ECE/CS/ME 532

Lecture 14

Lecture 14: Power Iterations & PageRank

SVD: $X = U \Sigma V^T \in \mathbb{R}^{p \times n}$

$$A := \underline{\underline{X^T X}} = V \Sigma \cancel{U^T U} \Sigma V^T$$

$$= V \Sigma^2 V^T$$

$$= V \Lambda V^T \qquad \left(\Lambda = \Sigma^2 = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \right)$$

eigenvectors

eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

start with random vector $\underline{b}_0 \in \mathbb{R}^n$

$$\underline{b}_0 = V\underline{c} = c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n v_n$$

for $k = 0, 1, 2, 3, \ldots$

$$\underline{b}_{k+1} = \frac{A\,\underline{b}_k}{\|A\,\underline{b}_k\|_2}$$

$$\left(\begin{aligned} A &= X^T X \\ &= \underline{\underline{V \Lambda V^T}} \end{aligned}\right)$$

end

$$\underline{b}_{k+1} = \frac{A^k\,\underline{b}_0}{\|A^k\,\underline{b}_0\|_2}$$

$$b_{k+1} = \frac{A b_k}{\|A b_k\|_2} = \frac{V \Lambda V^T b_k}{\|V \Lambda V^T b_k\|_2}$$

$$= \frac{(V \Lambda V^T)^k b_0}{\|(V \Lambda V^T)^k b_0\|_2}$$

$$= \frac{V \Lambda^k V^T b_0}{\|V \Lambda^k V^T b_0\|_2} = \frac{V \Lambda^k V^T V c}{\|V \Lambda^k V^T V c\|_2}$$

$$= \frac{V \Lambda^k c}{\|V \Lambda^k c\|_2}$$

$$(V \Lambda V^T)^2$$

$$= V \Lambda V^T V \Lambda V^T$$

$$= V \Lambda^2 V^T$$

$$b_0 = V c$$

$$V\Lambda^k c = \lambda_1^k \, V\left(\Lambda/\lambda_1\right)^k c \qquad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$= \underline{\lambda_1^k c_1} \; V \underbrace{\left(\Lambda/\lambda_1\right)^k}_{\shortparallel} c/c_1 \qquad c = [c_1 \; c_2 \; \cdots \; c_n]^T$$

$$\begin{bmatrix} 1 & & & & \\ & \lambda_2/\lambda_1 & & & \\ & & \lambda_3/\lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_n/\lambda_n \end{bmatrix}^k \Rightarrow \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}$$

$\lambda_i \leq \lambda_1$ for $i = 2, 3, \ldots, n$

$$\lambda_i/\lambda_1 < 1 \Longrightarrow \left(\lambda_i/\lambda_1\right)^k \xrightarrow{k\to\infty} 0$$

$$V\Lambda^k c = \lambda_1^k c_1 \, V \underbrace{\left(\Lambda/\lambda_1\right)^k}_{\to \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}} \left(\underline{c}/c_1\right)$$

$$\underbrace{\left(\Lambda/\lambda_1\right)^k \, c/c_1 \to \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

$$V \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underline{v}_1$$

$$V\Lambda^k c \longrightarrow \lambda_1^k c_1 \underline{v}_1$$

random b_0

$$b_{k+1} = \frac{V\Lambda^k c}{\|V\Lambda^k c\|_2} \xrightarrow{k\to 0} \frac{\lambda_1^k c_1 v_1}{\lambda_1^k c_1 \|v_1\|} = \frac{v_1}{\|v_1\|} = v_1$$

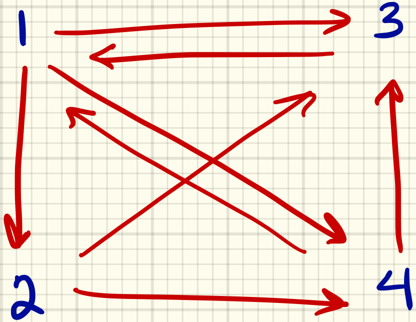
Power iteration method (to find 1^{st} right singular vector)

$$A = X^T X$$

rand. b_0

$$b_{k+1} = \frac{A b_k}{\|A b_k\|_2} \xrightarrow{k\to\infty} v_1$$

PageRank



Adjacency matrix

$$\tilde{A} = \begin{array}{c} \begin{matrix} \;\;1 & \;\;\;2 & \;\;\;3 & \;\;\;4 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

(unweighted)

$$A = \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

(weighted)

start w/ initial vector $b_0 = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

$$Av = \begin{bmatrix} .37 \\ .08 \\ .33 \\ .20 \end{bmatrix}, \quad A^2v = \begin{bmatrix} .43 \\ .12 \\ .27 \\ .16 \end{bmatrix} \ \ldots \ A^8v = \begin{bmatrix} .39 \\ .12 \\ .29 \\ .19 \end{bmatrix}$$

PageRank vector