JOURNAL OF EXPERIMENTAL CHILD PSYCHOLOGY **53**, 72–103 (1992)

Development of Calculation Abilities in Young Children

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This study investigates the development of skills for solving verbally and nonverbally presented calculation problems in children between 4 and 6 years of age. Identical addition and subtraction calculations were presented in three problem-type formats: nonverbal problems, story problems, and number-fact problems. The nonverbal problems involved presenting sets of physical referents that were then transformed either by adding or removing elements. The child saw the initial set and the number of elements that were added or removed, but not the final set. The task was to construct an array that contained the number of elements in the final set. The story problems and number-fact problems were presented orally, without props. Results indicate that children as young as 4 years of age have some success on the nonverbal problems, showing that they can transform sets by adding or subtracting elements. In contrast, children do not achieve comparable levels of success on the story problems or number-fact problems until 5½ to 6½ years of age. Moreover, throughout the age range tested, children performed better on nonverbal problems than on either story problems or number-fact problems. These results suggest that children's earliest ability to add and subtract is based on experiences combining and separating sets of objects in the world and that this ability precedes the development of conventional verbal methods of calculating. © 1992 Academic Press, Inc.

The authors thank the Spencer Foundation and the Smart Family Foundation for their generous support of this research. They also thank two anonymous reviewers for their input on an earlier version of the manuscript, Amy Taylor for her help with analyses, and Peter Rosenberger who long ago suggested the idea of a nonverbal calculation task to Janellen Huttenlocher. Address correspondence and reprint requests to Susan C. Levine, Department of Psychology, University of Chicago, 5848 S. University Avenue, Chicago, IL 60637.

0022-0965/92 $3.00

During the preschool years, children acquire a number of quantitative abilities that are relevant to formal addition and subtraction calculation skills. For example, they are able to determine which of two sets contains more elements than the other, they can discriminate between particular quantities (e.g., 2 vs 3) and they have some understanding that spatially rearranging a set of objects preserves quantity (Cooper, 1984; Gelman, 1972; Gelman & Gallistel, 1978). Further, young children can enumerate sets of objects and recognize that the final number used in counting represents the number of objects in the set (Fuson, Richards, & Briars, 1982; Gelman & Gallistel, 1978).

These abilities do not involve the essential requirement of calculation, i.e., the transformation of sets by adding or subtracting elements. In fact, there is some indication that preschoolers have difficulty adding and subtracting. In particular, it has been reported that young children have limited success in performing verbally presented addition and subtraction calculations, such as story problems (e.g., "Paul has 3 marbles. He got 2 more. How many marbles does he have altogether?") and number-fact problems ("How much is 3 and 2?"), at least when object referents are not present (e.g., Ginsburg & Russell, 1981; Williams, 1965).

Although these studies might be interpreted as evidence that preschoolers are not able to transform quantities by adding or subtracting, a number of factors other than a lack of understanding of the operations of addition and subtraction could limit the child's ability to solve story problems and number-fact problems. One of these is difficulty in comprehending particular linguistic terms used in verbally presented calculation problems (Gelman & Gallistel, 1978). For both story problems and number-fact problems the preschool child may have difficulty understanding number words, relational words (e.g., "more," "less"), or words for operations (e.g., "plus," "minus," "take away"). Such difficulties may result in the child failing to answer a story problem or number-fact problem correctly, even though he/she has the ability to transform quantities by applying addition or subtraction operations. Supporting the influence of linguistic factors on problem difficulty, variations in the syntactic and semantic characteristics of story problems have been shown to have significant effects on the ease of problem solution (e.g., Carpenter, Hiebert, & Moser, 1981; Riley, Greeno, & Heller, 1983). Memory limitations also may mask the young child's ability to calculate. In particular, Brainerd (1983) reports that working memory failures account for a greater proportion of 4- to 6-year-olds' calculation failures than processing errors. Moreover, he finds that improvements in calculation performance between children from 4 to 5 years of age and first graders are mainly attributable to improvements in working memory.

Accessing numerical representations of specific quantities from number words is another potential source of difficulty for young children trying to solve story problems and number-fact problems. For example, even if a child is able to label the numerosity of a set of four objects appropriately, he/she might have difficulty accessing a mental representation of four when presented with the word "four" in the absence of physical referents. Such a limitation would preclude solving story problems and number-fact problems unless the child had memorized the answer to a particular problem, such as $2 + 2 = 4$.

Thus, it is possible that the preschool child has some calculation abilities that are not apparent when story problems or number-fact problems are presented. Several studies indicate that children as young as 3 years of age understand that addition of elements increases the quantity of a set and subtraction of elements decreases the quantity of a set (Beilin, 1968; Brush, 1978; Cooper, 1984; Gelman, 1972; Mehler & Bever, 1967). In fact, one report in the literature presents data indicating that by 28 months of age, children are sensitive to addition or subtraction of a single element when very small sets are involved (Sophian & Adams, 1987). It should be noted, however, that all of these studies require the child to make a judgment of the relative numerosity of two sets and not to arrive at a quantitative solution to an addition or subtraction problem.

Several other studies report that preschool children are able to solve addition story problems when relevant sets of objects are presented along with the verbal input, although they are not able to do so in the absence of such props (Hebbeler, 1977; Ginsberg & Russell, 1981). Unfortunately, it is not clear that the children in these studies were actually calculating since the objects in both terms of the problem (i.e., augend and addend) were present simultaneously and could have been counted as a single set (Ginsburg & Russell, 1981). Thus, the essential requirement of calculation, transforming sets by adding or subtracting, may not have been tapped by these measures.

Starkey and Gelman (1982) avoided these problems by developing a calculation task with concrete referents that did not allow the child to view the initial array of objects and the objects to be added or subtracted at the same time. In their task, the examiner held a number of pennies in his/her hand and asked the child how many pennies were there. The examiner then closed his/her hand so that the child could no longer see the pennies. In the case of addition, the child then saw the examiner add a number of pennies to those already hidden, saying, "Now I'm putting *n* pennies in my hand." The child's task was to respond verbally to the question "How many pennies does this bunch have?" A similar procedure was used for subtraction. The child was not able to see the initial set of pennies and the pennies that were added or subtracted at the same time, ensuring that a calculation had to be performed to reach the correct answer.

Starkey and Gelman's data indicate that preschool children as young as 3 years of age can solve simple addition and subtraction problems when both verbal labels and physical referents are provided for the terms of the problem. Using a somewhat different task, Klein and Starkey (1988) report that children as young as 2 years of age can perform addition calculations involving sums of three or less. Consistent with these findings, a number of studies report that when referents are not provided, older children tend to create their own by using their fingers to calculate (e.g., Ilg & Ames, 1951; Siegler & Shrager, 1984). Considered together, these studies suggest that the ability to calculate by combining and separating sets of physical objects develops prior to the ability to solve verbally presented addition and subtraction problems in the absence of physical props.

The goal of the present study is to compare the development of skills for solving verbally and nonverbally presented calculation problems in children between 4 and 6 years of age. Identical addition and subtraction problems were presented to each child in the study in the form of nonverbal problems, story problems, and number-fact problems. On the nonverbal problems, we examined children's ability to add and subtract by presenting a set of physical referents that subsequently was increased by adding elements or decreased by removing elements. As in Starkey and Gelman's study (1982), the necessity of performing a calculation was ensured by not letting the child view the two terms of the problem simultaneously. However, in contrast to Starkey and Gelman's procedure, in the present study, both the presentation and the response modes were completely nonverbal. Recall that in Starkey and Gelman's procedure the numerosity of the first term of the problem was established verbally by asking the child how many pennies were in the experimenter's hand, the numerosity of the second term of the problem was stated by the experimenter, and the child provided a verbal answer. In the present study, the experimenter did not provide verbal labels for either of the terms of the problem nor was the child asked to generate them. Moreover, the child responded by laying out an appropriate number of disks rather than with a number word.

This procedure allows the comparison of performance on problems that are presented in a completely nonverbal format to those that are presented verbally, in the form of story problems or number-fact problems. The nonverbal calculation task removes some of the sources of difficulty, discussed above, that may mask the young child's calculation abilities. In particular, it removes the requirements of having knowledge of number words and relational terms and makes the numerosities involved in the original set and the transformation more readily available. This should make it easier for the child to represent the terms of the problem and the operation involved. Thus, the nonverbal task may make the calculation abilities that exist in young children more apparent.

