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SURVEY ARTICLE

Speaking with Shadows: A Study of Neo-Logicism

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ABSTRACT

According to the species of neo-logicism advanced by Hale and Wright, mathematical knowledge is essentially logical knowledge. Their view is found to be best understood as a set of related though independent theses: (1) *neo-fregeanism*—a general conception of the relation between language and reality; (2) *the method of abstraction*—a particular method for introducing concepts into language; (3) *the scope of logic*—second-order logic is logic. The criticisms of Boolos, Dummett, Field and Quine (amongst others) of these theses are explicated and assessed. The issues discussed include reductionism, rejectionism, the Julius Caesar problem, the Bad Company objections, and the charge that second-order logic is set theory in disguise.

The irresistible metaphor is that pure abstract objects [...] are no more than *shadows* cast by the syntax of our discourse. And the aptness of the metaphor is enhanced by the reflection that shadows are, after their own fashion, real. (Crispin Wright [1992], p. 181–2)

But I feel conscious that many a reader will scarcely recognise in the shadowy forms which I bring before him his numbers which all his life long have accompanied him as faithful and familiar friends; (Richard Dedekind [1963], p. 33)

1 Introduction

The neo-logicist claims that even though number-theory describes an extraordinary realm of abstract objects, an ordinary thinker may yet acquire knowledge of these objects, their properties and relations through reflection on logical and linguistic truths.[1,2,3] The neo-logicist offers a perspective—nigh on revolutionary—that lies squarely opposed to certain more familiar ways of conceiving the relationship between language and the world. We will not be in a position to evaluate the contribution made—evaluate the effectiveness of well-worn objections or understand just what would be required for its principled defence—until it is appreciated just how neo-logicism runs against the grain of more familiar lines of theorising.

2 Logical-historical preliminaries

Frege sought to demonstrate that we might acquire knowledge of finite cardinal numbers by deriving the general laws that govern them (laws equivalent to what we now call the 'Peano postulates') from logic and a definition of the term 'cardinal number'.[4] He argued that any adequate definition of this term must specify identity conditions for cardinal numbers and proposed Hume's Principle (HP) as a plausible first candidate (Frege [1953], §§62–5). (HP) says that the cardinal number belonging to the concept F is identical to the cardinal number belonging to the concept G if and only if the following condition obtains: the entities falling under the concept F figure in a relation of 1–1 correspondence with the entities falling under G.

(HP) $(\forall F)(\forall G)\,[(Nx : Fx = Nx : Gx) \leftrightarrow (F1{-}1\ G)]$

But Frege soon became dissatisfied with this first proposal (Frege [1953], §§66–7). He insisted that identity conditions for cardinal numbers must

[1] The term 'neo-logicism' may be applied to a variety of positions that draw inspiration from Frege's claim that arithmetical knowledge is *a priori* and apply techniques recognisably similar to those deployed by Frege in his attempt to vindicate this claim. Here I focus upon the neo-logicist position developed by Hale and Wright. See Wright ([1983]), Hale ([1987]), and Hale & Wright ([2001a]). The introduction and postscript to Hale & Wright ([2001a]) (that pinpoints 'eighteen problems' requiring further work) lay out the authors' current view of the territory. Regrettably, for reasons of space, this survey is unable to cover the full range of positions that in one way or another may be legitimately deemed to have taken their inspiration from Frege's central claims. Tennant ([1987], [1997a], [1997b], [forthcoming]) develops an independent programme but with significant affinities to the neo-logicism of Hale and Wright. Hodes ([1984], [1990a], [1990b]), Fine ([1998]) and Zalta ([1999], [2000]) develop further independent conceptions that draw inspiration from Frege and raise further issues. These subtle and intriguing views deserve sustained treatment. Unfortunately, a consideration of these different programmes will have to be postponed for another occasion (see MacBride [forthcoming]).

[2] For ease of exposition and the sake of philosophical perspicuity, the term 'neo-logicist' is used ambiguously in the text to stand for (variously) Hale, Wright, Hale & Wright, and a proponent of an idealised version of their view. Attention to context will allow straightforward disambiguation of the expression.

