?" and "What is 4 − 1?" They also can solve simple story problems such as: "John has four marbles. Harry gives him two more marbles (or, alternatively, "Harry takes away two of his marbles."); how many marbles does John have?" Clearly, the ability to solve such verbal calculation problems requires mastery of conventional arithmetic skills. Yet a child need not possess these conventional skills to be able to carry out calculations. Consider the following nonverbal problem. A set of four objects is shown. The set is then hidden. Next, one of two possible transformations is carried out, but the outcome cannot be seen; either a set of two items is added to the hidden array, or a subset of two items is removed from the hidden array. A child who can indicate the numerosity of the resultant array by producing an array with the correct number of items (for this and for comparable problems) is able to carry out numerical transformations. In the present article, we examine the development of the ability to solve such nonverbal problems.

A Mental Model for Arithmetic

It would be possible to solve a nonverbal number transformation problem by constructing a mental version of the initial (covered) array and then imagining the movement of items into or out of that array. The changed mental array resulting from the imagined movement of items in relation to the initial array would constitute the answer to the problem. That resultant mental array could then be used to produce an actual array of the proper numerosity, or, if a person knows the number words, to report the number of items in the resultant array. Such a problem-solving process would be comparable in certain ways to the use of physical models to "stand for" aspects of situations. For example, a person might construct a model spaceship and a simulated outer space to test whether an actual spaceship with certain features would be adequate for travel in outer space. To provide an adequate test, the model must preserve the critical features of the situation—the shape of the ship, its movement, and certain features of outer space. Other features, such as the color of the ship, are irrelevant. Mistakes may be made as to whether the actual spaceship would work if the features represented in the model are not the critical ones, or if the critical features are represented incorrectly or are not represented in the model (because they are not recognized as relevant) and so forth.

It is only necessary that the representation for number transformation, like that for the model spaceship, should preserve the relevant features of a situation. Like the physically realized model, the mental representation is not a replica of a situation; in fact, because the number of ways to conceptualize a situation is indefinitely large, the notion of a mental replica of a situation is not well defined. Although such representations do not involve arbitrary conventional symbols, they are symbolic in nature. Thus, for nonverbal calculation there is explicit representation of the number of countable entities in the initial set, the nature of the transformation (into or out of the set), and the number of entities involved in that transformation. Other features, such as the nature of the individual items, their spatial arrangement, and so forth, are irrelevant.

These representations are sometimes referred to as *mental models,* a phrase introduced by Gentner and Stevens (1983) and Johnson-Laird (1983). Barsalou (1992) described men-

Janellen Huttenlocher and Susan Cohen Levine, Department of Psychology, University of Chicago; Nancy C. Jordan, Department of Educational Psychology, Graduate School of Education, Rutgers—The State University of New Jersey.

This research was generously supported by a grant from the Smart Family Foundation to Janellen Huttenlocher and Susan Cohen Levine. We thank Susan Goldin-Meadow, Nora Newcombe, and Amanda Woodward for their helpful comments on the manuscript of this article.

Correspondence concerning this article should be addressed to Janellen Huttenlocher, Department of Psychology, University of Chicago, 5848 South University Avenue, Chicago, Illinois 60637.

tal models as follows: Their "attributes and relations are analogous to the physical parts and relations" in the situation they represent, and they are used to "produce quasi-continuous simulations" of changes in these situations. One sort of evidence for the use of mental models in adult problem solving comes from reaction time and error data (e.g., in spatial problems for which people predict the outcomes of rotating forms in space, of folding pieces of paper, etc.). When the physical situations to be represented are more complex, the problem solving process takes longer to carry out and is less accurate (e.g., Shepard & Metzler, 1971). For example, when asked to describe the mental processes in solving spatial problems, people frequently describe an imagined version of the initial situation and execution of imaginary actions that change that situation (cf. discussions in Huttenlocher, 1973, 1976).

In certain cases, mental models may provide formal mechanisms that, when applied correctly, necessarily yield the correct answer. These include mental rotation and paper-folding tasks, in which the question is whether an object will have a particular appearance when it is transformed in a certain way. These are determinate geometry problems—the information provided is sufficient to answer the question. Calculation tasks that involve sets of discrete items to which a number of items is added or subtracted also are determinate problems. The solution process we have described constitutes a formal mechanism for calculation that, when applied correctly, yields the number of elements resulting when a set has been transformed by movement of elements into or out of that set. For determinate problems of this sort, errors that are made when people possess a mental model (formal competence) are performance errors that are due to, for example, inattention. In the present article we use the term mental model to refer to the use of such a formal mechanism in calculation.

Mental Models in Development

A mental model for arithmetic, in which imagined entities and transformations are mapped onto actual objects and movements, is potentially of special interest in the study of cognitive development. In contrast to conventional symbolic schemes that are acquired from caregivers, such a model might, in principle, develop without input from caregivers. That is, a mental model for arithmetic—a formal competence (applicable to small numbers) that yields answers to calculation problems—might involve only an ability to abstract numerically relevant information from situations. Thus, whereas the ability to solve verbal arithmetic problems involves both conventional skills and an understanding of number transformation, the ability to solve nonverbal calculation problems might involve only an understanding of number transformation.

In fact, the acquisition of verbal arithmetic skills might involve the mapping of conventional symbols onto a preexisting mental model of number and number transformation. This model might be used in solving number–fact problems or simple story problems; that is, when children are given such a problem, they might imagine an array of the initial numerosity, transform that array by adding or subtracting items as described, and report the answer. The children also might use their fingers (a physical rather than a mental model) to represent the numbers and the transformation process (cf. Siegler & Shrager, 1984). A mental model also may be used in story problems in which the items are not physical objects and the transformations are not actual movements (e.g., in a problem in which a child "gives away" his "turns" in a game).

We have conducted two studies in which we have examined the relation between the development of children's ability to perform nonverbal calculation problems, story problems, and number–fact problems (Levine et al., 1992; Jordan et al., 1992). We used a nonverbal task like the one described above to assess calculation abilities without requiring the child to use conventional symbols. The child was shown a set of disks on a card that was then hidden with a cover. The hidden set was transformed either by inserting or removing disks through an opening in the side of the cover. The transformed set was not shown. The child then constructed an array with the same number of elements that was under the cover after the transformation. Rather than being described verbally, the task was shown, by having the experimenter do an example problem. Thus, we were able to determine whether nonverbal problems are solved at an earlier age than comparable problems involving conventional symbols.

In our initial study (Levine et al., 1992) we compared the ability of 4-, 5-, and 6-year-old children to do nonverbal addition and subtraction tasks with their ability to do comparable verbal tasks of two kinds: story problems, which describe actual objects and their movements, and number–fact problems, such as "How much is 3 and 2?" Children as young as 4 years could do nonverbal tasks involving small numerosities with great accuracy. However, it was only after 5 years of age that they became somewhat proficient on the corresponding story problems and number–fact problems. Story problems were easier than were number–fact problems. These findings are consistent with the view that children possess a mental model for arithmetic before they master conventional skills, and that they may use this model to solve story problems, mapping various aspects of the story onto elements in the mental model. The number–fact problems may be more difficult than the story problems because the possibility of mapping to imagined entities is not made explicit.