Children's performance levels on the nonverbal problems, story problems, and number-fact problems were compared to address this issue. Further, children's errors on the three types of problems were examined to determine whether they demonstrate a basic understanding of the effects of the operations of addition and subtraction on any or all of the problem types presented. Insofar as children have such knowledge, the errors they make should be in the right direction, that is, greater than the augend for addition and less than the minuend for subtraction. Better ability to calculate on the nonverbal task also should be reflected by errors that are closer to the correct answer. We also examined the frequency of repetition errors, which consist of giving the first or second term of the problem as the answer. We hypothesized that the frequency of certain types of repetition errors should decrease with increasing knowledge of calculation. In particular, this should be the case for repetitions of the first term of the problem, which are in neither the right or the wrong direction (e.g., $4 + 1 = 4$), and those second term repetitions that are in the wrong direction (e.g., $4 + 1 = 1$). These types of repetition errors may be the result of the child parroting one of the terms of the problem or attempting to calculate but displaying no real understanding of the nature of the addition operation. In contrast, although a second term repetition error that is in the right direction may reflect parroting (e.g., $1 + 3 = 3$), it may reflect some knowledge of calculation.

Finally, we observed the overt calculation methods employed by the children while performing verbal and nonverbal calculations. Siegler (1987, 1989) reports that children employ a wide variety of calculation strategies while they are acquiring arithmetic skills and that they tend to use each strategy most often on problems for which it is most advantageous. In the current study it seems likely that children will use different strategies on the different problem types given the differing demands of the tasks. We were particularly interested in how frequently children used their fingers when calculating. For nonverbal problems, fingers may be used relatively infrequently because of the availability of object referents. For story problems, which refer to specific object sets that are not physically present (e.g., "three pennies and two pennies"), the child's ability to construct a representation of the object sets may be enhanced, reducing the need to use fingers. Finally, for number-fact problems, in which specific object sets are neither provided or referred to, finger counting may be used relatively frequently. Thus, we predicted that children would use fingers in calculation most often for number-fact problems where there is no explicit reference to object sets, next often for story problems, and least often for nonverbal problems.

METHODS

Subjects

Sixty children participated in the study. They were divided into five age groups (years–months): (1) 4-0 to 4-5; (2) 4-6 to 4-11; (3) 5-0 to 5-5; (4) 5-6 to 5-11; and (5) 6-0 to 6-5. Previous studies as well as our own pilot work suggest that 3-year-olds can perform calculations when concrete referents are provided (Starkey & Gelman, 1982). In contrast, the ability to deal with verbal calculation problems, when no props are provided, develops somewhat later during the preschool years (Ginsburg & Russell, 1981). Thus, in an attempt to avoid floor effects on any of the tasks, 4-year-olds were the youngest age group included in this study. Within each age level, there were 12 children (6 boys and 6 girls). The children were drawn from three private preschools on the north side of Chicago. Subjects in the two oldest age groups attended kindergarten classrooms within the same private preschools and, according to their teachers, received some formal instruction in arithmetic calculation. All of the children came from middle-class homes where English was the primary language.

Materials and Procedure

Children were tested individually in their schools in late March and early April. Each child was given a set of six addition and six subtraction problems presented in three problem-type formats: (1) nonverbal problems; (2) story problems; (3) number-fact problems. All three tasks were given in one session that lasted approximately 20 min. The order of presentation of the tasks was counterbalanced across subjects within each age group. The same set of calculations, presented in the same fixed random order, was used in each of the three tasks. Addition and subtraction items were randomly intermixed with the constraint that there could be no more than three consecutive problems involving the same operation. For the addition calculations, the numerosities of the augends and addends were no greater than four and the sums were no greater than six. For the subtraction calculations, the numerosities of the minuends and subtrahends were no greater than six and the differences were no greater than four. One addition problem (1 + 1) and one subtraction problem (2 − 1) were used as demonstration items for all three tasks. Errors were corrected on the practice items for each task, but not on the test items. For all three conditions, an item was repeated once if requested by the child. The child's score on each calculation task was based on test items only, not on practice items.

The method of presenting each of the three calculation tasks is described below:

Nonverbal problems. Materials for the nonverbal calculation task included two 10″ × 10″ white cardboard mats, a set of 18 black disks (3/4 of an inch in diameter), a box for the disks, and a cover for the disks. One side of the cover had an opening so the examiner could easily put in or take out disks. The examiner and child sat at a small table facing each other, each with a mat in front of herself/himself.

For the addition demonstration item (1 + 1), the examiner placed one disk on her mat in full view of the child. This disk was then hidden under a cover. The examiner then slid another disk under the cover, making sure the child could see the disk while it was being moved but could not see the disk that was already hidden. Next, the examiner placed two disks in a horizontal line on the child's mat and lifted the cover to show the two disks on her mat, saying, "See, yours is just like mine." This demonstration item was then presented again, following the same procedure, except this time the child was asked to place the appropriate number of disks on the mat after the transformation had been made by the examiner. A verbal response was not required. If the child placed the wrong number of disks on the mat, the response was corrected and the item was repeated one more time. A parallel demonstration procedure was completed with a subtraction problem (2 − 1), but in this case the disk comprising the subtrahend was removed from under the cover.

Nonverbal test items were presented immediately following the demonstration items. For each addition problem, the examiner placed the set of disks comprising the augend on the mat and then covered it. The examiner then put the set of disks comprising the addend in a horizontal line next to the cover and slid them under the cover one by one. As in the demonstration procedure, the child then indicated how many disks were hiding under the cover by placing the appropriate number of disks on his/her mat. A comparable procedure was used for subtraction problems, but in this case the disks comprising the subtrahend were removed from under the cover, one by one. No verbal labels were provided on any of the problems.

Story problems. The verbal content of the story problems was intended to be as simple as possible (Hiebert, Carpenter, & Moser, 1982). The addition problems required subjects to join or combine two sets of objects (e.g., "Mike had *m* balls. He got *n* more. How many balls did he have altogether?"). The subtraction problems required subjects to separate a set of objects (e.g., "Kim had *m* crayons. She lost *n*. How many crayons did she have left?"). For all addition problems, the same verbs and syntactic structure were used, but subjects and objects were varied to sustain interest. This was also the case for all subtraction problems. The examiner read the story problems aloud and the child was required to respond to each problem with a number word. No physical props were provided.

As on the nonverbal problems, two practice items, 1 + 1 and 2 − 1, were administered prior to the test items. The examiner read the word problem associated with each practice item and if the child did not respond correctly then provided the answer and readministered the item.

Number–fact problems. The examiner read the addition number-fact problems to the children as "How much is *m* and *n*?" and the subtraction number-fact problems as "How much is *m* take away *n*?" The child responded by giving a verbal numerical response, and no physical props were provided. The same practice items, 1 + 1 and 2 − 1, were administered prior to the test items. As on the other tasks, if the child did not respond correctly to a practice item, the examiner provided the answer and readministered the item.

Children's calculation methods were recorded during the testing on a trial-by-trial basis. They were classified according to the following categories, similar to those described by Siegler and Shrager (1984): (1) counting-fingers strategy; (2) finger strategy; (3) counting strategy; and (4) unobserved strategy. Children were classified as using a *counting-fingers strategy* if they explicitly counted on their fingers either orally or by moving their fingers or head. A *finger strategy* was recorded if children held up their fingers for any term of the problem without counting them in an overt manner. Children were classified as using a *counting strategy* if they displayed counting behaviors without using their fingers (e.g., subvocalizing the number sequence and/or pointing with finger or head). An *unobserved strategy* was recorded when children answered without using their fingers and without counting overtly. In this case, children may have been retrieving the answer from memory or may have been using a covert algorithm (e.g., silent counting).

We observed another method of problem solving that was relevant only on nonverbal problems. In particular, in some instances children appeared to be imitating the examiner's actions. For example, on the nonverbal addition problem, 3 + 2, a child using such an imitation strategy put three disks on one part of the mat and two disks on another part. The child then slid the 2 disks over to the 3 disks, copying the examiner's transformation. Even more strikingly, on the nonverbal subtraction problem, 5 − 2, a child using such a strategy put 5 disks on the mat and then removed 2. In order to arrive at a correct answer using this method the child only needs to extract the numerosities of the two sets involved in a problem and apply the appropriate operation (combination or separation) to these numerosities. This can be done either by counting or by visually extracting numerosities without counting. Although children who "imitated" may have been copying the experimenter's actions, it should be noted that physically moving subsets together is an effective addition algorithm and physically separating subsets is an effective subtraction algorithm.