[3] Note that all references to Hale, Hale & Wright, Wright and Boolos will be to the reprints of their works in, respectively, Hale & Wright ([2001a]) and Boolos ([1998a]).

[4] Illuminating introductions to the historical and technical background sketched in this section are to be found in Resnik ([1980], pp. 161–234), Boolos ([1998b]), Weiner ([1999]) and Shapiro ([2000], pp. 107–39).

specify *which* objects are cardinal numbers. But (HP) conspicuously fails in this task. (HP) tells us whether objects described in numerical terms are identical to, or distinct from, one another (depending on whether the objects falling under their associated concepts figure in a 1–1 correspondence relation). But (HP) does not tell us whether objects described in the form 'Nx : Fx' are identical to or distinct from objects described in different ways. The principle is entirely *silent* concerning such matters; it says *absolutely nothing* about objects—for example, material objects—that are not already described in such terms. Since (HP) fails to inform us whether Nx : Fx really is, or really is not, the man Caesar, it thereby fails to inform us which objects are cardinal numbers.

In the light of this problem—the so-called 'Julius Caesar problem'—Frege proposed a second definition (Frege [1953], §§68–9). He assumed that there could be no doubt concerning which objects are the extensions of concepts. So if numbers are extensions then, according to Frege, there could be no doubt concerning which objects are the numbers and the Julius Caesar problem would not arise. In order to evade the Julius Caesar problem Frege therefore proposed a definition that identified numbers with the extensions of concepts (a definition that identifies the number belonging to the concept F with the extension of the concept *being in 1–1 correspondence with the extension of F*). According to that definition, numbers are just a species of a more fundamental kind of object (extensions). So Frege set about deriving the general laws of numerical objects from a principle specifying identity conditions for items of this more fundamental kind. This principle was later to be encoded in the infamous Axiom (V) of his *Basic Laws of Arithmetic*. According to (V), the extension of one concept is identical to the extension of another if and only if those concepts are co-extensive:

(V) $(\forall F)(\forall G)\,[(\mathrm{Ext} : Fx = \mathrm{Ext} : Gx) \leftrightarrow (Fx \leftrightarrow Gx)]$

Frege's grand design was to demonstrate that arithmetic inherits the epistemological status of logic. To realise this design he attempted to derive the truths of arithmetic from (V) in conjunction with second-order logic. But rather than being a logical truth or an innocent definition, (V) turned out to be inconsistent. The application of simple rules of proof to (V) generates Russell's Paradox. Moreover, (V) imposes impossible demands on the size of the domain of objects that it is intended to characterise (see Appendix 1). Recognition of (V)'s inconsistency left no doubt that his logicism had failed. Frege eventually judged that arithmetic could not possess the epistemological status of logic.

Neo-logicism is the doctrine that Frege's judgement was premature. His fatal error, the neo-logicist claims, was to suppose that the Julius Caesar problem which confronted (HP) could not be solved. Frege should not have abandoned (HP) in favour of a definition that identified numbers with extensions. If only Frege had persevered with his first proposal and derived the general laws of arithmetic straight from (HP), then arithmetic would have been set upon a sure epistemological footing. An investigation of the proof Frege actually constructed reveals how the laws of arithmetic might be derived in this way. The role of Frege's favoured definition in that derivation was simply to establish (HP) (Frege [1953], §73). Once (HP) was derived, the identification of numbers with extensions dropped away and performed no further function. Frege then went on to sketch *Frege's theorem*: the result that the Peano postulates can be interpreted and their interpretations proved in *Frege arithmetic*, the system that results when (HP) is adjoined to second-order logic (Frege [1953], §§73–83).[5]

After witnessing the spectacular logical collapse of Frege's account of arithmetic, it is tempting to perform a pessimistic induction and predict that the project of founding number theory on (HP) will also end in contradiction. But, the neo-logicist claims, such pessimism is unwarranted. Application of the simple rules of proof to (HP) that yielded Russell's Paradox from (V) fail to reveal an inconsistency. And rather than making the impossible demands on a domain that (V) imposes, (HP) requires only that the domain it characterises contain infinitely many objects (see Appendix 2).

