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Early Fraction Calculation Ability

Kelly S. Mix
Indiana University Bloomington

Susan Cohen Levine and Janellen Huttenlocher
University of Chicago

Three- to 7-year-olds' ability to calculate with whole-number, fraction, and mixed-number amounts was tested using a nonverbal task in which an amount was displayed and then hidden (J. Huttenlocher, N. C. Jordan, & S. C. Levine, 1994). Next, an amount was added to or subtracted from the hidden amount. The child's task was to determine the hidden amount that resulted from the transformation. Although fraction problems were more difficult than whole-number problems, competence on all problem types emerged in the early childhood period. Furthermore, there were striking parallels between the development of whole-number and fraction calculation. This is inconsistent with the hypothesis that early representations of quantity promote learning about whole numbers but interfere with learning about fractions (e.g., R. Gelman, 1991; K. Wynn, 1995, 1997).

Conventional fraction algorithms have been notoriously difficult for children to master (e.g., Behr, Wachsmuth, Post, & Lesh, 1984). Not only do school-age children make many more errors on written fraction calculation problems than they do on whole-number problems, but these errors often reflect a failure to grasp basic properties of fractions, such as the relation between the number of fractional parts and the size of these parts. Thus, children frequently add numerators and denominators together with apparent disregard for the part–whole relation these symbols are intended to represent (e.g., ¾ + ½ = 4/6; see Resnick & Ford, 1981). Even when children solve fraction calculation problems accurately, they often do so without appearing to understand the reasoning behind the symbol manipulations (Kerslake, 1986). For example, 12- to 14-year-olds who correctly solved problems like ¾ + ½ by finding a common denominator still could not explain why they did so. Instead, they seemed to apply by rote the procedure they had been taught in school without understanding the motivation for doing it.

To understand the source of these difficulties, researchers have begun to consider what children know about fractions before they receive formal instruction. This is an important line of inquiry because it addresses the difference between symbols and meaning. That is, one reason that children might err on conventional fraction problems is that they do not grasp the relations between fractional amounts. In this case, fraction symbols would be difficult to interpret because the conceptual referents would not be clear. In line with this explanation, several investigators have proposed that children's difficulty with learning fractions is rooted in the structure of their innate numerical representations (Gelman, 1991; Wynn, 1995, 1997). For example, Gelman argued, based on Gallistel and Gelman's (1992) preverbal counting model, that children represent numerosity as discrete blurs on a mental number line. This structure, Gelman contended, is isomorphic to that of counting (or natural) numbers and therefore helps children learn the verbal count words. However, because this structure is inherently discontinuous, Gelman predicted that children would have great difficulty understanding fractions and learning the verbal labels for them.

In support of this prediction, Gelman (1991) cited several experiments in which children's responses to fractions revealed misinterpretations based on whole numbers. For example, Gelman, Cohen, and Hartnett (1989, cited in Gelman, 1991) asked kindergarten, first-grade, and second-grade students to read and order written fractions, such as ½ and ¼. They found that children tended to read the written fractions in terms of whole numbers (e.g., reading "½" as "one and two," "one plus two," or "twelve" rather than "one-half"). Furthermore, most children incorrectly judged ¼ to be larger than ½, apparently because they treated these fractions like natural numbers. That is, children based their judgments on the fact that 4 is larger than 2. Gelman interpreted these errors as evidence that children's innate counting principles interfere with their ability to comprehend quantities beyond whole numbers. Indeed, her data suggest that it is not until third grade that children begin to overcome these difficulties.

However, there is at least one alternative explanation for children's poor performance on such tasks that does not invoke a preexisting representation that is limited to whole numbers. Instead, children may have trouble interpreting the conventional symbols for fractions because of interference from years of experience using these same symbols to stand for whole numbers. This alone could lead to the error pattern Gelman et al. (1989, cited in

Kelly S. Mix, Department of Psychology and Program in Cognitive Science, Indiana University Bloomington; Susan Cohen Levine and Janellen Huttenlocher, Department of Psychology, University of Chicago.

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Correspondence concerning this article should be addressed to Kelly S. Mix, Department of Psychology, Indiana University, Bloomington, Indiana 47405. Electronic mail may be sent to kmix@indiana.edu.

Gelman, 1991) observed, independent of whether children are capable of understanding and representing fractional amounts. Because the earlier tasks required children to read and interpret conventional symbols, they do not provide a clear test of the early-constraints hypothesis.

The same is true for another Gelman et al. (1989) experiment cited by Gelman (1991), although it may not be as immediately apparent. In this experiment, children were shown a special number line that was marked with sets of circles rather than written numerals. Thus, one circle, two circles, and three circles were placed at points corresponding to the whole numbers instead of the numerals 1, 2, and 3. Children were then given cards with various fractional amounts of a circle (e.g., 1½, 1⅓, ½, ¼, etc.) and were asked to place the cards "where [they] belonged" on the number line. The initial placements made by children just finishing kindergarten and first grade were rarely correct (approximately 10% and 20% correct responses, respectively). It is interesting to note that children most often erred by placing the cards on top of the whole-number landmarks. For example, children would place the card with 1½ circles on top of the two circle landmark because both amounts had two parts. Thus, children appear to err because of a bias to interpret fractions in terms of whole numbers.

However, as before, the source of this bias is unclear because prior experience with conventional number lines may have led to the observed whole-number errors. After all, kindergarten and first-grade children have probably encountered number lines in school—they are likely to know one when they see it. This prior experience might call to mind the written numerals that would normally be present even though these were not printed on the Gelman et al. (1989, cited in Gelman, 1991) version. In fact, this was virtually ensured by a series of pretest activities that highlighted the conventional whole-number characteristics of the number line. These included having the child place the two circle landmark on the number line between the one and three circle landmark, asking the child to name the landmarks "one, two, and three," and asking how many circles would appear to the left of "one" and the right of "three." Thus, it is unclear whether children's errors were due to confusion with conventional whole-number symbols or misinterpretations based on preexisting mental representations of quantity as Gelman (1991) and Wynn (1995, 1997) have argued.