RESULTS

Children's performance on each of the three problem types was scored for the number of problems solved correctly ($n = 6$ for addition and subtraction, respectively). The mean scores of children in each age group

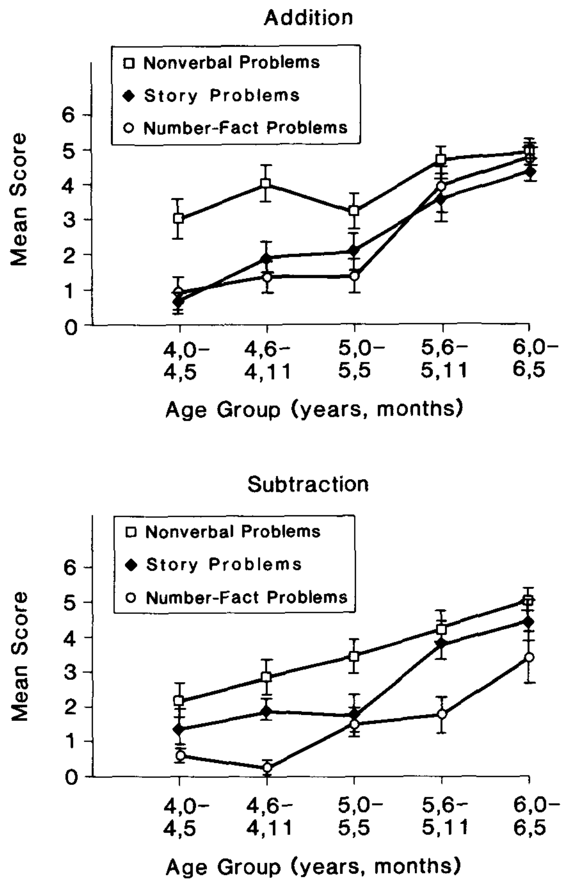

FIG. 1. Mean calculation scores by age group, operation, and problem type (bars denote standard errors).

broken down by problem type and operation are graphically displayed in Fig. 1. A preliminary analysis of variance indicated that neither the main effects nor the interactions involving sex of subjects or order of presentation of the different calculation tasks were significant. Thus, these factors were eliminated from subsequent analyses.

An analysis of variance on the mean number of problems correct with Age Group as a between-subjects factor and Problem Type (nonverbal problems, story problems, number-fact problems) and Operation (addition, subtraction) as within-subjects factors revealed a significant main effect of Age Group ($F(4, 55) = 13.7, p < .001$). A trend analysis showed a significant linear trend for Age ($F(1, 55) = 12.79, p < .001$) and no higher order trends. Pairwise comparisons (Tukey hsd tests) revealed a significant difference between the mean scores of the group from 5-0 to 5-5 years of age and the group from 5-6 to 5-11 years of age for all problem types ($p < .01$), and no significant differences for any other adjacent age groups.

The analysis also revealed significant main effects of Problem Type ($F(2, 110) = 55.7$, $p < .001$) and Operation ($F(1, 55) = 5.37$, $p < .02$). Tukey hsd tests revealed that nonverbal problems were significantly easier than story problems ($p < .01$), which in turn were significantly easier than number-fact problems ($p < .01$). The main effect of operation reflected the finding that addition problems were significantly easier than subtraction problems ($p < .01$, Tukey hsd test). However, the Problem Type × Operation ($F(2, 110) = 4.82$, $p < .009$) interaction showed that this was only significant for number-fact problems (tests of simple effects, $p < .002$), (see Fig. 1). Examination of Fig. 1 also reveals a tendency for nonverbal addition problems to be easier than nonverbal subtraction problems for the two youngest age groups, although this difference was not significant.

Tests of simple effects also revealed significant effects of problem type for addition ($p < .0001$) and subtraction ($p < .0001$). For addition, performance on nonverbal problems was significantly higher than that on story problems or number-fact problems (Tukey hsd tests, $p < .01$, in both cases), but story problems and number-fact problems did not differ significantly in difficulty. For subtraction, nonverbal problems were significantly easier than story problems ($p < .01$), which in turn were significantly easier than number-fact problems ($p < .01$).

We next examined the distributions of individual children's calculation scores by age group, problem type, and operation (see Table 1). These distributions indicate that the mean scores for each age group do not reflect extreme levels of performance characterized by a few children performing very well and most performing poorly. In particular, on the nonverbal problems most children in the youngest age group had some success, with performance level increasing steadily across the entire age range tested. Moreover, the individual data, mirroring the group means, suggest that children in the two youngest age groups performed somewhat better on nonverbal addition than nonverbal subtraction problems. On the story problems, the individual data show that only a few children younger than 5-6 to 5-11 years of age were successful on a majority of the problems for either addition or subtraction (scores of 4 or over in each case). This pattern again corresponds closely to the group means displayed in Fig. 1, which show a steep rise in performance for both addition and subtraction problems between the group including children 5-0 to 5-5 years of age and the group including children 5-6 to 5-11 years of age. Finally, the pattern of individual scores on the number-fact problems indicates that the majority of children received very low scores (0

TABLE 1
FREQUENCY OF CALCULATION SCORES BY AGE GROUP, PROBLEM TYPE, AND OPERATION

	Addition score							Subtraction score						
Age group	0	1	2	3	4	5	6	0	1	2	3	4	5	6
					Nonverbal problems									
4-0 to 4-5	1	3	0	3	2	2	1	5	1	0	3	0	2	1
4-6 to 4-11	0	1	3	0	2	3	3	1	3	1	1	4	2	0
5-0 to 5-5	2	0	1	2	5	2	0	1	0	2	1	6	2	0
5-6 to 5-11	0	0	1	1	2	3	5	0	1	1	1	2	5	2
6-0 to 6-5	0	0	1	0	3	2	6	0	0	1	0	2	4	5
					Word problems									
4-0 to 4-5	8	2	0	2	0	0	0	7	0	1	2	1	1	0
4-6 to 4-11	2	3	3	3	0	1	0	1	3	5	2	1	0	0
5-0 to 5-5	3	1	3	3	1	1	0	5	2	1	0	4	0	0
5-6 to 5-11	1	2	1	3	0	1	4	0	1	1	1	2	5	2
6-0 to 6-5	0	1	1	0	3	4	3	0	1	1	0	2	3	5
					Number-fact problems									
4-0 to 4-5	8	2	1	0	0	0	1	7	4	0	1	0	0	0
4-6 to 4-11	5	4	0	1	1	0	1	8	4	0	0	0	0	0
5-0 to 5-5	6	2	2	1	0	0	1	4	3	2	1	2	0	0
5-6 to 5-11	1	1	2	1	2	1	4	2	6	1	1	1	1	0
6-0 to 6-5	0	1	0	0	4	2	5	4	0	1	0	0	3	4

or 1) on addition up until 5-6 to 5-11 years and on subtraction up until 6-0 to 6-5 years. This again, corresponds closely to the group means for number-fact problems in Fig. 1.

Pearson product-moment correlations of individual children's scores on the three tasks revealed correlations of .63 between nonverbal problems and story problems, .65 between nonverbal problems and number-fact problems, and .78 between story problems and number-fact problems ($p < .001$ in all cases). These correlations were similar when calculated for addition and subtraction problems separately. Calculation of pairwise intertask correlations for individual subjects (ϕ coefficients) did not reveal a significant effect of age group on the magnitude of the correlations. However, the correlations tended to be higher for older than younger subjects, most likely because of floor effects on the word problems and number-fact problems in the younger age groups.

The relation of the three calculation tasks was further examined by comparing the rank order of item difficulty on each of the tasks (see Table 2). Spearman rank-order correlations showed that the ordering of item difficulty was positively correlated between each pair of tasks. The ranking of items for each task was determined on the basis of overall number correct for each item across all subjects. This correlation reached signif-

TABLE 2
RANK ORDER OF CALCULATION ITEMS BY PROBLEM TYPE (EASIEST TO HARDEST)

Nonverbal problems	Story problems	Number-fact problems
3 − 1	4 − 1	2 + 2
2 + 2	4 + 1	1 + 3
4 − 2	2 + 2	1 + 4
1 + 3*	3 − 1*	4 + 1
4 − 1	5 − 3	3 + 2
4 + 1	3 + 2	4 − 1
3 + 2	4 − 2*	2 + 4
1 + 4	1 + 4	5 − 3
5 − 2	5 − 2	5 − 2
5 − 3	1 + 3	3 − 1
2 + 4	6 − 4	4 − 2*
6 − 4	2 + 4	6 − 4

Note. An asterisk indicates a tie with problem difficulty directly above.

icance for nonverbal problems and story problems ($\sigma = .58$, $p < .05$), but not for nonverbal problems and number-fact problems ($\sigma = .31$) or for story problems and number-fact problems ($\sigma = .20$). For both nonverbal problems and story problems, easier items tended to involve smaller numerosities than more difficult items. For both of these problem types, addition and subtraction calculations occurred equally often in the easier and harder halves of the ranking. In contrast, for number-fact problems, addition calculations generally were easier than subtraction calculations.