In our second study we examined the performance of 5-year-old children in different social classes on verbal and nonverbal calculation tasks (Jordan et al., 1992). We believed that we would find social class differences in the ability to solve verbal calculation problems because the extent of verbal input differs among social-class groups (e.g., Bee, Van Egeren, Streissguth, Nyman, & Leckie, 1969; Kirk, Hunt, & Volkmar, 1975). However, if conventional symbols are not required for nonverbal calculation, then this ability might *not* vary across social class. The results of the study were striking and clear. Children from low-income families did considerably worse than children from middle-income families on both story problems and number–fact problems. However, low- and middle-income-family children did equally well on the nonverbal problems, even though their performances were far from ceiling.

In summary, the results of these two studies support the view that conventional calculation skills are not required for nonverbal calculation. First, children solved nonverbal problems at an earlier age than they solved verbal problems. Second, children from low-income groups did as well as children from middle-income groups on nonverbal problems but did less well on verbal problems, possibly because they had received less verbal input. We discuss below two further issues not previously addressed in our studies. One issue is when exact representations of number and the ability to calculate emerge. In considering this issue, we must discuss the infant literature, because some investigators believe that these abilities are innately available in infants. The other issue is whether the ability to represent information in the number domain reflects the same underlying processes as the ability to represent information in other domains, hence showing a substantial relation to measured general intelligence.

Emergence of a Mental Model for Arithmetic

It is not clear when the ability to reliably obtain answers on nonverbal calculation problems emerges. We have suggested that this ability involves symbolic processes—the mapping of imagined entities and their movements onto actual objects and their movements—although it does not involve conventional symbols. Indeed, the ability to solve the nonverbal calculation problems constitutes a formal mechanism for obtaining answers to problems involving small numbers. A possible prediction from this view is that children's ability to calculate should emerge at about 2 years of age and become increasingly well developed over the next year or so. It is in this age range that children exhibit a variety of symbolic abilities that do not involve conventional symbols. Their play comes to involve the use of substitute objects and activities to stand for real objects and activities (e.g., McCune-Nicolich, 1981). Also, the 2- to 3-year age range is when children become able to use physical models to provide information about actual situations. DeLoache (1987, 1991) found that children's ability to infer the location of an actual toy in a room from watching a model toy being hidden in a model room appeared between 2½ and 3 years (ability to use a picture as a representation of an actual spatial layout appeared somewhat earlier, by 2½ years).

In contrast to the possibility just discussed, some investigators have claimed that nonverbal numerical abilities such as those we have described are innately available in infants. Indeed, there is a set of studies that have shown that infants are remarkably sensitive to numerosity and to changes in numerosity. General questions have been raised in the literature as to the nature of the competencies exhibited by infants and their relation to the competencies of older children (cf. Fischer & Biddell, 1991; Kellman, 1988; Siegler, 1992). However, our present concern (i.e., infants' numerical competence) has not been specifically addressed in these discussions. Here we consider whether or not existing studies provide convincing evidence that infants' behaviors are based on representations sufficient to reliably produce exact answers.

Infant Studies

It has been found, through the use of a habituation paradigm, that over a sequence of trials infants become less attentive to a small set of a particular numerosity, but display looking time increases when they are shown a set of a different numerosity (e.g., Starkey & Cooper, 1980). Indeed, this is even true of neonates (Antell & Keating, 1983). The findings often have been interpreted as showing that infants represent small numbers exactly. Thus, Starkey (1992) said that the observed data show that "even infants can enumerate sets that are small in numerosity" (p. 93). In discussing a possible mechanism, Starkey suggested two possible alternatives, both involving exact representations of number—use of a tagging process (that does not involve conventional symbols), or use of one-to-one correspondence to directly compare the numerosities of sets. The results of infant studies have not provided support for such a strong conclusion. What has been found is a tendency, across infants and trials, to attend less to a particular numerosity and more to a different numerosity. Although this is a very important discovery, it does not necessarily indicate use of exact representations.

Even if one grants, as we do here, that infants represent arrays as sets of discrete entities that can vary in quantity, these representations might be inexact for small set sizes, as are adults' representations for large set sizes (if they do not count). That is, when an infant is presented an array of a particular size (even a very small one), number might be represented only approximately—as a generalization gradient centered at the true numerosity. The habituation arising over a series of trials would occur not only to the true numerosity but also—to a lesser extent—to surrounding numerosities. When the new number is presented on the dishabituation trial, it also might be represented as a generalization gradient centered at the true numerosity. Longer looking times would occur on dishabituation trials in which the representation of the new number differs sufficiently from the representation formed on the habituation trials. (Various process models might be developed to produce the observed behavior, as noted in the Discussion section. Our intention here is simply to point out that the data on infants' looking behaviors could be based on approximate rather than exact representations of the numbers shown.)

More recently, evidence has been presented that infants are sensitive to changes in numerosity. Wynn (1992), using a "surprise" paradigm, found that if infants are shown an array that is then hidden and are then shown an item being added to or taken away from that hidden array, they look longer if the wrong rather than the right size array is revealed. This result, like the results of habituation studies, is shown only in data that is averaged over trials and over infants. Wynn argues from her data that infants possess "true number concepts" and that they do "numerical reasoning." In short, she believes that exact representations of number and number transformation underlie their behavior. Possibly, Wynn is correct and infants calculate correctly from exact number representations. Alternatively, however, it is possible that infants tend to notice the direction of the transformation but form only approximate representations of the number of items in the initial set and the number of items in the transformation. That is, the existing data are consistent with the possibility that longer looking times to incorrect answers occur on only some percentage of trials, based on an approximating mechanism.

Looking paradigms may not be sufficiently sensitive to permit assessment of the hypothesis that infants can represent small numbers and number transformations exactly. That is, it may not be possible to determine from differential looking times that infants use exact representations of number even if they do so. Tasks that can be given to toddlers in which they indicate the exact answer by choosing between two arrays that differ by one or by constructing a set with the proper number of objects make it possible, at least in principle, to determine how accurately children calculate, whereas a looking time measure may not.

Toddler Studies

If infants do possess a mental model for arithmetic, then some time before 2 years of age, when they become able to do choice or construction tasks, they should be able to reliably produce correct answers. That is, if children have possessed exact numerical skills since infancy, they should be able to demonstrate those skills in choice or construction tasks. Two studies have examined the ability of children under 2 years to represent numerosity and to transform arrays (Sophian & Adams, 1987; Starkey, 1992). Neither of these studies provided convincing evidence of 2-year-olds' ability to calculate.

Sophian and Adams (1987) studied children from 14 to 24 months of age, using a technique similar to one introduced by Brush (1978) with older children. They presented subjects with two sets of objects and then covered the sets. It was assumed that children would choose the larger set if they knew the number of objects. On the baseline task, children chose between untransformed sets with one versus two objects. Children did not reliably choose the set with more (two) items at 24 months, although they did so at 28 months. On the other task, one of the sets was transformed by adding an object or by taking away an object. For some of the transformation problems the initial sets were equal (i.e., both contained one item), and for others they were unequal (i.e., two vs. none). The authors claimed that children in every age group tested could calculate, but the data do not support the claim that the children reliably answer correctly. Children up to 24 months tended to choose the set the experimenter manipulated, regardless of whether items were added or taken away and regardless of whether the resultant set was larger or smaller. The critical problem was one in which an object was added to a set, but in which the set remained the smaller of the two sets. Even the oldest subjects (28-month-olds) performed only slightly above chance on this problem (.60). Cooper (1984), using a similar method, concluded that 2-year-olds do not understand that the initial numerosity of a set is important for predicting the effect of the addition or subtraction of terms to that set.