To disentangle these possible explanations, one would need to measure children's understanding of fraction concepts while avoiding use of conventional symbols. In fact, when such tasks have been used, the beginnings of fraction knowledge have been revealed in relatively young children. For example, Goswami (1989) gave 4-, 6-, and 7-year-olds a series of a:b::c:d analogy problems based on shapes shaded in equivalent proportions (e.g., ½ of a circle:½ of a rectangle::¼ circle:?). Children chose the "d" term from among five choices that showed different shapes shaded in different proportions. Goswami found that all three age groups performed significantly above chance on these problems, although the scores for 4-year-olds were much lower than those for 6- and 7-year-olds (4-year-olds, 31% correct; 6-year-olds, 74% correct; and 7-year-olds, 86% correct). A simple proportion matching task also was given in which the a, b, and c terms showed the same proportion (e.g., ½ of a diamond:½ of a circle::½ of a square:?). The task was to complete the analogy by indicating which of four choice pictures showed the same proportion. Four-year-olds performed significantly above chance on this task (56% correct), and 6- and 7-year-olds performed near ceiling (86% and 91% correct, respectively). Thus, although the ability to recognize equivalence between proportions undergoes some development in early childhood, it clearly emerges before school age.

Spinillo and Bryant (1991) also reported relatively early emergence of fraction reasoning. They used a forced-choice matching task in which 4- to 7-year-olds were shown a picture of a box that was divided horizontally into white and blue sections. Then the children were asked to indicate which of two choices (larger boxes that each contained different proportions of white and blue) showed the equivalent proportion. This was not an easy task, and most of the 4- and 5-year-olds performed randomly. However, 6- and 7-year-olds performed significantly above chance, thus demonstrating proportional reasoning in relatively young children.

Studies of the ability to divide sets and amounts among several recipients have also revealed the beginnings of fraction knowledge in young children (e.g., Frydman & Bryant, 1988; Hunting & Sharpley, 1988). For example, Hunting and Sharpley asked 4- to 7-year-olds to distribute amounts of both continuous and discrete material among several dolls. They compared children's performance on this distribution task with their ability to divide materials into specific fractions in response to a verbal request (e.g., "Cut this in half"). Although children were unable to interpret the verbal commands and perform specific divisions, most of them could systematically divide the amounts into equal shares by age 5 years. These results, taken together with the results of equivalence matching experiments (Goswami, 1989; Spinillo & Bryant, 1991), provide evidence of fraction reasoning emerging around the time children enter school—earlier than they can interpret conventional symbols and verbal labels for fractions and long before they have received formal instruction in fraction concepts.

The extant studies on early fraction reasoning have focused primarily on equivalence relations—dividing equally or recognizing equal proportions. However, there are many other skills that could be tested. In fact, the literature on whole-number concepts has investigated a rich array of quantitative concepts in preschool children, including equivalence relations (Gelman, 1972; Gelman & Tucker, 1975; Huttenlocher, Jordan, & Levine, 1994; Mix, in press; Mix, Huttenlocher, & Levine, 1996), ordinal relations (Bullock & Gelman, 1977; Estes, 1976), and calculation (Huttenlocher et al., 1994; Jordan, Huttenlocher, & Levine, 1994; Levine, Jordan, & Huttenlocher, 1992). In these studies, methods have been devised for studying quantitative concepts that do not require knowledge of conventional symbols or verbal labels. Use of such methods would allow a more direct investigation of children's ability to reason about fractions and extend what is known about the earliest emergence of these skills.

One such method was developed for testing whole-number calculation in very young children (Huttenlocher et al., 1994; Jordan et al., 1994; Levine et al., 1992). In this procedure, a set of objects is displayed for a few seconds and then hidden. Next, objects are shown being added to or removed from the hidden set, but the outcome cannot be seen (thus the initial set and the transformation must be mentally represented). The child's task is to indicate the numerosity of the resultant set, either by producing an equivalent array (Levine et al., 1992) or by pointing to a picture of an equivalent array (Jordan et al., 1994). Levine et al. compared children's performance on nonverbal calculation problems to performance on verbal story problems and number fact problems. They found that children successfully solved the nonverbal problems by 4 years of age, whereas children did not achieve comparable levels of success on either of the verbal problems until 5½ to 6½ years of age. A subsequent study with younger children revealed that the ability to solve nonverbal calculation problems first emerges around 3 years of age (Huttenlocher et al., 1994). Furthermore, the response mode does not affect children's performance: Children respond just as accurately when they produce an equivalent set as when they choose an equivalent set from among four pictures (Jordan et al., 1994).

In the present study, we used a modified version of the Levine et al. (1992) task to evaluate children's ability to calculate with fractions and mixed numbers. In our version, fractional amounts were substituted for objects. Thus, for each problem the amount comprising the first term was placed in a shallow hole using circular pieces of sponge (e.g., half was filled in). Then the hole was hidden. Next, semicircles of sponge were either added to or subtracted from the hidden amount (e.g., one quarter was added), but the outcome could not be seen. Children chose the resultant amount (e.g., three quarters) from among four pictures.

This procedure has several advantages. First, it measures fraction reasoning without using verbal labels. Specifically, it involves representing fractional parts and reasoning about transformations of these parts without requiring knowledge of conventional fraction notation. To be correct, children must attend to the size of the pieces involved, not just the number of pieces. However, understanding fractions not only involves attention to pieces and amount but also the ability to interpret these amounts in relation to some unit. The term *three quarters* only has meaning in relation to a particular whole, such as three quarters of a pizza, a cup, or an hour. This unit need not be a standard unit of measurement—it could be defined as any whole thing or bounded mass (e.g., three quarters of a rock or three quarters of a blob of whipped cream). Similarly, the unit could be defined beyond the level of a single object (e.g., three quarters of a dozen, a gross, or the United States population). The important thing is that fractions derive their meaning in relation to a whole unit. Gelman (1991) contended that fractions are difficult for young children to understand because they cannot represent amounts between whole units. The present procedure tests whether this is so by requiring children to reason about quantities that fall between whole circles. Success on this task would argue against the idea that early quantitative representations are restricted to whole numbers.

Another advantage of the nonverbal calculation task is that it allows a direct comparison between development of whole-number concepts and fraction concepts. If whole-number concepts are privileged, then development of fraction calculation should be dramatically different from development of whole-number calculation. That is, either children should not be able to solve fraction calculation problems like those described earlier, or development of this ability should be idiosyncratic for fractions versus whole numbers. However, if development of fraction calculation parallels development of whole-number calculation, it would add to the evidence that early quantitative reasoning ability is not limited to whole numbers. Instead, it would show that children can attend to either whole numbers or fractional amounts when reasoning about quantity.