A subset of the problems in Starkey and Gelman's (1982) study, described earlier, was the same as the problem set used in the present study. The ordering of problem difficulty of that subset in Starkey and Gelman's study was marginally significantly correlated with the ordering of problem difficulty of the nonverbal problems ($\sigma = .57$, $p < .06$), significantly correlated with the ordering of problem difficulty of the story problems ($\sigma = .69$, $p < .02$), and positively, but not significantly, correlated with the ordering of problem difficulty of the number-fact problems ($\sigma = .45$, n.s.).

Subsequent analyses examined the effects of numerosity on children's calculation success in more detail. Problems were divided into two sets, one involving very low numerosities (no numerosity greater than four involved in the terms of the problem or the answer) and the other involving somewhat higher numerosity problems (a numerosity of five or six involved in the terms of the problem or the answer). Figure 2 shows the mean percentage correct on "small" and "big" numerosity problems for each of the problem types. An analysis of variance on percentage of problems correct with Age as a between-subjects factor and Problem Type and

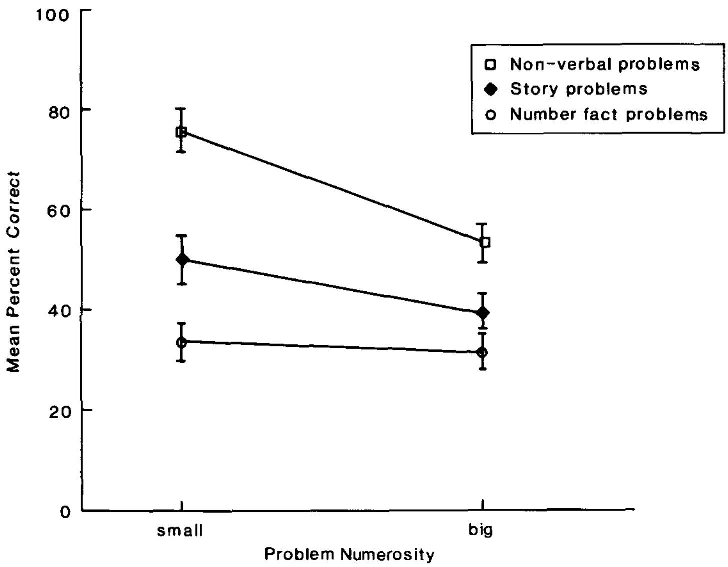

FIG. 2. Mean percentage correct on low and high numerosity problems by problem type (bars denote standard errors).

Numerosity as within-subjects factors was performed. As in previous analyses, the main effects of Age ($F(4, 55) = 13.10, p < .0001$) and Problem Type ($F(2, 110) = 62.77, p < .0001$) were highly significant. There also was a significant main effect of Numerosity ($F(1, 55) = 29.23, p < .0001$) and a significant Problem Type × Numerosity interaction ($F(2, 110) = 15.26, p < .0001$). The main effect of numerosity reflected better overall performance on problems involving lower numerosities. However, the Problem Type × Numerosity interaction showed that the numerosity effect was significant on nonverbal problems ($p < .0001$) and story problems ($p < .001$) but not on number-fact problems ($p = .38$) (tests of simple effects). Moreover, the effect of numerosity was significantly larger on nonverbal problems than on story problems ($p < .01$) or on number-fact problems ($p < .01$), which did not differ significantly from each other (Tukey hsd tests).

It might be argued that the absence of a numerosity effect on the number-fact problems is attributable to the lower overall performance level on these problems. However, this does not appear to be the case. In particular, the performance level of the youngest two age groups (4-0 to 4-11) on the nonverbal problems is comparable to that of the oldest two age groups (5-6 to 6-5) on the number-fact problems (53% vs 55%, respectively). An analysis of variance on percentage of problems correct with Problem Type (nonverbal problems, youngest two age groups vs number-fact problems, oldest two age groups) as a between-subjects factor and Numerosity as a within-subjects factor revealed a significant Problem Type × Numerosity interaction ($F(1, 46) = 5.92$, $p < .02$). Tests of simple effects showed a highly significant numerosity effect for nonverbal problems in the youngest two age groups ($p < .001$; 64% correct on small numerosity problems vs 42% correct on large numerosity problems) but not for number-fact problems in the oldest two age groups ($p = .27$; 58% correct on small numerosity problems vs 53% correct on large numerosity problems). Thus, the absence of a significant numerosity effect on the number-fact problems does not appear to be attributable to performance level.

Error Analyses

Analyses of error type. We examined the frequency of particular types of errors as certain types errors may reflect more knowledge of the effects of addition and subtraction operations than others. Table 3 shows the percentages of children's errors that were in the right direction, in the wrong direction, repetitions of the first term of the problem, repetitions of the second term of the problem, and no response errors by age group, problem type, and operation. Right direction errors consist of errors that are larger than the augend for addition (e.g., 3 + 1 = 5) or smaller than minuend for subtraction (4 − 1 = 2). Wrong direction errors consist of errors that are smaller than the augend for addition (e.g., 3 + 1 = 2) or larger than the minuend for subtraction (e.g., 4 − 1 = 6). Right direction errors may reflect more knowledge of addition and subtraction operations than wrong direction errors.

Repetition errors consist of giving an answer that is either the first or the second term of the problem. A child making a repetition error may merely be parroting one of the terms of the problem. However, a second term repetition error may reflect more knowledge of calculation than a first term repetition error. That is, first term repetition errors are never in the right or wrong direction (e.g., 4 + 1 = 4). In contrast, on subtraction problems involving only positive numbers, second term repetitions are always in the right direction and may even be the correct answer (e.g., 4 − 2 = 2 in the present study). On addition problems, second term repetition errors may be in either the right or the wrong direction. Consider the following examples: On the problem 4 + 1, a child who responds "one" may be parroting the second term of the problem or attempting to calculate but arriving at an answer that displays no understanding of the nature of the addition operation; in contrast, on the problem 1 + 3, a child who responds "three" may either be parroting the second term of the problem or may be calculating but obtaining an erroneous answer that is in the right direction and only different from the right answer, four, by one number. Thus, such a response may reflect

TABLE 3

DIRECTIONALITY OF CHILDREN'S ERRORS BY AGE GROUP, OPERATION, AND PROBLEM TYPE (EXPRESSED AS PERCENTAGES)

Age group	Addition							Subtraction				
	RD	WD	R1	R2		NR	O	RD	WD	R1	R2(RD)	NR
				RD	WD							
						Nonverbal problems						
4-0 to 4-5	42	6	17	28	6	0	3	24	26	38	12	0
4-6 to 4-11	75	4	4	17	0	0	0	34	13	34	18	0
5-0 to 5-5	79	0	6	15	0	0	0	45	13	13	29	0
5-6 to 5-11	60	7	13	20	0	0	0	32	23	27	18	0
6-0 to 6-5	79	7	7	7	0	0	0	33	0	8	58	0
Across ages	**65**	**4**	**10**	**19**	**2**	**0**	**1**	**33**	**17**	**28**	**22**	**0**
						Story problems						
4-0 to 4-5	24	14	17	17	5	14	8	37	6	20	24	13
4-6 to 4-11	39	0	10	31	6	14	0	40	21	15	6	17
5-0 to 5-5	60	6	2	26	2	0	4	41	29	6	20	4
5-6 to 5-11	27	7	10	43	3	7	3	44	15	15	19	7
6-0 to 6-5	79	0	5	16	0	0	0	41	29	6	24	0
Across ages	**41**	**7**	**10**	**26**	**4**	**9**	**4**	**40**	**19**	**13**	**18**	**10**
						Number-fact problems						
4-0 to 4-5	26	5	6	13	8	34	8	14	30	20	9	27
4-6 to 4-11	36	13	7	0	0	40	4	24	18	15	10	34
5-0 to 5-5	41	18	7	12	2	16	4	34	34	9	6	17
5-6 to 5-11	40	8	12	12	4	16	8	16	26	34	10	14
6-0 to 6-5	80	0	7	13	0	0	0	13	35	35	16	0
Across ages	**38**	**10**	**8**	**9**	**3**	**26**	**5**	**21**	**27**	**21**	**10**	**21**

Note. RD, Right direction errors (e.g., $4 + 1 = 6$; $5 - 2 = 1$); WD, wrong direction errors (e.g., $4 + 1 = 2$; $5 - 2 = 7$); R1, first term repetition errors (e.g., $4 + 1 = 4$; $5 - 2 = 5$); R2(RD), second term repetition errors that are in the right direction (e.g., $1 + 3 = 3$; $5 - 2 = 2$) [Note that on subtraction problems all second term repetition errors are in the right direction]; R2(WD), second term repetition errors that are in the wrong direction (e.g., $4 + 1 = 1$); NR, no response; O, Ambiguous errors (i.e., $2 + 2 = 2$). All percentages were rounded to the nearest whole number some understanding of the addition operation. Finally, no response errors consist of not responding or saying "I don't know."