Starkey (1992) gave young children nonverbal calculation problems. In his task, a child placed a set of objects one at a time into an opaque box and then watched items being added to or removed from that set. The child's task was to remove all the objects from the box. The test of whether a child could calculate was whether the number of reaches equaled the number of objects in the set. On each trial, the set of items was secretly removed from the box and a single item was replaced before the child reached each time. Presumably this procedure was initiated so the child could remove only one object at a time and would not discover, during a reach, that more objects were present in the box. However, this procedure gives rise to a potential problem. That is, children could determine by touch, during a reach, that there were no other objects in the box to be removed (which would lead them not to reach again if they used touch as a cue).

The data from Starkey's calculation task provide considerable grounds to worry that touch cues (i.e., discovering that no other objects are present during a reach) led to score inflation. The calculation task was given to children from 18 months to 4 years of age. Across age, problems were easier when the answer was smaller, even when this would not be expected, because the numerosities involved were the same (e.g., for $4 - 3$, the percent correct was 64%, whereas for $4 - 1$, it was only 14%). Such a result would be puzzling if children were calculating but would be expected if children were using touch cues, at least in part, to determine the number of reaches. Starkey's results showed that by 24 months of age subjects did very well on the easiest calculation problems, $1 + 1$ and $2 - 1$. Given the problems with his procedure, the question arises of whether children's high scores could reflect use of touch as a cue together with approximate number abilities. Suppose children note that the initial array has some items, but not the exact number, and that they further note the direction but not the exact number of items involved in the transformation. For $1 + 1$, they should expect more than one item in the final array and therefore might not use the cue from touch on the first reach. Having no further expectations, they might rely on touch on the second reach, leading to the correct answer. For $2 - 1$, they should not expect more than one item in the final array, and hence might use the cue from touch on the first reach, leading to the correct answer. This hypothetical scenario may or may not explain children's success on these problems, but the fact that it *might* do so suggests the need for further work with less ambiguous tasks.

Mental Models and Intelligence

We have suggested that nonverbal calculation is based on a domain-general symbolic ability that makes it possible to abstract information from situations. In this case, nonverbal calculation ability should be substantially related to general intelligence level. In contrast, if nonverbal calculation is a modular domain-specific competence that is innately available to the species, there would be no reason to expect a strong relation to general intelligence level. This argument has been made for syntactic development. Lenneberg (1967) first noted that, insofar as language acquisition is based on a distinct and species-wide "device" or capacity, it should be exhibited in a similar fashion across people who vary in ability. In support of this argument, he presented data showing syntactic competence even in children with severe mental disabilities. This argument continues to motivate research on syntactic development in populations of varying intelligence (e.g., Curtiss, 1988). There is not much empirical evidence concerning the relation between general intelligence and the sort of symbolic process we posit to underlie nonverbal calculation (a mental model). There *is* some evidence that children with general cognitive delays (in particular, children with Down's syndrome) are impaired in the development of symbolic play (Hill & McCune-Nicolich, 1981).

The Studies

We present below three studies in which we assessed young children's ability to calculate nonverbal tasks. We also assessed their ability to represent numerosity in the absence of a transformation. In the first two studies we assessed the development of nonverbal numerical abilities in children across the age period from 2 to 4 years. In the third study we compared the performance of 3-year-olds in regular preschool classes with that of 3-year-olds in special education classes for children with mild intellectual delays.

Study 1

As indicated above, existing findings do not provide clear evidence as to when children can reliably represent numerosity and calculate (with small numbers). Two issues must be addressed to deal adequately with the question. The first issue concerns the task. It is important to use a task that does not bias responding toward correct answers in the absence of calculation. The task we used in our earlier studies is adequate in this regard; children can put out any number of items on any trial, and no irrelevant cues are provided. In addition, it is important to use a task that does not introduce unnecessary complexities that might lead to failures in children who can calculate. Our task captures just the essential features of calculation: (a) the ability to represent the numerosity of an initial set and retain that representation at least briefly when the set is hidden (allowing an unseen transformation to be performed), (b) the ability to modify the initial representation to correspond to the addition or subtraction of a particular number of elements, and (c) the ability to demonstrate behaviorally their representation of the outcome. (In our task this involves constructing an array. It also is possible to have the child choose from a set of arrays that vary in number or to report the number verbally. Verbal reports, however, require knowledge of the conventional symbols of arithmetic as well as calculation, which is problematic for children from low socioeconomic groups; see Jordan, Huttenlocher, & Levine, in press).

The second issue is to distinguish between the pattern of responses that would be expected if children have an exact numerical competence to that which would be expected if their competence is one that does not, in principle, reliably yield correct answers. In characterizing an exact mechanism, consider the case in which there are no performance errors. An exact mechanism should reliably yield correct answers across some range of numerosities. At the least, it should apply to problems that involve numerosities up to 2, because changes from 0 to 1 or vice versa involve just presence versus absence. Thus, a child with an exact mechanism should get the problems $1 + 1$ and $2 - 1$ correct. If the exact mechanism applies across a greater range of numerosity, say, if it encompasses numerosities of 3, then in addition to getting $1 + 1$ and $2 - 1$ right, children should get the following problems right as well: $1 + 2$, $2 + 1$, $3 - 1$, and $3 - 2$, and so on for larger numerosities. If most children in an age group show the strict correspondence just described between the numerosity of a problem and whether they get problems correct, and if few of them get scores of 0, that would constitute evidence for a mental model in which number and number transformation are represented exactly.

When representation is only approximate, as in animals, the probability of error also increases with numerosity (cf. Gallistel & Gelman, 1992). However, there would be many errors on all problems, and because of the inexactness across the entire range, the relation of problem difficulty to numerosity would be imperfect. In fact, given fairly wide generalization gradients for both the number in the original set and the number in the transformation, many children will get scores of 0. Hence if few children in a particular age group get all problems correct up to a particular numerosity and many of them receive scores of 0, there would be no reason to claim more than an approximate mechanism in that age group.

Finally, in order to calculate, children must be able to represent number exactly in the absence of transformation. Thus, children who can calculate up to some numerosity should be able to match up to at least that numerosity on a nontransformation task in which they are shown an array that is then hidden (e.g., if they calculate up to numerosity 3, they should also match sets of at least three items). As for calculation problems, they should show a strict ordering of matching problem difficulty by numerosity. In contrast, children in an age group that represents number only approximately should not show such a strict ordering, and there is no reason to expect that matching and calculation scores should be tightly linked, as should be the case with an exact mechanism.