We present two experiments in which we evaluate children's ability to calculate with fractional amounts in a nonverbal task. In Experiment 1, we focus on simple fraction calculation problems with solutions that are less than or equal to one (e.g., ½ + ¼ = ¾). Because these problems are relatively low in complexity, they are likely to reveal early fraction calculation ability if it exists. Furthermore, these problems are analogous to the range of whole-number problems young children are known to solve (Huttenlocher et al., 1994). Thus, they provide the most direct possible comparison between development in fraction and whole-number calculation in this task. In Experiment 2, we focus on mixed number problems with solutions that are less than or equal to three (e.g., 1½ + ¾ = 2¼). Because these problems involve larger and more variable quantities than the simple fraction problems, they provide a stronger test of early fraction reasoning. In particular, because these problems involve up to three whole units, they allow us to test directly whether children can represent amounts between whole numbers.

Experiment 1

Method

Participants

Seventy-two children participated in the experiment. They were divided evenly into three age groups: 3-year-olds (mean age = 3 years 7 months; range 3 years 1 month to 3 years 11 months), 4-year-olds (mean age = 4 years 6 months; range 4 years to 4 years 11 months), and 5-year-olds (mean age = 5 years 3 months; range 5 years to 5 years 10 months). Each age group included 12 boys and 12 girls. The children were drawn from preschools that served a predominantly White, middle-class population. All came from homes in which English was the primary language.

Materials

Each child completed two tasks: fraction calculation and whole-number calculation. These were counterbalanced for order of presentation across children. Materials for the fraction calculation task included a set of four white circular sponges (6 cm in diameter; 0.5 cm thick), a black tray (10 × 30 cm) with a shallow circular hole in the center (6 cm diameter; 0.5 cm deep), a paper bag (15 cm high), and a three-paneled cardboard screen (28 × 21 cm per panel). Each sponge was cut into one-quarter pieces. Strips of Velcro® were attached to the edges so that the pieces could be rejoined to form all or part of a circle. Materials for the whole-number calculation task included a set of 15 black disks (1.9 cm in diameter), a box, a cover, and a white cardboard mat (25 × 25 cm). The cover had an opening on one side so that the experimenter could easily add or remove disks.

A response book was used in both calculation tasks. Each page of the response book showed four amounts that were arranged on an imaginary 2 × 2 grid. One of the amounts was the correct solution to the calculation problem, and the other three were foils. For fraction calculation items, the four choices were ¼, ½, ¾, and 1. Each amount was presented as a white circle (3 cm in diameter) or semicircle centered in a black rectangle. The rectangles were separated from one another by 4 cm of white space. Note that the size difference between the sponge stimulus circles and the printed response circles prevented children from responding correctly by matching the total amounts. Also, unlike the sponge pieces that were divided and rejoined in quarters, the response amounts were presented as seamless white shapes (see Figure 1). This helped to ensure that children were not responding on the basis of the number of quarter pieces in each solution. For the whole-number calculation items, the four amounts were arrays of one, two, three, and four black dots. Each array was presented in a horizontal line running across the center of a white rectangle that was

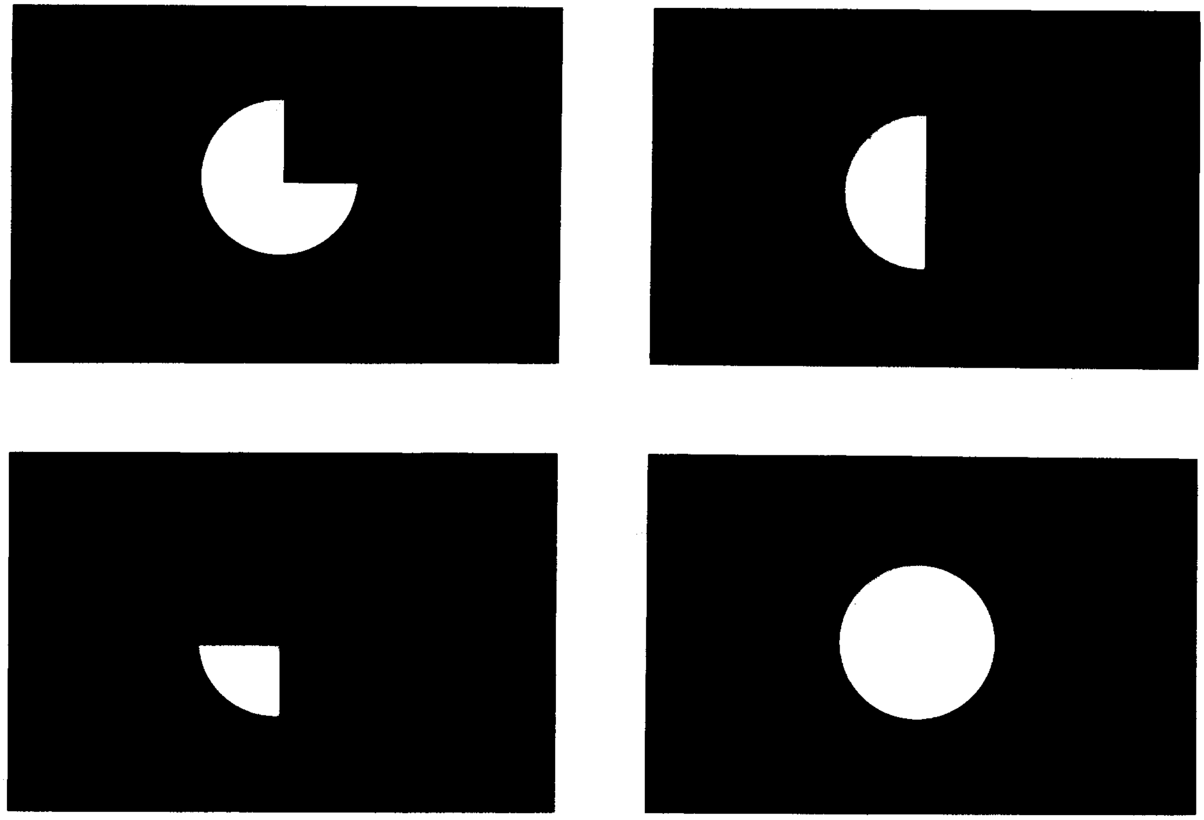

Figure 1. Sample response page used for the fraction and mixed-number calculation tasks.

surrounded by a narrow black border. The rectangles were separated from one another by 4 cm of white space. The spatial positions of the response amounts were counterbalanced across items so that the correct response appeared in each quadrant equally often. This was true both within and across tasks. Similarly, the magnitude of the correct response in relation to the foils was counterbalanced across items so that the correct response was equally often the smallest, next to smallest, largest, and next to largest amount, both within and across tasks.