Our first error type analysis contrasted right direction with wrong direction errors. Given that children more frequently responded correctly on the nonverbal problems than on the other problem types, we predicted that the frequency of right direction errors would be higher on the nonverbal problems than on the other problem types. Error direction was examined separately for addition and subtraction problems as the opportunity to make right direction errors by chance is greater for addition than for subtraction. This is because zero serves as a boundary for right direction errors for subtraction but not for addition. For each subject, we calculated the proportion of total errors that were in the right direction and in the wrong direction. Errors that were repetitions of one of the terms of a problem, e.g., $5 - 2 = 2$, were not counted as right or wrong direction in this analysis. Further, in this analysis, problems on which children made no response were not considered in calculating the percentage of total right and wrong direction errors as these errors are ambiguous with respect to whether the child understands calculation. That is, a child may not respond to a problem for a variety of reasons related to performance (e.g., inattention) rather than competence. Finally, only children who made some errors other than no response errors on each problem type were included in this analysis. It was only possible to include the youngest three age groups in this analysis as well as in subsequent analyses on error type as many children in the two oldest age groups made no errors on one or more of the problem types or made only one or two errors on at least one problem type, making their error type data unreliable.

For addition problems, we performed an analysis of variance with Age as a between-subjects factor and Error Direction (percentage of total errors that were right direction vs wrong direction) and Problem Type as within-subjects factors. The number of subjects included in each of the three youngest age groups was as follows: 4-0 to 4-5, $N = 7$; 4-6 to 4-11, $N = 8$; 5-0 to 5-5, $N = 10$. The analysis revealed a main effect of Error Direction ($F(1, 44) = 64.85$, $p < .0001$), reflecting the greater frequency of right direction errors than wrong direction errors (56% right direction vs 9% wrong direction errors across the three problem types). The Problem Type by Error Direction interaction also was significant ($F(2, 44) = 5.00$, $p < .02$), reflecting the finding of a higher percentage of total errors in the right direction on nonverbal problems than on either of the other problem types (Tukey hsd tests, $p < .05$ in both cases), which did not significantly differ from each other. The percentage of total errors that were in the right direction on each problem type was as follows: nonverbal problems 72.2%; story problems 48.5%; number-fact problems: 48.3%. The percentage of total errors that were in the wrong direction did not differ significantly across the three problem types (nonverbal problems 3.6%; story problems 8.0%; number-fact problems 14.9%). No other main effects or interactions were significant. Thus, across the age range from 4-0 to 5-5, right direction errors on addition problems were significantly more frequent on nonverbal problems than on the other problem types, consistent with our finding of better overall performance on the nonverbal task.

A parallel analysis was performed on the percentage of total errors that were in the right and wrong direction for subtraction problems, again including only subjects in the youngest three age groups (age group 4-0 to 4-5, $N = 9$; age group 4-6 to 4-11, $N = 10$; age group 5-0 to 5-5, $N = 11$). Table 3 shows that there were more right than wrong direction responses for subtraction on nonverbal problems and story problems, but not on number-fact problems. Nonetheless, the analysis revealed a main effect of Error Direction ($F(1, 54) = 14.46$, $p < .001$) and the Problem Type × Error Direction interaction did not reach significance ($F(2, 54) = 1.54$, $p = .22$).

Our next analyses examined the percentage of total errors that were first and second term repetition errors on the different problem types. An analysis of variance on percentage of total errors that were repetition errors with Age (youngest three age groups) as a between-subjects factor and Repetition Error Type (first term, second term), Problem Type, and Operation as within-subjects factors was performed. Data from one addition problem ($2 + 2 = 4$) and one subtraction problem ($4 - 2 = 2$) were omitted. On the addition problem it is not possible to distinguish between a first vs second term repetition error and on the subtraction problem a repetition of the second term is the correct answer. The analysis revealed a significant Repetition Error Type × Operation interaction ($F(1, 21) = 6.19$, $p < .03$), such that there were significantly more second term than first term repetition errors for addition problems ($p < .001$, test of simple effects) but not for subtraction problems ($p = .70$). While the percentage of first term repetitions was significantly smaller for addition than subtraction ($p < .003$), the percentage of second term repetitions did not significantly differ for addition vs subtraction ($p = .27$).

Consistent with the hypothesis that second term repetitions reflect more knowledge of calculation than first term repetitions, the analysis also revealed a significant interaction of Repetition Error Type × Problem Type ($F(2, 42) = 3.76$, $p < .04$). In particular, the frequency of second term repetitions was higher than first term repetitions for story problems (test of simple effects, $p < .04$) and for nonverbal problems, although this difference was not statistically significant ($p = .17$). In contrast, the percentage of first term repetitions was higher than the percentage of second term repetitions for number-fact problems, although these percentages did not differ significantly ($p = .25$) (see Table 3). The greater frequency of second than first term repetition errors on nonverbal problems and story problems may reflect some understanding of calculation. This would be supported if second term repetition errors on these two problem types are more often in the right than the wrong direction on addition problems, for which it is possible to make this comparison.

The next analysis examined this question. In order to compare the relative frequency of right and wrong direction second term repetition errors, each child's number of right direction second term repetitions on addition problems was divided by 3 and his/her number of wrong direction repetitions was divided by 2, to obtain a per problem average of right and wrong direction second term repetition errors. This was done because in the problem set administered, a second term repetition error was in the wrong direction on two addition problems, 4 + 1 and 3 + 2, and in the right direction on three addition problems, 1 + 3, 1 + 4, and 2 + 4. Data from the one remaining addition problem, 2 + 2, were not included in this analysis as on this problem it is not possible to distinguish between first and second term repetition errors, and these errors are neither in the right nor the wrong direction. An analysis of variance with Error Direction (percentage of total errors on addition problems that were right vs wrong direction second term repetitions) and Problem Type as within-subjects factors and Age (youngest three age groups) as a between-subjects factor was performed. Consistent with the hypothesis that second term repetition errors reflect some knowledge of calculation, the analysis revealed a significant main effect of Error Direction ($F(1, 22) = 13.57$, $p < .002$), reflecting a higher frequency of right than wrong direction second term repetition errors. There also was a significant Error Direction × Problem Type interaction ($F(2, 44) = 4.35$, $p < .02$), such that subjects made consistently more right than wrong direction second term repetition errors on nonverbal problems and story problems ($p < .001$, nonverbal problems, $p < .002$ story problems, tests of simples effects), but not on number-fact problems ($p = .70$).

In sum, we have suggested that right direction errors and right direction second term repetition errors reflect more knowledge of calculation than wrong direction errors, wrong direction repetition errors, and first term repetition errors. At least for addition problems, the results of our error analyses generally support the percentage correct data, indicating that subjects perform better on the nonverbal calculation problems than on the other problem types. In particular, subjects in the youngest three age groups made more right direction errors on nonverbal addition problems than on the other problem types. Further, right direction second term repetition errors were more frequent than wrong direction second term repetition errors on nonverbal and story problem addition problems but not on number-fact addition problems. For subtraction problems, subjects made more right direction than wrong direction errors on nonverbal prob-

TABLE 4
MEAN DISTANCE FROM CORRECT ANSWER BY AGE GROUP, OPERATION, AND PROBLEM TYPE (EXPRESSED AS ABSOLUTE VALUES)

Age group	Nonverbal problems	Story problems	Number-fact problems
		Addition	
4-0 to 4-5	1.69	2.67	2.85
4-6 to 4-11	1.33	1.67	1.97
5-0 to 5-5	1.18	2.00	2.43
5-6 to 5-11	1.33	1.75	1.95
6-0 to 6-5	1.14	2.05	1.27
Across ages	**1.37**	**2.08**	**2.27**
		Subtraction	
4-0 to 4-5	2.36	1.66	3.00
4-6 to 4-11	1.74	2.21	2.38
5-0 to 5-5	1.90	2.53	2.77
5-6 to 5-11	1.82	1.80	2.40
6-0 to 6-5	1.67	3.12	2.65
Across ages	**1.96**	**2.18**	**2.64**

lems and story problems, but not on number-fact problems. However, the Problem Type × Error Direction interaction did not reach significance.