Method

Subjects. Subjects were 180 children, 30 in each of six age groups (years; months): 2;6–2;8, 2;9–2;11, 3;0–3;2, 3;3–3;5, 3;6–3;8, and 3;9–3;11. The children in each age group were drawn from a broad range of social classes and ethnicities.[1] They were from urban and suburban areas within four geographical regions: East, South, Midwest, and West. Many were recruited at day care centers and preschools. There was an equal number of girls and boys in each age group. The nonverbal matching and calculation tasks were included in a larger set of tasks for a study of the development of a range of cognitive abilities in young children.

Materials and procedure. Children were tested individually. First they were given a nonverbal matching task and then a nonverbal calculation task. Materials for the matching and calculation tasks included two 25 cm × 25 cm white cardboard mats, a set of 20 black disks (1.9 cm in diameter), a box, and a cover. The cover had an opening on one side so that the experimenter could easily add or remove disks during the calculation task. The experimenter and the child sat on opposite sides of a table, each with a mat in front of her- or himself.

Matching task. Two matching demonstration items were provided before the matching test items. On the first demonstration item, the experimenter took 1 disk from the box and placed it on her mat in full view of the child. The disk was then hidden with the cover. The experimenter then put 1 disk on the child's mat and lifted the cover from her own mat so the child could see that the 2 mats had the same number of disks. The experimenter stated "See, yours is just like mine," pointing to the disks on both mats. The demonstration item was presented again, following the same procedure, except this time the child was asked to place the appropriate number of disks on his or her mat after the disk was hidden. If the child did not respond or placed the wrong number of disks on the mat, he or she was corrected and the item was repeated one more time. The same demonstration procedure was used with 2 disks.

After the demonstration items, the child matched the quantities of five test items, ranging in numerosity from 1 to 5. The matching test items were presented in a fixed random order. For each test item, the experimenter took a set of disks from the box and placed it on her mat in full view of the child. The disk set was then hidden with the cover. The child's task was to indicate how many disks were under the cover by placing the appropriate number of disks on his or her mat. Each set was presented in a horizontal linear array. The total possible score on the matching task ranged from 0 to 5.

Calculation task. There was one demonstration item for addition and one for subtraction. For the addition demonstration item (1 + 1), the experimenter placed 1 disk on her mat in full view of the child. This disk was then hidden with the cover. The experimenter then slid another disk under the cover, in full view of the child. Next, the experimenter placed 2 disks in a horizontal line on the child's mat and lifted the cover to show the 2 disks on her mat, saying "See, yours is just like mine." The addition demonstration item was presented again, following the same procedure, except this time the child was asked to place the appropriate number of disks on his or her mat after the experimenter made the transformation. If the child did not respond or placed the wrong number of disks on the mat, he or she was corrected and the item was repeated one more time. A parallel demonstration procedure was completed with a subtraction problem (2 − 1), but in this case the disk was removed from under the cover. (We initially believed that demonstrations were necessary to explain the task. We realized that this might elevate scores on the test items 1 + 1 and 2 − 1 that also had been demonstrated. Study 2, however, in which no demonstrations were used, shows that this was not a substantial variable.)

The calculation test items were presented after the demonstration items. For the addition test items, the experimenter first placed the set of disks comprising the augend in full view on the mat of the child and then covered it. She then put the set of disks comprising the addend in a horizontal line next to the cover and then slid them under the cover, all at once. The augend and the addend were never in view at the same time. The child indicated how many disks were under the cover by placing the appropriate number of disks on his or her mat. A comparable procedure was used for subtraction, but in this case the disks comprising the subtrahend were removed from under the cover all at once. There were 7 addition problems and 5 subtraction problems. For addition problems, the numerosities of the sums were no greater than 5. For subtraction problems, the numerosities of the minuends were no greater than 4. The order of presentation proceeded from problems of lower numerosity to problems of higher numerosity. The addition and subtraction problems were intermixed. The total possible score on the calculation task ranged from 0 to 12.

Results

Matching task. Table 1 shows the rank order of difficulty of matching items, indicating the percentage of children who answered correctly for each item within each age group. For all age groups, the proportion of matching items that were answered correctly was greater for items with smaller numerosities (1 and 2) than with larger numerosities (3, 4, and 5). Table 2 (second column) shows the mean matching scores for each age group. An analysis of variance (ANOVA) on children's matching scores with age group and sex as between-subjects variables revealed a significant main effect of age group, $F(5, 168) = 4.51$, $p < .001$. Contrasts revealed a significant linear trend, ($p < .0001$), and no higher order trends. There was no main effect or interaction involving sex.

Table 2 also shows the number of children in each age group who received a score of 0 and the extent to which children in each age group could match numbers up to some numerosity (i.e., the child made no errors on problems involving that numerosity as well as no errors on problems involving any lower numerosities). The columns on the right side of the table show the number of children whose correct answers encompass numerosities of at least 1, 2, 3, 4, and 5. In all age groups, most children could match up to a numerosity of at least 2. The number encompassed increased with age. More than half of the 3-year-olds and about one third of the 2-year-olds could match up to a numerosity of at least 3. The data clearly indicate that the majority of children in these age groups can represent small number sets exactly when transformations are not involved.

Calculation task. Table 3 shows the rank order of difficulty of the calculation problems, indicating the percent-

[1] Susan Cohen Levine, Janellen Huttenlocher, and Peter R. Huttenlocher used this set of tasks to construct an instrument to assess early cognitive development. Detailed information concerning this instrument and participant recruitment is presented in the final report of a National Institutes of Health Phase 1 Small Business Grant, No. 1R43HD30034-01.

Table 1
Rank Order of Matching Items (Easiest to Most Difficult) and Percentage of Items That Were Solved Correctly in Study 1

Numerosity of matching item	Age group 2;6–2;8	2;9–2;11	3;0–3;2	3;3–3;5	3;6–3;8	3;9–3;11	Total group
2	90	77	87	93	97	97	90
1	77	77	90	90	97	93	87
3	37	40	50	60	60	84	55
4	27	30	37	40	30	57	37
5	20	7	23	13	27	30	20

age of trials that were solved correctly for each item in each age group. The numerosities the children could deal with successfully increased with age. The data show that problem difficulty for the group as a whole is ordered by numerosity. That is, across age groups, all problems of numerosity 2 are easier than all problems of numerosity 3, and these are in turn easier than all problems of numerosity 4.

Table 4 shows the mean calculation scores for each age group in the second column. An ANOVA on children's calculation scores, with age and sex as between-subjects variables, revealed a significant main effect of age group, $F(5, 168) = 15.44$, $p < .0001$. Contrasts revealed a significant linear trend, ($p < .0001$), and no higher order trends. There was no main effect or interaction involving sex. We examined whether children's errors consisted of reproductions of the initial array (e.g., $3 + 1 = 3$). Thirty-three percent of children's errors consisted of laying out the number of disks in the initial array, and this did not vary significantly with age.

To examine the effects of addition versus subtraction on children's scores, we performed an ANOVA with age as a between-subjects variable and operation as a within-subjects variable. In this analysis, we compared subtraction problems ($2 - 1$, $3 - 1$, $3 - 2$, $4 - 1$, and $4 - 3$) to addition problems of corresponding numerosities ($1 + 1$, $2 + 1$, $1 + 2$, $3 + 1$, and $1 + 3$), excluding the two addition problems that involved higher numerosities. There was no effect of operation for any age group (overall $M = 1.7$ for addition vs. 1.5 for subtraction), indicating that whether the transformation involves movement into versus out of an array was not critical to problem difficulty for these tasks.