Procedure

Fraction calculation task. Twelve fraction calculation problems were presented in one of two fixed random orders. The number of addition and subtraction problems was equal. Fraction calculation problems began when the first term was placed in the shallow hole of the black test tray using pieces of sponge (e.g., three quarters were filled in). This was left in full view of the child for a few seconds, and then a screen was raised so that the tray was hidden. Next, a piece of sponge was shown either entering or emerging from behind the screen—that is, being added to or subtracted from the hidden amount (e.g., half was subtracted)—but the outcome could not be seen. Children chose the resultant amount (e.g., one quarter) from among the four pictures shown in the response book. Note that portions were added or subtracted altogether. So, for example, ½ would be subtracted as two ¼ pieces attached to form a single piece, not as two separate pieces. During the fraction calculation problems, the pool of sponge pieces was kept together in a short paper bag that was placed in front of the screen and off to one side. This was intended to keep the pool of pieces hidden, yet allow children to see every time the experimenter moved pieces to or from the pool.

To introduce the fraction calculation block, the experimenter showed the first problem while prompting the child to look closely at both the initial amount and the amount used in the transformation. Then, the corresponding page in the response book was revealed. The experimenter said, "Now, which one of these [gesturing to the response page] looks just like mine [pointing behind the screen]?" The child's response was recorded, but no feedback was given on this or subsequent trials. Pilot testing revealed that this procedure sufficiently conveyed the task without the need for practice trials.

Whole-number calculation task. A block of eight problems was presented in one of two fixed random orders. (Order 1 was presented along with Order 1 of the fraction problems and Order 2 was presented along with Order 2 of the fraction problems.) The block included an equal number of addition and subtraction problems chosen from the pool of 12 problems with sums or minuends of 4 or fewer items (e.g., 1 + 3, 2 + 1, 1 + 1, 4 − 3, 3 − 2). These problems are among the first to be solved by young children as they enter their 3rd year (Huttenlocher et al., 1994). Only eight problems were included in the present study to provide an adequate sample of this ability without causing undue fatigue, particularly in the youngest participants.

The whole-number calculation procedure was parallel to the fraction calculation procedure, except that sets of individual disks were used instead of continuous sponge semicircles. Whole-number calculation problems began when the first term was placed on the mat as a horizontal line of black disks (e.g., three disks were placed on the mat). These disks were left in full view of the child for a few seconds, and then they were hidden underneath a cover. Next, disks were shown either entering or emerging from the cover as a group, that is, being added to or subtracted from the hidden amount (e.g., two disks were subtracted), but the outcome could not be seen. Children chose the resultant amount (e.g., one disk) from among

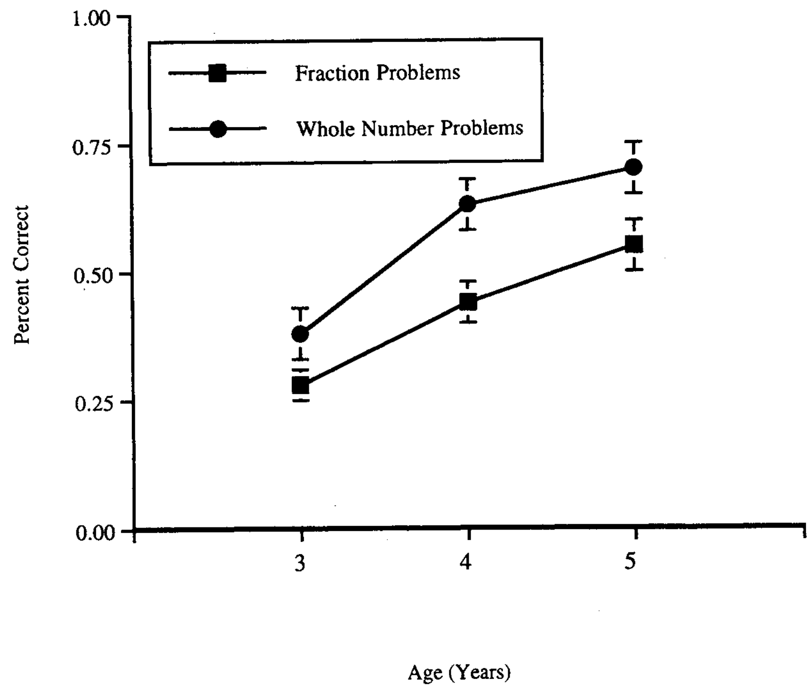

Figure 2. Proportion of correct responses for each age group on the whole-number and fraction calculation tasks, Experiment 1. Note that these are discontinuous categories. Vertical lines depict standard errors.

the four pictures in the response book. During the whole-number calculation problems, the pool of disks was kept together in a box that was placed off to one side of the mat. This was intended to keep the pool of disks hidden, yet allow children to see every time the experimenter added or removed disks from the pool.

The whole-number calculation block was introduced the same way as the fraction calculation block. The experimenter showed the first problem while prompting the child to look closely at both the initial amount and the amount used in the transformation. Then, the corresponding page in the response book was revealed. The experimenter said, "Now, which one of these [gesturing to the response page] looks just like mine [pointing behind the screen]?" The child's response was recorded, but no feedback was given on this or subsequent trials. Pilot testing revealed that this procedure sufficiently conveyed the task without the need for practice trials.

Results and Discussion

Every child completed all the problems in each block. Figure 2 shows the mean proportion correct for each age group on both calculation tasks. We used two-tailed t tests to compare the proportions correct for each age group to chance (i.e., 0.25 in a four-choice task). As expected based on previous work (e.g., Huttenlocher et al., 1994), children in all three age groups performed significantly above chance on the whole-number calculation problems: 3-year-olds, $t(23) = 2.60$, $p < .05$; 4-year-olds, $t(23) = 7.50, p < .0005$; and 5-year-olds, $t(23) = 9.34, p < .0005$. On the fraction calculation problems, 4- and 5-year-olds, but not 3-year-olds, performed significantly above chance: 3-year-olds, $t(23) = 1.15$, *ns*; 4-year-olds, $t(23) = 4.45$, $p < .0005$; and 5-year-olds, $t(23) = 6.56, p < .0005$. Thus, whole-number calculation emerges about a year earlier than fraction calculation, but at least some competence on both tasks is present by 4 years of age.