Nevertheless, pointing out the proof and model-theoretic differences between (HP) and (V) does not show that the former is consistent. It may be, for all that has been shown, that (HP) is inconsistent, but its inconsistency must be drawn out in a different way from (V)'s. But it is *almost* inconceivable that (HP) is inconsistent (Boolos [1986], p. 175, [1987], p. 191, [1993], p. 230; Wright [1998a], p. 235). For it has been proved that if classical

[5] Whether Frege himself was aware of the character and significance of the proof he sketched is a contentious issue. According to Dummett ([1993a], p. 123), Frege had already established Frege's theorem in *Foundations*. But the sketch Frege provides there does not in fact suffice for a proof of that result. On the other hand, Heck ([1993a]), and Boolos & Heck ([1998]) show how to reconstruct the proofs of Frege [1893–1904] resting on (HP) alone. Geach noted the possibility that an account of arithmetic might proceed directly from (HP) (rather than a principle governing extensions): 'We still have to discuss Frege's view that numbers are classesextensions of concepts. He himself attached only secondary importance to this (*Foundation* §107); rejection of it would ruin the symbolic structure of his *Grundgesetze*, but not shake the foundations of arithmetic laid down in the *Grundlagen*' (see Geach [1951], p. 541; see also Geach [1955], p. 569). (I am grateful to Professor Geach for discussion of these passages.) The derivability of Frege's Theorem was first explicitly noted in Parsons ([1965], p. 19). See also (Smiley [1981], pp. 54–5) for a related result (providing support for 'a bastard offspring of logicism') that derives the Peano Postulates from (HP) and the further non-logical postulate that numbers exist. But it was Wright (see his [1983], pp. 158–69) who, having independently rediscovered Frege's theorem and actually sketched the proof in detail, influentially argued for its philosophical significance and made the theorem the basis of a going concern within contemporary philosophy of mathematics. See Boolos ([1987]) for a discursive treatment, and Boolos ([1990], pp. 217–8), for an outline of the derivation. Clark ([1993b]) scrutinises the non-constructive assumptions of the proof, and Boolos ([1995], [1996]) and Bell ([1999]) provide further detailed analyses of Frege's theorem.

analysis is consistent then the system that results from adding (HP) to second-order logic—*Frege arithmetic*—is consistent too.[6] So if we are to countenance the possibility that (HP) is inconsistent we must also countenance the possibility that vast swathes of pure and applied mathematics as currently practised also harbour contradiction. The discovery of Russell's paradox precipitated a crisis in the foundations of mathematics. To seriously contemplate the possibility that (HP) is inconsistent is to envisage a future crisis in mathematics, a crisis of such proportions as would make the former appear almost without consequence.

3 The linguistic turn

Frege's theorem establishes that the fundamental laws of arithmetic may be interpreted and proved in the system that results from a single, simple principle whose consistency we may barely doubt. But the neo-logicist claims that this result is more than mathematically insightful. He also claims that it is epistemologically significant. For, according to neo-logicism, (HP) is no more than a stipulation about how to use words, and Frege's Theorem reveals how we may advance from knowledge of this stipulated truth to knowledge of arithmetic by means of logical reflection alone.

Neo-logicism appears an extraordinary view. Its blunt statement invites—and receives—damning and apparently conclusive replies: if (HP) is committed to the existence of infinitely many objects, its truth can hardly be a matter for stipulation; you can't stipulate a single object into existence, never mind infinitely many. But there is a theoretical perspective, from which it does not appear so extraordinary and the criticisms it elicits seem far from conclusive. To attain that perspective it needs to be appreciated that neo-logicism is comprised by three distinct bodies of doctrine: i) a general conception of language and reality; ii) a particular method for introducing novel expressions into language; iii) a specific understanding of the scope of logic.