Table 4 also shows the number of children who received a score of 0 for each age group and also the number of children in each age group who calculated correctly up to some numerosity (i.e., the child made no errors on problems involving that numerosity as well as no errors on any problems involving any lower numerosities). For the 3;9–3;11 group, no children received scores of 0, and 21 of the 30 children met the criterion for an exact mechanism that encompassed a numerosity of at least 2. Of these, 6 children also could do problems up to a numerosity of 3, of whom, in turn, 2 could also do problems to a numerosity of 4 and 1 could also do problems to a numerosity of 5 (perfect score). For the 3;6–3;8 group and the 3;3–3;5 age group, only 3 children got a score of 0 and at least half could do problems through numerosity 2. Even in the 3;0–3;2 group, only 4 children received a score of 0, and 13 children could do problems through numerosity 2, two of whom could also do problems through numerosity 3. Some children in an age group with only approximate representations might get $1 + 1$ and $2 - 1$ right. However, almost half the children got these two problems correct, so it would seem that an exact mechanism is emerging. A major drop-off in performance occurred in children under 3 years. For children aged 2;9–2;11, fourteen children got a score of 0, and only 3 children could do problems through numerosity 2. For the 2;6–2;8 group, 15 children received a score of 0, and only 4 children could do problems through numerosity 2. It is possible that these few children possess an exact mechanism. However, because only about 10% of children in these age groups showed evidence of exact representation up to numerosity 2, it is not clear whether we should attribute a mental model to any children in this study who were under 3 years of age.

The ability to calculate depends on the ability to represent the number of items in the initial array. Thus, a child who can calculate up to numerosity 2 (solving $1 + 1$ and $2 - 1$ correctly) should be able to match at least up to that numerosity. In Table 5 we examine the relation of children's calculation and matching scores. The clustering of data in the upper right triangle of Table 5 shows that the vast majority of children (97%) were able to match at least up to the numerosity through which they could calculate. Of the 72 children who could calculate up to at least numerosity 2, only 5 received matching scores that were less than their calculation scores. Children's better matching than calcula-

Table 2
Exact Representations on Matching Problems by Age Group in Study 1

Age	No. correct *M*	No. correct *SD*	No. zero scores	No. children whose correct answers encompass a numerosity of at least: 1	2	3	4	5
2;6–2;8	2.5	1.2	2	23	23	10	03	02
2;9–2;11	2.3	1.3	5	24	23	12	03	01
3;0–3;2	2.9	1.3	1	26	23	14	07	04
3;3–3;5	3.0	1.0	1	25	24	15	07	01
3;6–3;8	3.1	1.2	0	29	28	17	07	05
3;9–3;11	3.6	1.1	0	28	28	25	16	06

Note. $N = 30$ in each age group.

Table 3
Rank Order of Calculation Items (Easiest to Most Difficult) and Percentage of Items That Were Solved Correctly in Study 1

Calculation item	Age group 2;6–2;8	2;9–2;11	3;0–3;2	3;3–3;5	3;6–3;8	3;9–3;11	Total
1 + 1	23	33	60	77	67	97	60
2 − 1	30	20	60	63	57	70	50
3 − 2	13	17	30	50	47	67	37
2 + 1	13	10	27	37	43	70	33
1 + 2	7	7	37	40	47	53	32
3 − 1	10	13	20	37	37	50	28
1 + 3	17	13	37	10	20	43	23
3 + 1	3	3	23	33	30	40	22
2 + 2	7	13	23	17	30	27	20
4 − 1	7	10	13	30	27	30	20
4 − 3	3	7	13	17	23	40	17
4 + 1	0	3	17	7	17	30	12

tion performance also can be seen by comparing the most frequent calculation score (0) with the most frequent matching scores (2 and 3). It should be noted that all 7 children under 3 years of age who could calculate up to numerosity 2 could match at least up to that numerosity. These findings provide further support for the view that children who met our criterion for calculation possess an exact mechanism. That is, one would not expect such a tight linkage between matching and calculation if childrens' representations of numerosity were only approximate. (However, the results should be interpreted cautiously, because the matching data involve only one trial per numerosity, whereas the calculation data usually involve several trials. Thus, the probability of meeting the criteria for a particular numerosity is not equal for matching and calculation.)

Study 2

In this study, we examined nonverbal calculation in children under 3 years of age, using somewhat different conditions than in our first study. That study differed from a typical experimental study in that children were given a whole set of cognitive tasks. It is possible that children were less attentive to the matching and calculation tasks because of this. Furthermore, the study differed from many experimental studies in the literature in that children were drawn from a national sample that included children of a wide range of social class and ability levels. We wanted to examine whether attentive children under 3 years of age could calculate when they were tested only on the nonverbal task and not on other cognitive tasks.

Table 4
Exact Representations on Calculation Problems by Age Group in Study 1

Age	No. correct *M*	No. correct *SD*	No. zero scores	No. children whose correct answers encompass a numerosity of at least: 2	3	4	5
2;6–2;8	1.3	1.7	15	4	0	0	0
2;9–2;11	1.5	1.7	14	3	0	0	0
3;0–3;2	3.6	2.9	4	13	2	0	0
3;3–3;5	4.2	2.7	3	17	4	0	0
3;6–3;8	4.4	3.4	3	15	6	1	1
3;9–3;11	6.2	2.9	0	21	6	2	1

Note. $N = 30$ in each age group.

Method

Subjects. The subjects were 96 middle-class children, ranging in age from 2 years, 0 months to 2 years, 11 months. There were 24 children in each of the four age groups (years;months): 2;0–2;2, 2;3–2;5, 2;6–2;8, and 2;9–2;11. In each age group, half of the children were boys and half were girls. The subjects were drawn from a single academic community (Hyde Park in Chicago) rather than from a heterogeneous national population of children. They were recruited from playgrounds and were eager to participate in our "game."

Materials and procedure. Children were given six calculation problems. Only small numerosity items were used (sums or minuends of 3 or less). The calculation procedure was identical in format to the one described in Study 1. However, in Study 2 no demonstration items were repeated as test items. We used matching items (with numerosities of 1 and 3, respectively) to demonstrate the general procedure for the nonverbal task. Our previous work indicates that demonstration of matching is sufficient for children to learn the procedure for calculation (Jordan et al., 1992).

Results

Let us consider whether children under 3 years of age can calculate. Table 6 shows the rank order of calculation problems over the entire 2;0–2;11 age range and the percentage of correct answers for each calculation item by age group. Note that the order of problem difficulty for the group as a whole corresponds to numerosity (i.e., problems involving sums or minuends of 2 were easier than problems involving sums or minuends of 3). When we consider the age groups

Table 5
Number of Children Across All Age Groups Showing an Exact Representation on Matching and Calculation Problems Up to a Particular Numerosity

Calculation	Matching						Total
	0	1	2	3	4	5	
0	20	6	45	24	7	6	108
2	2	0	10	23	12	9	56
3	1	0	1	5	4	2	13
4	0	0	0	0	0	1	1
5	0	0	0	0	1	1	2
Total	23	6	56	52	24	19	180

Note. $N = 180$.

separately, however, this finding is consistent only for the 2;9–2;11 children, suggesting the use of an approximate rather than an exact mechanism by many children before this age.