An inspection of Figure 2 indicates that fraction calculation problems were more difficult than whole-number problems, but clearly the patterns of development for the two tasks were quite similar. That is, performance improved in parallel over the 3- to 5-year age range. An analysis of variance (ANOVA) confirmed that this was the case. A preliminary ANOVA showed no effects of gender or task order on children's calculation scores (all $ps >$ .15), so these variables were eliminated from subsequent analyses. A second ANOVA with age group as a between-subjects variable and task (fractions vs. whole numbers) as a within-subject variable revealed a significant main effect of task, $F(1, 69) = 29.62$, $p <$ .0001, that reflected better performance on the whole-number problems (.57 correct vs. .42 correct). There also was a significant main effect of age group, $F(2, 69) = 15.70$, $p < .0001$. Pairwise comparisons (Scheffé's S, $p < .05$) revealed that this was due to significantly better performance by 4- and 5-year-olds when compared with 3-year-olds (3-year-olds, $M = .33$; 4-year-olds, $M =$ .53; and 5-year-olds, $M = .62$). The difference in performance between 4-year-olds and 5-year-olds was not significant. Importantly, the interaction between age and task was not significant ($p > .25$). Thus, although significant age and task differences were obtained, developmental changes on these tasks occurred in parallel.[1]

This parallel pattern of development was reflected in two additional respects. First, performance on the fraction problems was

[1] Because the data in this experiment were proportional, parallel analyses were carried out on arcsine transformations of the calculation scores to ensure that skew was not affecting the results. These analyses yielded the same pattern of findings as the analyses using raw data, thus confirming the robustness of the results reported here.

significantly correlated with performance on the whole-number problems, $r(71) = .59$, $p < .0001$. This indicates that a common underlying ability was tapped by both tasks. Second, neither task appeared to be extremely easy or extremely difficult—scores for both were in the middle range of performance. Thus, both problem types could be solved in the preschool age range, but there was still room for growth in both abilities. Such parallels are not predicted by the early-constraints hypothesis.

The present results indicate that there are important similarities between early whole-number and fraction calculation. But does that mean that children were using the same strategy to solve both problem types? Because all fraction problems were presented using one-quarter pieces, it might be argued that children used a whole-number strategy to solve them. Even though the pieces were joined to form semicircles, the divisions between pieces were still perceptible. Thus, it is possible that children solved fraction calculation problems by representing the number of individual quarter pieces as in the whole-number problems rather than representing the continuous amount of the semicircles.

Although this possibility cannot be ruled out entirely, several factors argue against it. First, the format of the response book impedes the use of this strategy because the response choices for fraction problems are presented as continuous semicircles. Therefore, to use a whole-number strategy, children would have to mentally translate the attached pieces into separate pieces, find the solution to the whole-number problem, and then translate this solution back into continuous unitized amounts through division. This approach does not seem to be the most direct way to solve the fraction problems. If children have an alternative strategy they could use, it seems likely that they would.

Second, fraction problems did not order for difficulty according to maximum number of pieces as whole-number problems do (Huttenlocher et al., 1994; Levine et al., 1992). For example, the problem $3 + 1$ encompasses a higher numerosity than $1 + 1$ because to calculate $3 + 1$ nonverbally one must represent up to four things, whereas to calculate $1 + 1$ one must represent a maximum of only two things. Similarly, the problem $4 - 2$ encompasses a higher numerosity than $3 - 2$ because to calculate $4 - 2$ nonverbally one must represent a maximum of up to four things, whereas to calculate $3 - 2$ one must represent a maximum of only three things. In previous studies using the nonverbal calculation procedure, whole-number problems consistently ordered for difficulty on the basis of the highest numerosity encompassed by the problem. Thus, $3 + 1$ was more difficult to solve than $1 + 1$ and $4 - 2$ was more difficult than $3 - 2$. Whole-number problems in the present study also ordered for difficulty according to numerosity, although not quite as consistently as in past research (see Table 1).

In contrast, fraction calculation problems did not order for difficulty according to the maximum number of one-quarter pieces. The analogous contrast to the above addition example would be ¾ (three quarter pieces) + ¼ (one quarter piece) versus ¼ (one quarter piece) + ¼ (one quarter piece). If children used a whole-number strategy to solve the fraction problems, then ¾ + ¼ should be more difficult than ¼ + ¼. However, it was not (see Table 2). In fact, maximum numerosity generally bore little relation to difficulty on the fraction problems. Numerosity four problems tended to be less difficult than lower numerosity problems, which is the opposite of the pattern found for whole-number problems (Huttenlocher et al., 1994; Levine et al., 1992). Furthermore, the most difficult fraction calculation problem also involved the fewest pieces. On the basis of this pattern, it does not seem likely that children solved fraction problems through a whole-number strategy.

It should be noted, however, that even if some children attended to the individual quarter pieces on the fraction calculation problems, they would still be reasoning about the amounts as fractions because the answers were expressed in terms of a unit. If children saw the problems in terms of continuous quantity, the answers were expressed in relation to a whole circle. If they saw them in terms of individual pieces, the answers were expressed in relation to sets of four—as in reasoning about fractions of a dozen, a gross, or some other discrete set. It may not be possible to determine with certainty which approach children used from the present data, but the important point is that in either case they clearly responded in terms of a fractional quantity.

In summary, the results of Experiment 1 indicate that fraction calculation, like whole-number calculation, is an emerging ability in the preschool age range. Although it is true that children in the present study performed better on the whole-number task than the fraction task, the difference is much smaller than the early-constraints hypothesis would predict. In fact, the patterns of development for both tasks are remarkably similar. Thus, the development of fraction concepts does not appear to be idiosyncratic. Furthermore, competence on the fraction calculation task appears long before these concepts are taught in school and also earlier

Table 1
Rank Order (Easiest to Most Difficult) of Whole-Number Problems in Experiment 1 Based on Overall Proportion Correct

			Age		
Problem	Numerosity	Overall	3-year-olds	4-year-olds	5-year-olds
2 − 1	2	.68	.46	.71	.88
1 + 1	2	.68	.50	.79	.75
2 + 1	3	.64	.46	.75	.71
1 + 3	4	.63	.38	.67	.83
3 + 1	4	.61	.38	.71	.75
3 − 2	3	.51	.34	.63	.58
4 − 1	4	.40	.21	.50	.50
3 − 1	3	.39	.34	.25	.58