Distance of errors from the correct answer. Our next analyses examined the mean distance of subjects' responses from the correct answer. The premise underlying these analyses is that errors that are closer to the correct answer reflect more knowledge of calculation. Table 4 summarizes the mean distance of subjects' errors from the correct answer by problem type, operation, and age group. It should be noted that when a child responded to a story problem or number-fact problem with answers higher than would be possible on the parallel nonverbal problem because of the finite number of disks available, the child's answer was coded as the highest response possible on the nonverbal problem. That is, if the child's response was 14 on the number-fact problem 3 + 2, his answer was coded as 13, as only 13 of the 18 disks were available to the child after the experimenter had placed 5 disks (2 + 3) under the box on the parallel nonverbal problem.

Separate analyses were carried out for addition and subtraction problems because of the differential effects of the boundary of zero on the different operations. The first analysis examined mean distance of all addition errors from the correct answer (calculated in absolute value, e.g., 2 + 3 = 4, distance = 1) with Age (youngest three age groups) as a between-subjects factor and Problem Type as a within-subjects factor. The results of the analysis revealed a significant main effect of Problem

TABLE 5
PERCENTAGE OF ERRORS ADJACENT TO CORRECT ANSWER BY AGE GROUP, OPERATION, AND PROBLEM TYPE

Age group	Nonverbal problems	Story problems	Number-fact problems
		Addition	
4-0 to 4-5	58	28	29
4-6 to 4-11	79	64	39
5-0 to 5-5	82	53	34
5-6 to 5-11	80	50	48
6-0 to 6-5	86	58	87
Across ages	**75**	**48**	**41**
		Subtraction	
4-0 to 4-5	40	55	32
4-6 to 4-11	61	59	40
5-0 to 5-5	58	57	30
5-6 to 5-11	64	68	33
6-0 to 6-5	58	47	26
Across ages	**54**	**57**	**32**

Note. All percentages were rounded to the nearest whole number.

Type ($F(2, 44) = 12.10$, $p < .0001$), reflecting the smaller mean distance of addition errors from the correct answer on nonverbal problems than on story problems or number-fact problems (Tukey hsd tests, $p < .01$, in both cases). Neither the main effect of age nor the Age × Problem Type interaction was significant. However, the pool of errors as well as the number of children making errors decreased with age. Thus, the children in the older age groups who were still making errors were not significantly better at coming close to the correct answer than the younger children who, in general, made more errors.

A parallel analysis was performed on subtraction errors. As for addition problems, a child's mean distance from the correct answer on a verbal subtraction problem was limited by the highest distance possible on the parallel nonverbal subtraction problem. This analysis also revealed a significant main effect of Problem Type ($F(2, 54) = 3.19$, $p < .05$) and no main effect of Age or Age × Problem Type interaction. The main effect of problem type reflected the smaller mean distance from the correct answer of errors on nonverbal problems than on number-fact problems (Tukey hsd test, $p < .05$). The mean distance from the correct answer of errors on story problems fell between the other two problem types, but did not differ significantly from either of them.

We also examined our data to determine the percentage of errors that were adjacent to the correct answer (within plus or minus one of the correct answer) (see Table 5). Adjacency errors are consistent with the child having some knowledge of calculation. Such errors may be attributable to the child erroneously making an error in representing the exact numerosity of one or both terms of the problem and/or making an error in combining the terms. An analysis of variance was performed on the proportion of total errors (excluding no response errors) that were adjacent to the correct answer with Age (youngest three age groups) as a between-subjects factor and Problem Type and Operation as within-subjects factors. Again, any subject who made no errors or made all "no response" errors on any problem type was excluded from the analysis. The results of this analysis revealed a highly significant main effect of Problem Type ($F(2, 44) = 14.47, p < .0001$) and a significant interaction of Problem Type × Operation ($F(2, 44) = 4.16, p < .03$). For addition, Tukey hsd tests show that the percentage of total errors adjacent to the correct answer was significantly higher for nonverbal problems than for story problems ($p < .01$) or number-fact problems ($p < .01$), which did not significantly differ from each other. In contrast, for subtraction, although the percentage of total errors adjacent to the correct answer was higher on nonverbal problems and story problems than on number-fact problems, these differences were not significant according to Tukey tests.

The percentage of total errors adjacent to the correct answer was higher for nonverbal addition problems than for nonverbal subtraction problems (test of simple effects, $p < .02$). However, there was no effect of operation on the two other problem types. The finding of a higher percentage of adjacency errors for nonverbal addition than nonverbal subtraction is consistent with the finding of a nonsignificant performance level advantage on nonverbal addition compared to nonverbal subtraction in the youngest two age groups (see Fig. 1).

Finally, we examined our error data for the occurrence of "counting-string associate" errors. Siegler and Shrager (1984) have reported that the most frequent error on verbally presented addition problems in which the addend is equal to or larger than the augend (e.g., $2 + 4$) is a counting-string associate, that is, the number that is one higher than the second term in the problem. In our study, counting-string associates may be expected to occur more frequently on number-fact problems than on nonverbal problems where no number words are stated or on story problems where the number words are embedded in a context. This possibility is supported by our finding that $1 + 3$ and $1 + 4$, two problems on which the counting-string associate is the correct answer, are among the easier problems in the difficulty rank ordering of number-fact problems, but not of word problems or nonverbal problems (see Table 1).

However, the error data suggest that the relative ease of $1 + 3$ and $1 + 4$ on the number-fact problems may not be attributable to the use of a counting-string associate strategy. In particular, counting-string associate errors were not made more often on number-fact problems than on the other problem types. This was particularly true for the type of problem on which counting-string associates were the most frequent type of error in Siegler and Shrager's (1984) study, i.e., problems on which the second term of the problem was greater than or equal to the first term. In our problem set there were only two problems of this type, 2 + 4 and 2 + 2. In fact, counting-string associates on these problems were most frequent on the nonverbal problems on which no number words were expressed by the examiner (49% on nonverbal problems; 23% on story problems; 19% on number-fact problems). This may be attributable to the fact that the counting-string associates on these two problems are also adjacency errors. The less frequent occurrence of counting-string associate errors on our verbal problems than in Siegler and Shrager's study may be attributable to their use of more difficult, higher numerosity problems.

Calculation Methods

Table 6 summarizes the percentage of trials on which each strategy was used (for each age group and problem type), as well as the percentage of trials on which a particular strategy produced a correct answer. Our first analyses of strategies focussed on children's use of finger strategies (count fingers and fingers in Table 6 were combined for this analysis). The mean number of trials on which children in each age group used these finger strategies on each problem type is displayed in Fig. 3. An analysis of variance on the use of these strategies with Age and Problem Type as factors revealed significant main effects of Age ($F(4, 55) = 4.83$, $p < .002$) and Problem Type ($F(2, 110) = 13.25, p < .001$) and a significant Age × Problem Type interaction ($F(8, 110) = 11.18$, $p < .008$). As predicted, the effect of problem type reflected the finding that fingers were used most often for number-fact problems, next often for story problems, and least often for nonverbal problems. Tests of simple effects showed that this ordering is significant only for the two oldest age groups ($p < .01$ and $p < .001$ for age groups 5-6 to 5-11 and 6-0 to 6-5, respectively), most likely because finger and finger counting strategies were seldom employed by children under 5½ years of age on any problem type. The Age × Problem Type interaction is not readily interpretable because of floor effects in the use of finger strategies in the younger age groups.