Table 7 displays data for Study 2 in a manner parallel to that for Study 1. The mean calculation scores are shown in the second column. An ANOVA on children's calculation scores with age and sex as between-subjects variables revealed a significant effect of age, $F(3, 88) = 11.3$, $p < .0001$. Contrasts revealed a significant linear trend ($p < .0001$) and no higher order trends.

Consider now the evidence relevant to whether any of these children exhibit a mental model. The number of 2½- to 3-year-old children in Study 2 who received a score of 0 was substantially less than in Study 1. In the 2;9–2;11 group, 3 (out of 24) children got scores of 0, and in the 2;6–2;8 group, 4 (out of 24) children got scores of 0. Furthermore, the percentage of 2½- to 3-year-old children in Study 2 who met the criterion for a mental model was greater than in Study 1. Table 7 shows that for the 2;9–2;11 group, over half of the children got 1 + 1 and 2 − 1 right, and 1 of these children met the criterion for a mental model up to a numerosity of 3. In the 2;6–2;8 group, close to one third of the children got 1 + 1 and 2 − 1 correct. Hence, the data provide evidence that some children between 2½ and 3 years of age possess a mental model.

In contrast to children over 2½ years of age, the data do not provide evidence that children under 2½ years possess a mental model. Table 7 shows that for the 2;3–2;5 group, 8 (out of 24) children received a score of 0, and only 1 child got 1 + 1 and 2 − 1 correct. For the 2;0–2;2 group, well over half (16 out of 24) had a score of 0, and none got both 1 + 1 and 2 − 1 correct.

Finally, let us consider further the data for children aged 2;0–2;5 where evidence of an exact mechanism was absent. Because even infants exhibit approximate representations, we wanted to determine if our subjects showed use of such an approximate mechanism rather than simply responding at chance. First, if children used an approximate mechanism rather than choosing at random, their success should be greater on lower than on higher numerosity problems. Therefore, we divided the problems into those involving sums and minuends of 2 and those involving sums and minuends of 3. A repeated measures ANOVA on percent correct with numerosity (2 vs. 3) as a variable showed that children performed significantly better on problems involving a numerosity of 2 than those involving a numerosity of 3, $F(1, 47) = 4.21$, $p < .05$. Second, if children used an approximate mechanism rather than responding at chance, the number of disks laid out on individual problems should vary systematically with the correct answer. Thus, we sorted the items according to their answers (i.e., 1, 2, and 3) and conducted a repeated measures ANOVA to determine if the mean number of disks children put out for answers increased with the numerosity of the answer. There was a significant effect of numerosity, $F(2, 90) = 9.62$, $p < .002$. Contrasts showed no significant difference in the number of disks put out when the answer was 1 versus 2. However, the mean number of disks put out was greater when the answer was 3 than when the answer was 1, $p < .003$, or 2, $p < .002$. The two analyses above indicate that children between 2;0 and 2;5 were not simply responding randomly on these tasks.

Study 3

We have proposed that nonverbal calculation involves abstraction of the numerically relevant aspects of situations—namely, the number of entities in the initial array, the number in the transformation, and the direction of movement with respect to the initial array. The ability to calculate on the nonverbal task provides an index of young children's intelligence in the number domain. In fact, as we noted in the introduction, nonverbal calculation might provide a better index of intelligence than does verbal calculation, because the latter requires the acquisition of conventional symbols as well as the ability to calculate. In our previous work, social class did not affect performance on nonverbal calculation problems (Jordan et al., 1992). Ginsburg and Russell (1981) have reported similar findings. Although the literature on measured intelligence shows a relation to social class, that relation is far from perfect and may reflect, at least in part, differences in the environments of children in different social classes. If nonverbal calculation constitutes an innately available modular competence rather than a general symbolic capacity, then the ability might be similar across a wide range of intelligence, as Lenneberg (1967)

Table 6
Rank Order of Calculation Items (Easiest to Most Difficult) and Percentage of Items That Were Solved Correctly in Study 2

Calculation item	Percentage correct				Total group
	2;0–2;2	2;3–2;5	2;6–2;8	2;9–2;11	
1 + 1	13	25	67	63	42
2 − 1	13	33	38	63	37
3 − 1	8	25	58	42	33
2 + 1	21	13	29	42	26
3 − 2	4	21	21	46	23
1 + 2	8	17	17	29	18

Table 7
Exact Representation on Calculation Problems by Age Group in Study 2

Age	No. correct		No. zero scores	No. children whose correct answers encompass a numerosity of at least:	
	M	*SD*		2	3
2;0–2;2	0.7	0.9	16	0	0
2;3–2;5	1.3	1.2	8	1	0
2;6–2;8	2.3	1.5	4	7	0
2;9–2;11	2.8	1.8	3	13	1

Note. $N = 24$ in each age group.

argued for syntactic development. According to this view, there would be no reason to expect a strong relation to general intelligence. To address this issue, we next compared performance of 3-year-olds in regular preschool classes with that of 3-year-olds in special education classes for children with mild general intellectual delays.

Method

Subjects. Subjects were 100 children between 3 and 4 years of age. All children were from a single middle-class suburban community. Half of the children (the special group) were in a special program designed for preschool children identified as having mild developmental delays on the basis of informal criteria (e.g., teacher recommendation, scores on screening measures); children with moderate to severe intellectual delays were served separately. The other half of the children (the control group) were in regular preschool programs. All of the children were tested as a part of a larger study the purpose of which was to determine those aspects of cognitive function that differentiate preschool children suspected as having mild intellectual delays from normally functioning preschool children. We tested all children in the special program for 3-year-olds during a single academic year. The mean age of the group was 3 years, 9 months ($SD = 4.6$ months). There were more boys than girls in the special program (34 boys and 16 girls). The control children were matched with the children in the special program for age and sex and came from regular preschools in the same community.

Materials and procedure. Children were given the nonverbal matching and calculation tasks. The test procedure was the same as the one described in Study 1. Children also were given other cognitive tasks, such as tests of language and neuromotor ability. For the matching task, test items included sets of 3, 4, and 5 disks. For the calculation task, the problems involved sums of 5 or less and minuends of 4 or less.

Results

The rank order of the matching items is shown in Table 8. Although almost all children in regular classes could match 3 items (the easiest problem), only 66% of children in special classes could do so. The mean matching scores were 1.8 ($SD = 0.9$) for the control children and 1.1 ($SD = 1.0$) for the children in the special program. The difference between these means was significant, $t(98) = 3.73$, $p < .0001$, two-tailed.

The rank order of the calculation items is shown in Table 9. The calculation item involving the smallest numerosity ($3 - 1$) was easiest for the control children but not for the special children. The mean nonverbal calculation scores (out of 5) were 2.3 for the control children and 1.1 (*SD*s = 1.4 and 1.0, respectively) for the special children. The difference between these means was significant, $t(98) = 4.95$, $p < .0001$, two-tailed.