Table 2
Rank Order (Easiest to Most Difficult) of Fraction Problems in Experiment 1 Based on Overall Proportion Correct

			Age		
Problem	Numerosity	Overall	3-year-olds	4-year-olds	5-year-olds
½ − ¼	2	.57	.54	.58	.58
¼ + ½	3	.54	.29	.58	.75
½ + ½	4	.51	.38	.46	.71
¼ + ¾	4	.50	.29	.54	.67
1 − ¼	4	.50	.13	.67	.71
¾ + ¼	4	.46	.38	.46	.54
1 − ¾	4	.43	.33	.46	.50
¾ − ¼	3	.39	.42	.38	.38
¾ − ½	3	.36	.25	.33	.50
½ + ¼	3	.32	.13	.29	.54
1 − ½	4	.28	.21	.25	.38
¼ + ¼	2	.19	.04	.25	.29

than children succeed on the symbolic and quasisymbolic tasks used in previous research (e.g., Gelman et al., 1989; cited in Gelman, 1991).

Experiment 2

In Experiment 1, 4- and 5-year-olds demonstrated an emerging ability to solve simple fraction calculation problems (i.e., problems with solutions less than or equal to one whole). In Experiment 2, we investigated whether this ability extends to more complex mixed-number problems, such as 3 − 1¼ = 1¾ or ¾ + 1¾ = 2½. Such problems provide a stronger test of the ability to reason about fractional amounts in that they involve larger and more variable quantities than the simple fraction problems. These problems also involve up to three whole units rather than one. Thus, to respond accurately, children must be able to represent amounts between whole numbers. For example, to solve 3 − 1¼, the child would have to recognize that this leaves enough stuff to make one whole with some amount left over.

Method

Participants

One hundred eighty-six children participated in the experiment. They were divided into four age groups: 4-year-olds (mean age = 4 years 5 months; range 4 years to 4 years 11 months); 5-year-olds (mean age = 5 years 4 months; range 5 years to 5 years 11 months); 6-year-olds (mean age = 6 years 4 months; range 5 years 8 months to 7 years); and 7-year-olds (mean age = 7 years 5 months; range 6 years 10 months to 8 years). Children were tested between the spring and fall of 1996. The 4- and 5-year-olds all attended preschool or day-care programs. All of the 6-year-olds were between kindergarten and first grade in school, and all of the 7-year-olds were between first and second grade. Testing was stopped in mid-October to minimize the difference in educational experience between children tested in the fall and children tested in the spring and summer. At that time, the age and gender groups were nearly but not exactly equal. The children were drawn from preschools and elementary schools that served a predominantly White, middle-class population. All came from homes in which English was the primary language. A teacher questionnaire confirmed that none of the children in this study had been exposed to instruction on fraction calculation or mixed numbers in school.

Materials

The materials and procedure for mixed-number problems were identical to those used for fraction calculation problems in Experiment 1, except that a three-hole tray was used to present the problems (10 × 30 cm overall with holes 6 cm diameter; 0.5 cm deep; spaced 4 cm apart).

Procedure

As in the fraction problems of Experiment 1, children were shown an amount of sponge circle that was the first term of each problem, and then this amount was hidden with a screen. Next, the experimenter added or subtracted a portion of sponge circle in full view of the child, but the outcome could not be seen. The child's task was to point to the amount in the response book that showed the resulting amount.

The 12 test problems were chosen from the pool of possible problems with sums and minuends of 3 (i.e., ¹²⁄₄) or less that result from various combinations of fractions and mixed numbers in one-quarter increments (i.e., ¼, ½, ¾, 1, 1¼, 1½, 1¾, 2, 2¼, 2½, 2¾, 3). As before, four choices were presented on each trial: the correct answer and three foils. The foils were other fractions, whole numbers, or mixed numbers that differed from the correct response in one-quarter increments. For example, if the correct response was 1½, the foils might include 1¼, 1¾, and 2. The magnitude of the correct response in relation to the foils was randomly determined and counterbalanced across items so that the correct response was equally often the smallest, next to smallest, largest, and next to largest magnitude. Thus, the relative magnitude of the correct response and the restriction to one-quarter increments determined what the foils would be on any given trial. For example, if the correct response was 1½ and the relative magnitude was largest, the foils would necessarily be 1¼, 1, and ¾.

Children were introduced to the mixed-number task in one of two conditions. One condition used the same instructions as in Experiment 1 and was included to parallel the procedure used in previous nonverbal calculation studies. Children were shown the first problem and then asked, "Which one of *these* [gesturing toward the pictures on the first response page] looks just like mine?" If a child did not point immediately, they were given the prompt, "Point to the one that looks just like mine." For children in this condition, this trial was followed immediately by the next problem without further explanation or feedback.

In the other condition, we tested children on the same set of mixed-number problems but introduced the task with a block of demonstration and practice trials. These trials illustrated how pieces could be recombined with respect to the whole units to match one of the pictures in the response

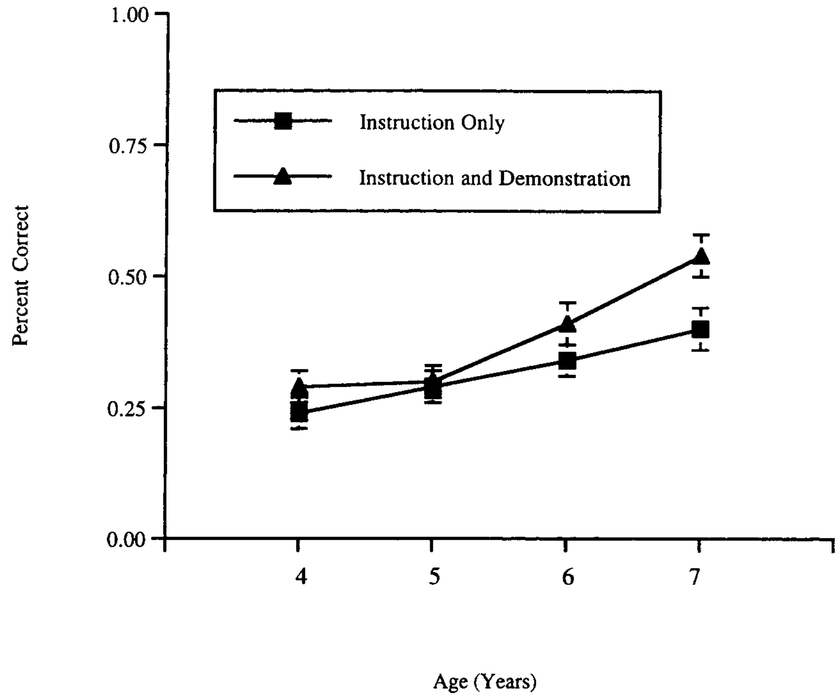

Figure 3. Proportion of correct responses for each age group in each condition of the mixed-number calculation task, Experiment 2. Note that these are discontinuous categories. Vertical lines depict standard errors.