On all problem types, children in the two oldest age groups (5-6 to 6-5) tended to reach correct solutions to calculation problems more often when they used their fingers than when they did not. Accuracy on the nonverbal problems was 90% when fingers were used compared to 78% when they were not, accuracy on the story problems was 83% when fingers were used compared to 66% when they were not, and accuracy on the number-fact problems was 75% when fingers were used compared to 54% when they were not. In fact, among children in the two oldest age groups, the percentage correct on story problems and number-fact problems when

TABLE 6
CHILDREN'S CALCULATION STRATEGIES, BY AGE GROUP, PROBLEM TYPE, AND OPERATION

	Addition		Subtraction	
Strategy	No. of trials on which strategy used	Correct answers (%)	No. of trials on which strategy used	Correct answers (%)
Nonverbal problems				
Age group 4-0 to 4-5 years				
Unobserved	68	46%	59	37%
Counting	3	100%	3	0%
Count fingers	0	*	0	*
Fingers	0	*	0	*
Imitation	1	100%	10	80%
Age group 4-6 to 4-11 years				
Unobserved	65	72%	56	46%
Counting	4	25%	6	16%
Count fingers	0	*	0	*
Fingers	0	*	0	*
Imitation	3	100%	10	70%
Age group 5-0 to 5-5 years				
Unobserved	56	53%	54	59%
Counting	4	25%	3	67%
Count fingers	0	*	0	*
Fingers	0	*	0	*
Imitation	12	58%	15	53%
Age group 5-6 to 5-11 years				
Unobserved	47	83%	55	77%
Counting	10	60%	4	75%
Count fingers	4	75%	2	50%
Fingers	2	100%	1	100%
Imitation	9	78%	10	90%
Age group 6-0 to 6-5 years				
Unobserved	55	76%	52	88%
Counting	6	83%	4	75%
Count fingers	2	100%	3	67%
Fingers	3	100%	2	100%
Imitation	6	100%	11	63%
Story problems				
Age group 4-0 to 4-5 years				
Unobserved	72	11%	72	24%
Counting	0	*	0	*
Count fingers	0	*	0	*
Fingers	0	*	0	*
Age group 4-6 to 4-11 years				
Unobserved	69	29%	70	33%
Counting	2	100%	1	100%
Count fingers	0	*	0	*
Fingers	1	100%	1	100%

TABLE 6—*Continued*

Strategy	Addition: No. of trials on which strategy used	Addition: Correct answers (%)	Subtraction: No. of trials on which strategy used	Subtraction: Correct answers (%)
Age group 5-0 to 5-5 years				
Unobserved	71	34%	72	29%
Counting	0	*	0	*
Count fingers	1	0%	0	*
Fingers	0	*	0	*
Age group 5-6 to 5-11 years				
Unobserved	60	43%	59	66%
Counting	1	0%	4	75%
Count fingers	4	75%	3	33%
Fingers	7	71%	6	83%
Age group 6-0 to 6-5 years				
Unobserved	42	62%	40	58%
Counting	5	100%	8	100%
Count fingers	4	50%	4	100%
Fingers	21	85%	20	100%
Number-fact problems				
Age group 4-0 to 4-5 years				
Unobserved	70	9%	70	6%
Counting	0	*	0	*
Count fingers	2	100%	1	0%
Fingers	0	*	1	0%
Age group 4-6 to 4-11 years				
Unobserved	70	19%	67	37%
Counting	1	100%	0	*
Count fingers	1	100%	0	*
Fingers	0	0	5	*
Age group 5-0 to 5-5 years				
Unobserved	61	13%	58	24%
Counting	5	60%	5	40%
Count fingers	5	100%	9	33%
Fingers	1	0%	0	*
Age group 5-6 to 5-11 years				
Unobserved	50	56%	51	67%
Counting	0	*	3	0%
Count fingers	14	79%	3	67%
Fingers	8	100%	15	30%
Age group 6-0 to 6-5 years				
Unobserved	29	69%	22	37%
Counting	10	90%	23	67%
Count fingers	16	75%	12	58%
Fingers	17	94%	15	70%

Note. Total per cell is 72 (12 children per age group × 6 trials per problem type and operation).

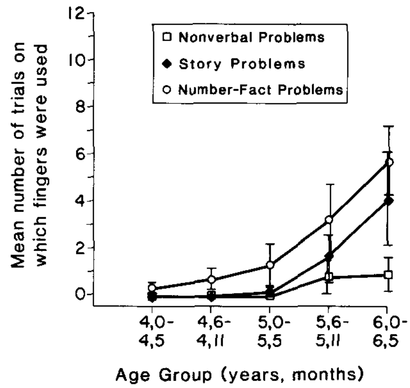

FIG. 3. Children's use of fingers by age group and problem type (bars denote standard errors).

finger or finger counting strategies were used was similar to the overall percentage correct on nonverbal problems (84% correct using fingers on story problems and 71% correct using fingers on number-fact problems compared to 76% correct on nonverbal problems).

Finally, we examined whether use of the imitation strategy on the nonverbal problems affected performance on these problems. A child was classified as an imitator if the imitation strategy was used on at least one problem. Only 15 of 60 children in the sample met this criterion. Inclusion in this group was not related to age (3 children in age group 1; 4 in age group 2; 3 in age group 3; 3 in age group 4; and 2 in age group 5). The

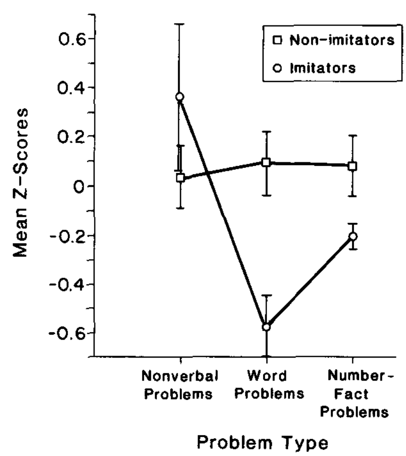

FIG. 4. Mean z scores of imitators and nonimitators by problem type (bars denote standard errors).

nongeneralizability of this strategy is shown by the fact that imitators performed better on the nonverbal problems, but worse on the other two problem types than nonimitators (see Fig. 4). In order to compare the performance level of imitators and nonimitators on the three problem types, we converted each child's raw score to a z score because the distribution of ages among the imitators and nonimitators was not identical. Each child's z scores were calculated using the mean and standard deviation of the child's age group for a particular problem type. An analysis of variance on subjects' z scores with Group (Imitator, Nonimitator) and Problem Type as factors revealed a significant Group by Problem Type interaction ($F(2, 116) = 4.99, p < .008$). Tests of simple effects showed that the effect of problem type was significant for imitators ($p < .01$) but not for the nonimitators. This reflected the finding that the imitators' mean z score on nonverbal problems was significantly higher than their mean z score on story problems ($p < .001$) or number-fact problems ($p < .01$), whereas the nonimitators' mean z scores on the three problem types were nearly equivalent (see Fig. 4). The nonimitators performed significantly better than the imitators on the story problems ($p < .02$), but the group differences on the other two problem types did not reach statistical significance.

Because the imitators performed somewhat better than the nonimitators on the nonverbal problems, it is important to determine whether the nonverbal problems remain significantly easier than the other problem types with imitators excluded. In order to address this question, we performed an analysis of variance with Age, Problem Type, and Operation as factors, excluding the 15 "imitators." Results were identical to the analysis of variance in which all subjects were included, indicating that nonverbal problems were significantly easier than the other problem types for nonimitators as well as for imitators.

DISCUSSION

Our study compared young children's ability to calculate on three different problem types: nonverbal problems, story problems, and number-fact problems. Throughout the age range tested, performance level on the nonverbal problems was higher than that on story problems and number-fact problems. This was true for both addition and subtraction problems, as well as for both small and somewhat larger numerosity problems. Whereas children as young as 4 to $4\frac{1}{2}$ years of age achieved some success on nonverbal calculation problems, comparable levels of performance were not achieved until $5\frac{1}{2}$ to 6 years of age on addition and subtraction story problems and on addition number-fact problems, and not until 6 to $6\frac{1}{2}$ years of age on subtraction number-fact problems.

Many studies examining the quantitative abilities of infants and young children report that performance with very low numerosities is better than performance with higher numerosities or that their quantitative abilities are limited to very low numerosities (e.g., Gelman & Gallistel, 1978; Sophian, 1987, 1988; Strauss & Curtis, 1981; Wynn, 1990; Starkey & Gelman, 1982). Consistent with these findings, our subjects performed significantly better on very low numerosity problems than on the somewhat higher numerosity problems on both the nonverbal task and the story problem task, throughout the age range tested. In contrast, numerosity did not have a significant effect on children's success on number fact problems, at least within the range of numerosities included in this study.

As noted under Results, the absence of a numerosity effect on number-fact problems does not appear to be attributable to lower overall performance on this task. It is possible that the child's frequent use of finger strategies on number-fact problems is a factor. When using fingers to perform a calculation the child is able to view the physical referents for both terms of a problem simultaneously without having to hold one in memory as is necessary on the nonverbal task. The simultaneous availability of physical referents for both terms of the problem makes recounting and checking more feasible, possibly decreasing the effect of numerosity on children's problem solving success (Brainerd, 1983). Retrieving a previously memorized number fact in response to a number-fact problem also may be a factor. The difficulty of retrieving the answer to a higher numerosity problem such as 4 + 1 may not differ significantly from the difficulty of retrieving the answer to a lower numerosity problem such as 2 + 1. This may be particularly true by $5\frac{1}{2}$ to 6 years of age, the age at which children begin to have any success on number-fact problems. These same factors may account for the less dramatic effect of numerosity on story problems than on nonverbal problems.