In interpreting the relation of nonverbal calculation and matching tasks to intellectual level in 3-year-old children, one should note that the decision to place children in special preschool classes did not include observations of their calculation ability. The placement was based instead on children's language, motor, social abilities, or some combination of these. Interestingly, the language task (including vocabulary and sentence comprehension) and the neuromotor task, each of which contained many more items than the calculation or matching tasks, showed somewhat less striking (although significant) differences between these two groups of children.

General Discussion

In the present studies, we examined young children's ability to do a nonverbal calculation task in which they were shown items being added to or subtracted from a hidden array. To reliably get the answers, children must preserve the exact number of items in the original array and determine the effects on set size of moving particular numbers of items into or out of that array. In our previous work we showed that the ability to do these tasks does not require mastery of the conventional symbols of arithmetic. Children could do such nonverbal calculation tasks at a younger age than they could do verbal calculation tasks (story problems and number–fact problems; Levine et al., 1992). Furthermore, although verbal calculation ability varied with social class, nonverbal calculation ability did not (Jordan et al., 1992). In a more recent study, Jordan et al. (in press) found that there are young low-income children who could do low numerosity nonverbal calculation problems even though they could not count or answer with a number word.

We have suggested that the ability to solve these problems reliably involves a representation (a mental model) in which imagined entities and transformations are mapped onto actual objects and their movements. Such a model

Table 8
Rank Order of Matching Problems and Percentage of Items That Were Solved for Each Ability Group in Study 3

Numerosity	Ability group	
	Control	Special
3	94	66
4	52	22
5	34	24

Table 9
Rank Order of Calculation Items and Percentage of Items That Were Solved Correctly for Each Ability Group in Study 3

Calculation item	Ability group	
	Control	Special
3 − 1	66	30
3 + 1	54	32
4 − 2	40	14
4 − 1	34	18
3 + 2	34	14

provides a formal mechanism, which, when correctly applied, necessarily yields the answer. We argued that nonverbal calculation is similar in certain ways to a variety of other skills requiring symbolic processes (but not conventional symbols) that arise in the 2- to 3-year age period and would not emerge until that time. Furthermore, we expected that nonverbal calculation ability would be related to general intelligence level.

In contrast to such predictions, it has been claimed that the ability to represent number and even change in number, is an innately available modular ability. In this view, nonverbal calculation ability should be present in infancy and in children across a broad range of intelligence levels. Although the existing literature on infants shows that they are sensitive to number, the results are consistent with either of two possibilities—that infants represent number exactly but are inattentive subjects, or that they represent number only approximately. Furthermore, the existing literature does not even provide evidence that toddlers can calculate.

In the first two experiments in the present article we assessed nonverbal calculation ability in children from 2 to 4 years of age, using the task from our earlier studies. Evidence for a reliable ability to calculate only became clear-cut after 2½ years. Before 2½ years, there was evidence that children possess an approximate mechanism. In the third experiment, we assessed nonverbal calculation ability in young children who varied in general intellectual competence. We found substantial differences in performance on nonverbal calculation tasks among 3-year-olds whose intelligence levels differed. If nonverbal calculation ability indeed emerges from approximate representations in infancy, it is important to understand what those approximations consist of and what is involved in the development of an "exact" mechanism (mental model). Let us discuss these issues briefly.

Approximate Representation

In a recent article, Gallistel and Gelman (1992) suggested that the innately available number ability in infants might be comparable to that in animals. Let us consider what the number abilities of animals consist of and what mechanisms have been proposed to explain these abilities. One should note at the start that there are striking differences between the procedures in infant studies, which generally involve small spatial arrays, and those in animal studies. In the animal experiments discussed by Gallistel and Gelman, number ability was assessed for temporal arrays with fairly large numbers (animals must respond to a certain number of bursts of sound or must execute a certain number of bar presses). The findings of these animal studies indicate a sensitivity to number across a broad range of numerosities, up to at least 24. As Gallistel and Gelman reported, Mechner (1958) showed that rats can approximate a target number (of bar presses) across values ranging from 4 through 16, and Platt and Johnson (1971) showed that rats can approximate a target number across a range from 4 through 24. Although the animals' modal responses lie at the true number, there is a gradient of responses around the true number, and the variability of the responses increases with the numerosity of a set.

Gallistel and Gelman (1992) suggested that a model of the number ability of animals proposed by Meck and Church (1983) might be applicable to infants. In this model, numerosity is represented by an accumulator of mental magnitudes. This mechanism, which is hypothesized to underlie the discrimination of quantity, does not distinguish between discrete and continuous quantities. The mechanism accumulates quantity by "jumping ahead" a step after particular numbers of items or durations. Inexactness in the representation of discrete quantities might arise either in the process of extracting number (i.e., the accumulator might sometimes fail to register an item or might jump ahead by two) or in the translation process (i.e., there might be failures in establishing a correct durable mental representation from the output of the accumulator; cf. Broadbent, Rakitin, Church, and Meck, 1992).

Emergence of a Mental Model

The claims of the accumulator model are of interest in considering the relation between approximate and exact representations of quantity that may distinguish infants' numerical abilities from those of early childhood. First, the notion of a mechanism that does not distinguish between continuous and discrete quantities provides a context for thinking about approximate representation. For continuous quantity, it is, in principle, only possible to approximate amount to some degree of precision. The notion of an exact representation of quantity applies only to discrete entities; it rests on a one-to-one mapping between those entities and mental representatives of those entities. Hence, it is plausible to argue that continuous and discontinuous quantity become differentiated only when such a mapping process emerges. Second, the notion of a translation process that forms a durable representation from the output of the accumulator provides a context for thinking about the emergence of exact representation. That is, the direct output of the accumulator may be approximate unless a symbolic translation process is available to map that output onto a set of distinct mental elements. Thus, continuous and discontinuous quantities may become differentiated only when such a symbolic process emerges.

Despite the plausibility of the notion that it is symbolic processes that underlie the differentiation between continuous and discontinuous quantity, this may not be quite right. Although it seems possible that discrete versus continuous sound bursts might not be differentiated, it seems unlikely that small sets of discrete items are not differentiated from amounts of a continuous quantity. Infants appear to be sensitive to objects as discrete entities. Thus, infant representations may well involve discrete objects, even if, as for adults' representations of large sets when they do not count, the number of discrete objects is not exactly represented. Our claim is that it is the appearance of the ability to do nonverbal calculation reliably in this age range that is driven by newly emerging symbolic processes. The notion in the accumulator model of a translation process requiring symbols is clearly consistent with this view.

Finally, one should note that, although our findings strongly support the claim that a mental model underlies the acquisition of exact nonverbal calculation ability, one could argue that this ability does not necessarily depend on the emergence of a new set of symbolic processes. One alternative, as we have noted, is that a one-to-one mechanism operates even in infancy, but that infants make many errors because they are inattentive subjects. Another alternative is that number ability remains approximate in the age range when children's calculation becomes accurate for small numbers, but that the generalization gradient around the true numerosity becomes increasingly narrow, so that the distributions for those small numbers become nonoverlapping (distinct). In any case, as Gallistel and Gelman (1992) argued, the approximate numerical competence of infancy provides "the framework—the underlying conceptual scheme—that makes it possible for the young child to understand and assimilate verbal numerical reasoning" (pp. 65–66). The present studies show that there is an intermediate ability between approximate competencies in infancy and the acquisition of conventional symbols, namely, an ability to do nonverbal calculation with small numbers.