book. During this block, the experimenter first took out a few sponge pieces and showed them to the child. She explained that these were "sticky" so she could stick them together or pull them apart. Next, she presented the addition problem ¾ + ¾ without screening the result from view. On this trial, the experimenter said, "Let's say I start with this much (pointing to the ¾ piece) and then I get this much more (adding another ¾ piece but placing it in a separate hole). Now, I have all this (gesturing toward the two amounts). If I wanted to, I could put them all together (recombining the amounts into 1½) and then they would look like this." The same problem was demonstrated again, but this time the result was screened and the experimenter pointed to the correct response in the response book. Finally, the problem was presented a third time, screened, and the child was prompted to point to a picture in the response book. The experimenter provided feedback and demonstrated why the child's response was right or wrong by lowering the screen, combining the pieces, and comparing them to the response choices. A parallel set of demonstration and practice trials was then presented on the subtraction problem 1¼ − ½ and this was followed immediately by the block of test problems.

Results and Discussion

All children completed the entire block of problems. Figure 3 shows the mean proportion correct for each age group in each condition. We used two-tailed t tests to compare these proportions to chance (i.e., 0.25 in a four-choice task) and found that regardless of how the task was introduced, 6- and 7-year-olds solved mixed-number calculation problems significantly above chance, lowest $t(21) = 3.37$, $p < .005$; however, 4- and 5-year-olds did not, highest $t(23) = 2.00$, $p < .10$. Thus, the early fraction calculation ability observed in Experiment 1 does extend to more complex mixed-number problems, but not until somewhat later in development. Still, even at age 6 and 7 years, children are solving mixed-number problems far earlier than they are taught formal algorithms for these problems in school and also earlier than children succeed on symbolic and quasisymbolic fraction tasks (e.g., Gelman et al., 1989; cited in Gelman, 1991).

A preliminary ANOVA ruled out the effect of gender on children's mixed-number calculation scores ($p > .15$), so this variable was eliminated from subsequent analyses. A second ANOVA conducted on the children's calculation scores, with age group and condition (instruction vs. demonstration) as between-subjects variables, revealed a significant main effect of age, $F(3, 177) = 17.48$, $p < .0001$. Pairwise comparisons (Scheffé's S, $p < .05$) revealed that this was due to reliably higher scores for 7-year-olds than for the other age groups, as well as significantly higher scores for 6-year-olds than for 4-year-olds. None of the other age comparisons were significant. There also was a significant effect of condition that reflected higher scores for the children who received demonstration trials, $F(1, 177) = 9.37$, $p < .005$. Although the amount of improvement was greatest for 7-year-olds (.14), the interaction between age and condition did not reach significance ($p > .20$). These results suggest that the demonstration and practice trials improved scores for some age groups but did not change the basic pattern of findings (i.e., emerging ability by age 6 with improvement by age 7).[2]

[2] As in Experiment 1, because the data in this experiment are proportional, we conducted parallel analyses using arcsine transformations of children's calculation scores. These tests revealed the same pattern of results, confirming the robustness of the results reported here.

Table 3
Rank Order (Easiest to Most Difficult) of Mixed-Number Problems in Experiment 2 Based on Overall Proportion Correct

			Age			
Problem	Numerosity	Overall	4-year-olds	5-year-olds	6-year-olds	7-year-olds
¼ + 2	9	.51	.19	.50	.58	.79
3 − 2½	12	.50	.29	.40	.58	.75
3 − 1¼	12	.48	.38	.35	.54	.68
2¾ − ¾	11	.47	.29	.48	.51	.62
1½ + 1½	12	.46	.44	.38	.40	.62
1½ − ¾	6	.36	.42	.40	.30	.30
¾ + ½	5	.34	.23	.23	.42	.49
1½ + 1¼	11	.28	.29	.25	.26	.32
1¼ + ¾	8	.25	.19	.13	.26	.43
2¼ − 1½	9	.23	.21	.19	.30	.23
2¾ − 1½	11	.20	.13	.13	.21	.36
¾ + 1¾	10	.12	.17	.10	.12	.09

As noted previously, one might argue that children solve the fraction calculation problems by means of whole-number strategy rather than by representing continuous amount. That is, children might count up the number of one-quarter pieces and calculate the total of these pieces. However, an examination of the order of difficulty for mixed number problems provides strong evidence against this hypothesis (see Table 3). Unlike whole-number calculation problems in this and previous studies, mixed-number problems did not order for difficulty according to the number of individual quarter pieces. Several of the easiest problems involved the greatest number of quarter pieces (i.e., 12 pieces). The rest of the distribution seemed to order randomly with respect to numerosity, with items involving 11 quarter pieces ordering between items with 5 and 8 quarter pieces, and so forth.

General Discussion

To explain why children have difficulty mastering fractions in school, some researchers have hypothesized that the structure of early quantitative representations is isomorphic to that of counting numbers and therefore does not readily apply to fractions (Gelman, 1991; Wynn, 1995, 1997). This implies that the course of development for fraction concepts should be dramatically different from the course of development for whole-number concepts. That is, children should have a firm grasp of the whole-number system before acquisition of conventional algorithms but extremely limited, if any, understanding of fractions during this same period. However, the present results did not reveal this to be the case.

We assessed children's understanding of fractions using a nonverbal procedure that has previously revealed whole-number calculation ability in preschool children (Huttenlocher et al., 1994; Levine et al., 1992). Our results demonstrate that there are striking parallels between development of whole-number and fraction calculation. First, there was the same gradual rise in performance over time on both tasks rather than an abrupt shift at any particular age. Second, although whole-number calculation scores were higher, scores for both tasks were in the middle range of performance. Thus, neither task was particularly easy or difficult. These findings reveal nothing idiosyncratic about the development of fraction concepts in comparison with whole-number concepts. Furthermore, fraction calculation ability was clearly emerging in the early childhood age range. Children as young as 4 years old could calculate with fractional amounts less than or equal to one. Somewhat older children (6- and 7-year-olds) could accurately solve more complex mixed-number problems. This was true even though none of these children had been exposed to written fraction algorithms in school. In fact, many of them had not even been taught the verbal labels for common fractions. Thus, understanding of fractions develops before acquisition of conventional fraction skills.