Young children's advantage in solving the nonverbal calculation problems was reflected by the nature of their errors as well as by their performance level. On addition problems, children's errors on the nonverbal task were more frequently in the right direction and were closer to the correct answer than on the other problem types, as indexed by both mean distance from the correct answer and the percentage of total errors adjacent to the correct answer. Further, second term repetition errors on addition problems were more frequently in the right than in the wrong direction on nonverbal problems and story problems, but not on number-fact problems. The error pattern on subtraction problems was less conclusive. In particular, errors on subtraction problems were more frequently in the right than in the wrong direction, but this did not interact with problem type. Analyses on distance of errors from the correct answer showed that errors on nonverbal problems were significantly closer to the correct answer than those on number-fact problems. Although the mean distance of errors from the correct answer on story problems was intermediate between the other two problem types, it did not differ significantly from either of them. Further, the percentage of total errors adjacent to the correct answer was greater for nonverbal problems and story problems than for number-fact problems, but these differences were not significant.

Multiple factors may contribute to young children's early calculation abilities being more apparent on nonverbal problems than on the other problem types. Several considerations suggest that the availability of physical referents plays an important role in children's better performance on nonverbal problems. On the nonverbal problems, the numerosities involved in the terms of the problem are provided by the disks. Further, the operation of adding or subtracting is provided by the physical act of combining or separating sets. These physical referents may make it easier for the child to represent the terms of the problem and the operation involved in the calculation than on verbal problems. In particular, on story problems and number-fact problems, the child must access and represent the numerosities and operation involved in a problem from linguistic input. Even if the child is able to undertake this representational task, he/she may be more likely to make an error in representing one or more of the basic elements involved in the calculation than when the numerosities and operation are provided by physical referents, as is the case for the nonverbal problems.

Children's significantly better performance on subtraction story problems than on subtraction number-fact problems also may be related to the difficulty of creating representations for the numerosities and operation involved in a problem. Story problems may be easier both because they provide a more meaningful context and because they explicitly refer to concrete referents (e.g., "three apples"), whereas number-fact problems do not ("three"). Interestingly, number-fact problems become easier than story problems later during development, most likely because children attend less to meaningful aspects of story problems as they memorize number-fact problems (Carpenter & Moser, 1982).

Variations in the frequency with which different calculation methods were used on the three problem types also support the importance of physical referents in children's problem solving success. In particular, 5- and 6-year-old children used their fingers (counting fingers and finger strategies) least frequently for nonverbal problems which provide object referents, at an intermediate level for story problems which refer to specific object sets that are not physically present, and most frequently for number-fact problems which provide no explicit reference to object sets. Thus, the use of fingers appears to increase with decreasing availability of other concrete referents. Moreover, our data indicate that children are not very successful in solving story problems and number-fact problems until sometime between 5 and 6 years of age, when they start using their fingers with some frequency. In fact, we find that when finger strategies are used to solve story problems and number-fact problems, performance is as good as on the nonverbal problems. Thus, the importance of the availability of object referents to young children's success in performing calculations is underscored not only by their better performance on the nonverbal task but also by the role of finger strategies in their success on story problems and number-fact problems.

This raises the question of why children are able to solve at least some of the nonverbal problems at an earlier age than when they start to use their fingers spontaneously as calculation aids on the verbal problems. While the nonverbal problems provide physical representations of the numerosities (disks) and the operation of adding or subtracting (putting disks in or taking them out), using fingers on story problems and number-fact problems requires the child to introduce a calculation devise. This developmental pattern is similar to that observed when children compare the lengths of two entities. In particular, by 4 years of age children can make judgments about the relative lengths of two entities when each has been compared to the same physically present standard (Bryant & Trabasso, 1971), but they do not spontaneously introduce a unit measure such as a ruler to compare two lengths until age 7 (Piaget, Inhelder, & Szeminska, 1960). Both the absence of finger use in calculating and the absence of the use of a ruler in measuring may stem from young children not realizing that such intervening representations would be helpful.

The presence of physical referents on the nonverbal problems also may increase the likelihood that children will use counting algorithms in calculating (Starkey & Gelman, 1982). The one-by-one addition or subtraction of disks in the second term of the nonverbal problems may, in fact, have encouraged the application of counting algorithms. Although overt counting with or without the use of fingers on the nonverbal task was infrequent, several considerations suggest that children may have been counting silently. First, the tendency for children in the two youngest age groups to perform better on the nonverbal addition task than on the nonverbal subtraction task is consistent with children using counting on the nonverbal task. For the nonverbal addition problems "counting on," i.e., determining the size of the initial set and then counting each additional item as it is added to that set, may be a relatively easy method to apply. It should be noted, that "counting on" is a mode of calculation because it involves the transformation of a set through the addition of elements. In contrast to counting on, a "counting down" strategy would be difficult to apply, particularly for children in our youngest age groups, as it would involve counting backward (Fuson & Willis, 1988; Thornton, 1990). Thus, it does not seem that the application of counting strategies can fully account for better performance on the nonverbal task as young children perform significantly better on nonverbal subtraction problems as well as on nonverbal addition problems than on the other problem types.

Second, we found that children's wrong answers on the nonverbal addition problems deviated from the correct answer by + or −1 frequently, and more often than on the other problem types. Such deviations have been interpreted as reflecting inaccurate counting (Gelman & Gallistel, 1978), although other interpretations are possible. Third, the finding that children in the two oldest age groups who use finger and finger counting strategies on the verbal problem types perform as well as children in these age groups on nonverbal problems supports the use of counting on the nonverbal problems. Thus, physical referents, whether they are objects or fingers, may promote the use of counting algorithms in calculating. Finally, in another study (Jordan, Levine, & Huttenlocher, 1991), we found that children as young as 3 years of age perform equally well on the nonverbal task when they respond with a number word or by laying out disks. This suggests that by 3 years of age children are mapping the numerosities involved in the nonverbal task onto the verbal number system, most likely by counting.

Children's performance on the nonverbal problems also may be affected by their ability to apply the "imitation" strategy to these problems. This strategy was observed on at least one problem in 25% of the children tested. The specificity of this algorithm to the nonverbal problems is underlined by our finding that performance on nonverbal problems was higher but performance on story problems and number-fact problems was lower for children who used this strategy than for those who did not. However, it should be noted that application of this method is not sufficient to account for children's better performance on the nonverbal task as performance remained significantly higher on the nonverbal task when the imitators were excluded from the analysis comparing performance levels on the three calculation tasks.

Finally, removing linguistic demands (e.g., Carpenter et al., 1981; Riley et al., 1983) and/or decreasing memory demands (Brainerd, 1983) may contribute to the ease of the nonverbal task. Although we are not able to specify the relative contributions of linguistic factors, memorial factors, and the availability of physical referents for numerosities and operations to the ease of the nonverbal task, it is clear that one or more of these factors makes it possible to observe the child's calculation abilities at a much younger age than is possible with verbal story problems and number-fact problems.

In conclusion, our results are consistent with findings in the literature indicating that children have a rich array of quantitative abilities prior to developing conventional verbal methods of operating on quantities (e.g., Gelman and Gallistel, 1978; Sophian & Adams, 1987; Starkey & Gelman, 1982). Just as young children have knowledge of low numerosities prior to learning the relevant count words (e.g., Strauss & Curtis, 1981; Starkey & Cooper, 1980), our findings, together with a few previous studies (Starkey & Gelman, 1982; Sophian & Adams, 1987), indicate that children are able to calculate prior to being able to solve the simplest word problems or number-fact problems. The finding that nonverbal problems are easier for young children than story problems and number-fact problems supports the view that the child's earliest ability to add and subtract is based on experiences combining and separating sets of objects in the world. As the child develops, the use of physical referents to represent the quantities involved in a calculation may be replaced by an increased ability to create representations for quantities referred to linguistically and/or by an increased reliance on memorization of number facts and schooled calculation algorithms. Whether the child's early nonverbal calculation skills predict his/her later success with story problems, number-fact problems, or other aspects of mathematical functioning remains an open question.

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RECEIVED: April 4, 1990; REVISED: August 20, 1991.