Conclusions

In conclusion, the results of the present study, together with earlier findings, suggest the following view of the development of number ability. From the start, human beings can form approximate representations of very small numbers. These representations might involve either an accumulation of quantity that is not yet differentiated into discrete objects or an accumulation of discrete objects that is not yet enumerated. In either case, these representations may be thought of as the output of something like Meck and Church's (1993) accumulator. As children acquire symbolic processes, the reading off of the output of the accumulator becomes exact for very small numbers. These processes make it possible to form an explicit representation of the number of items in an array as well as of the exact effects on number of the movement of items into or out of an array. The emergence of this symbolic ability (a mental model) does not depend on the acquisition of conventional symbols; it emerges after the approximate skills of infancy and before the conventional skills of school-age children. We have found that the development of this intermediate ability is related to the underlying intellectual competence of the child but not to conventional training as reflected by such variables as social class. Development of the ability to deal with large numbers surely requires the conventional symbols of arithmetic, such as mastery of counting to establish how many items are present, the ability to "carry" numbers by adding or subtracting, and so on. Thus, even though a basic understanding of number and the operations that transform number seem to originate without conventional input, further development is no doubt closely linked to the child's exposure to relevant conventional input.

References

Antell, S. R., & Keating, D. (1983). Perception of numerical invariance by neonates. *Child Development, 54,* 695–701.

Barsalou, L. (1992). *Cognitive psychology: An overview for cognitive scientists.* Hillsdale, NJ: Erlbaum.

Bee, H. L., Van Egeren, L. F., Streissguth, A. P., Nyman, B. A., & Leckie, M. (1969). Social class differences in maternal teaching and speech patterns. *Developmental Psychology, 1,* 726–734.

Broadbent, H., Rakitin, B. C., Church, R. M., & Meck, W. H. (1992). Quantitative relations between timing and counting. In S. T. Boysen & E. J. Capaldi (Eds.), *The development of numerical competence: Animal and human models* (pp. 171–187). Hillsdale, NJ: Erlbaum.

Brush, L. R. (1978). Preschool children's knowledge of addition and subtraction. *Journal for Research in Mathematics Education, 9,* 44–54.

Cooper, R. G. (1984). Early number development: Discovering number space with addition and subtraction. In C. Sophian (Ed.), *Origins of cognitive skills* (pp. 157–192). Hillsdale, NJ: Erlbaum.

Curtiss, S. (1988). Abnormal language acquisition and the modularity of language. In F. Newmeyer (Ed.), *Linguistics: The Cambridge survey: Vol. II. Grammatical theory: Extensions and implications* (pp. 96–117). Cambridge, England: Cambridge University Press.

DeLoache, J. S. (1987). Rapid change in the symbolic functioning of very young children. *Science, 238,* 1556–1557.

DeLoache, J. S. (1991). Symbolic functioning in very young children: Understanding of picture models. *Child Development, 62,* 736–752.

Fischer, K., & Biddell, T. (1991). Constraining nativist inferences about cognitive capacities. In S. Carey & R. Gelman (Eds.), *The epigenesis of mind: Essays on biology and cognition* (pp. 199–236). Hillsdale, NJ: Erlbaum.

Gallistel, C. R., & Gelman, R. (1992). Preverbal and verbal counting and computation. *Cognition, 44,* 48–74.

Gentner, D., & Stevens, A. (Eds.). (1983). *Mental models.* Hillsdale, NJ: Erlbaum.

Ginsburg, H. P., & Russell, R. L. (1981). Social class and racial influences on early mathematical thinking. *Monographs of the Society for Research in Child Development, 46*(6, Serial No. 193).

Hill, P. M., & McCune-Nicolich, L. (1981). Pretend play and patterns of cognition in Down's syndrome children. *Child Development, 52,* 611–617.

Huttenlocher, J. (1973). Language and thought. In G. Miller (Ed.),

Language and communication (pp. 172–184). New York: Basic Books.
Huttenlocher, J. (1976). Language and intelligence. In L. Resnick (Ed.), *New approaches to intelligence* (pp. 261–281). Hillsdale, NJ: Erlbaum.
Johnson-Laird, P. N. (1983). *Mental models.* Cambridge, MA: Harvard University Press.
Jordan, N. C., Huttenlocher, J., & Levine, S. C. (1992). Differential calculation abilities in young children from middle- and low-income families. *Developmental Psychology, 28,* 644–653.
Jordan, N. C., Huttenlocher, J., & Levine, S. C. (in press). Assessing early arithmetic abilities: Effects of verbal and nonverbal response types on the calculation performance of middle- and low-income children. *Learning and Individual Differences.*
Kellman, P. (1988). Theories of perception and research in perceptual development. In A. Yonas (Ed.), *Minnesota Symposium on Child Psychology* (pp. 267–281). Hillsdale, NJ: Erlbaum.
Kirk, G. E., Hunt, J., & Volkmar, F. (1975). Social class and preschool language skill: V. Cognitive and semantic mastery of number. *Genetic Psychology Monographs, 92,* 131–153.
Lenneberg, E. (1967). *Biological foundations of language.* New York: Wiley.
Levine, S. C., Jordan, N. C., & Huttenlocher, J. (1992). Development of calculation abilities in young children. *Journal of Experimental Child Psychology, 53,* 72–103.
McCune-Nicolich, L. (1981). Toward symbolic functioning: Structure of early pretend games and potential parallels with language. *Child Development, 52,* 785–797.
Mechner, F. (1958). Probability relations within response sequences under ratio reinforcement. *Journal of the Experimental Analysis of Behavior, 1,* 109–122.
Meck, W. H., & Church, R. M. (1983). A mode control model of counting and timing processes. *Journal of Experimental Psychology: Animal Behavior Processes, 9,* 320–334.
Platt, J. R., & Johnson, D. M. (1971). Localization of position within a homogeneous behavior chain: Effects of error contingencies. *Learning and Motivation, 2,* 386–414.
Shepard, R., & Metzler, S. (1971). Mental rotation of three-dimensional objects. *Science, 171,* 701–703.
Siegler, R. S. (1992). Cheers and lamentations. In C. E. Granrud (Ed.), *Visual perception and cognition in infancy* (pp. 333–344). Hillsdale, NJ: Erlbaum.
Siegler, R. S., & Jenkins, E. (1989). *How children discover new strategies.* Hillsdale, NJ: Erlbaum.
Siegler, R. S., & Shrager, J. (1984). Strategy choices in addition and subtraction: How do children know what to do? In L. Sophian (Ed.), *Origins of cognitive skills* (pp. 227–287). Hillsdale, NJ: Erlbaum.
Sophian, C., & Adams, N. (1987). Infants' understanding of numerical transformations. *British Journal of Developmental Psychology, 5,* 257–264.
Starkey, P. (1992). The early development of numerical reasoning. *Cognition, 43,* 93–126.
Starkey, P., & Cooper, R. G., Jr. (1980). Perception of numbers by human infants. *Science, 210,* 1033–1035.
Wynn, K. (1992). Addition and subtraction by human infants. *Nature, 27,* 749–750.

Received May 19, 1993
Revision received April 8, 1994
Accepted April 11, 1994 ■