The present results do not provide support for the early-constraints hypothesis as an explanation for children's difficulty learning conventional fraction symbols and algorithms. This leaves open the question of why children continue to fail conventional fraction tasks so late in development—indeed, these problems persist for many years after children have mastered conventional whole-number algorithms. As noted in the introduction, one possibility is that children are confused by the symbols themselves rather than by the conceptual referents. For example, children may have difficulty using the same written symbols to stand for both whole numbers and fractional amounts. The whole-number interpretation might be privileged because children typically have years of experience with whole-number symbols before they are introduced to fraction symbols. The present results do not provide direct evidence for this explanation, but they suggest that this may be a fruitful direction for future research.

The results of the present study also have implications for the way early quantitative representations should be characterized. It is unlikely that children solved the nonverbal fraction and mixed-number problems using conventional symbols or algorithms given that these had not yet been taught in school. Indeed, an inspection of popular mathematics texts reveals that children are not even introduced to conventional fraction calculation algorithms until the end of second grade (e.g., *Addison-Wesley Mathematics,* 1992; *Exploring Mathematics,* 1991; *Heath Mathematics: Connections,* 1992). If children were not using conventional symbols to solve the fraction and mixed-number calculation problems in the present study, then what process were they using? In other words, when they were successful, how did they represent the hidden amounts and the transformations?

One proposal is that infants and young children use preverbal counting—counting without linguistic tags—as described briefly in the introduction (Gallistel & Gelman, 1992). Preverbal counting is thought to operate using the same mechanism as Meck and Church's (1983) model of timing and counting. In this model, there is an endogenous pacemaker that emits pulses at a constant rate. To begin timing or counting, a switch is closed that gates pulses into an accumulator. The resulting fullness of the accumulator represents the total duration or quantity, depending on which mode has been operated. Gallistel and Gelman argued that this process conforms to the principles that define counting (as outlined in Gelman & Gallistel, 1978) and therefore constitutes a legitimate counting system. They further proposed that the magnitudes produced by the accumulator are represented on a mental number line that preserves the ordinal relations among them.

If preschool children use preverbal counting to represent quantity, as Gallistel and Gelman (1992) have proposed, then it is not at all clear how they could solve the fraction and mixed-number calculation problems. That is precisely why Gelman (1991) predicted that learning fractions would be difficult given this type of representation. First, correct responses on these problems would require the ability to recognize solutions that fall between whole circles. As Gelman (1991) pointed out, the preverbal counting process and resulting number line representation does not lend itself to computing or remembering such amounts because it does not allow for amounts between the whole-number units. Second, it is unclear how preverbal counting could be used to represent the hidden amounts in the initial arrays. Although it is theoretically possible for children to count up the number of individual quarter pieces, the order of difficulty analyses reported previously provide no evidence that this is what children were actually doing. If instead, children represented the continuous amounts, preverbal counting per se would not work. It is conceivable that the timing mode of the accumulator could be used to "measure" the amounts; however, the accumulator has not been discussed or tested in this way to our knowledge.

Huttenlocher et al. (1994) proposed an alternative view of early quantitative competence that may provide a better account of the present results. In this view, children solve quantitative tasks by constructing a mental model that preserves critical information about the situation involved while eliminating irrelevant details, such as the color or texture of the objects involved. For example, to solve the whole-number calculation problems, children would construct a mental version of number of items in the initial (hidden) array and then imagine items moving into and out of it. The changed mental array would constitute the answer to the problem. Children might use a similar process to solve the fraction and mixed-number problems. That is, children could construct a mental version of the initial (hidden) amount and then imagine amounts being removed or combined with it. One possibility is that children imagine the spatial transformations that occur as the amounts are combined in relation to the whole unit. Another possibility is that they use the shapes of the pieces to imagine the recombinations without explicit reference to the whole. This may be how children come to understand fractions through everyday experience. For example, a child might see an apple cut into quarters. The child could gain insight into the nature of fractions by either (a) imagining the whole apple being reformed as the quarter pieces came back together or (b) using the shapes of the pieces to suggest how they would recombine and noticing that they make a whole apple.

Although early fraction and whole-number calculation abilities develop in parallel, they do not emerge simultaneously—fraction calculation appears somewhat later. One potential explanation for this lag, particularly if children use a mental model, is that fraction and mixed-number problems calculation may depend on the development of more complex spatial skills than whole-number problems require. As noted earlier, solving whole-number problems through a mental model requires only the mental movement of items into or out of a space. In contrast, the fraction problems would require not only these movements, but also rotation, separation, and recombination of various amounts. These added demands on spatial ability may make the task too difficult for children under 4 years old to perform accurately. Even after children are able to perform the task, they may need further development in spatial skills before they can accurately solve the more complex mixed-number problems.

However, it should be noted that the transformation was not the only potential source of error in the present task. For each problem, children also needed to remember the initial amount and the added or subtracted amount. It is possible that improved memory for these amounts led to the observed changes in calculation ability. Indeed, previous research on nonverbal calculation has shown that children's memory for the initial amount in whole-number problems improves significantly with age (Huttenlocher et al., 1994). Further research is needed to determine whether improvement in either memory for precise quantities, ability to perform spatial transformations, or both led to the developmental progression reported here.

Finally, let us consider what the present study implies about the origins of quantitative representation. The findings reported here might be seen as evidence that the early-constraints hypothesis is incorrect because innate numerical representations can apply to quantities other than whole numbers. In this view, humans would be born with representations that support reasoning about both whole or fractional amounts. However, there is no reason to conclude that innate representations underlie children's performance on the present task. First, fraction calculation ability, like whole-number calculation ability, is not evident until preschool age. Then, it continues to improve and expand during early childhood. Second, whole-number and fractional amounts are ubiquitous in children's environments. Therefore, reasoning about both types of quantity could develop gradually on the basis of accrued experiences (Huttenlocher, 1994). Unless children are prevented from assimilating fraction experiences by an innate constraint, as some have proposed, then these concepts should develop in parallel with whole-number concepts. The present results add to a growing body of evidence that suggests this is the case.

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