The general conception of language and reality (hereby dubbed 'neo-fregean') reverses a more familiar style of theorising that assumes that the

[6] Consistency proofs that show the domains ω and $\omega+1$ are models for (HP) were provided independently by Burgess ([1984]), Hodes ([1984], p. 138) and Hazen ([1985a]). But these proofs suffer from a dialectical weakness: to establish that these domains model (HP) they make appeal to an informal version of (HP), the very principle of whose consistency we are attempting to assure ourselves. To overcome this shortcoming Boolos ([1987]) developed a proof of the equi-consistency of classical analysis and Frege arithmetic. See Boolos & Heck ([1998], pp. 334–6) for further elaboration.

nature of reality is fixed independently of language.[7] By contrast, the neo-fregean claims, reality and language are so related that, if we speak truly, the structure of reality inevitably mirrors the contours of our speech. The neo-fregean appeals to what he calls the 'syntactic priority' thesis to secure the desired community of language and reality.[8] But really the best way to understand this thesis is to distinguish the following distinct components:

(SP1) *Syntactic Decisiveness*: if an expression exhibits the characteristic syntactic features of a singular term, then that fact decisively determines that the expression in question has the semantic function of a singular term (reference).

(SP2) *Referential Minimalism*: the mere fact that a referring expression figures in a true (extensional) atomic sentence determines that there is an item in the world to respond to the referential probing of that expression.

(SP3) *Linguistic Priority*: linguistic categories are prior to ontological ones; an item belongs to the category of objects if it is possible that a singular term refer to it.

Suppose that singular terms exhibit some set of distinctive syntactic features. It follows that if an expression t_1 exhibits those features then it has the semantic function of a singular term (*Syntactic Decisiveness*). So if t_1 figures in a true atomic sentence S_1, there exists an item in the world to which t_1 refers (*Referential Minimalism*). Since t_1 is a singular term, the item to which it refers belongs to the category of objects (*Linguistic Priority*). Together these doctrines establish that the syntactic form of our (true) sentences cannot deceive us; reality cannot fail to include the objects and concepts which these sentences apparently describe.[9]

[7] The terms 'neo-logicism', 'neo-fregeanism' and 'fregean platonism' have been used interchangeably in the literature. I have chosen to regiment usage and take advantage of the superfluity of terms by using 'neo-logicism' to describe a composite doctrine about ontology, stipulation and second-order logic. The term 'neo-fregeanism' is reserved for the component view about ontology. I have dropped the term 'fregean platonism'. It will become apparent that neo-logicism is neither exactly fregean or platonic (Sections 4 and 5).

[8] See Wright ([1983], pp. 7–8, pp. 13–4, p. 25, pp. 51–2, p. 129, [1990], pp. 153–4, [1992], pp. 28–9, pp. 178–82, pp. 192–3, [1998a], pp. 239–40); Hale ([1987], pp. 3–4, pp. 10–14), and also Dummett ([1956], pp. 38–41, [1981a], pp. 494–8 and [1981b], pp. 381–5). The doctrine is (in part) inspired by the linguistic turn taken by Frege when he converted an epistemological enquiry about numbers into an investigation of the senses of numerals: 'How then, are numbers to be given to us, if we cannot have any ideas or intuitions about them? Since it is only in the context of a proposition that words have any meaning, our problem becomes this: To define the sense of a proposition in which a number word occurs' (Frege [1953], §62).

[9] These doctrines may appear to commit the neo-fregean to an unacceptably bloated ontology. Take, for example, fictional objects: just consider a true sentence featuring a singular term for your favourite fictional character and neo-fregeanism appears to license an entailment to the existence of the character in question (Williamson [1994], Divers & Miller [1995]). The neo-fregean may respond either by i) accepting the consequent fictionalist ontology or by ii) denying that the contexts in question are properly true and content bearing (Wright [1983], pp. 25–8, [1994], pp. 327–30).

Neo-fregeanism (don't forget: the term is being used for the underlying conception of ontology) thus stands opposed to the thesis that the structure of a true sentence may fail to mirror the structure of a state of affairs that makes it true. Different versions of this opposing thesis may be discerned. According to one version, further (empirical) investigation may be required subsequent to the discovery that a sentence is true to determine what configuration of objects and properties is actually responsible for its truth (see the *a posteriori* realism developed by Armstrong [1993], pp. 429–30). According to another version, terms in language may be empty and simply fail to pick out any worldly item even when these expressions figure in true sentences (witness the behaviour of singular terms in Scott's development of positive free logic in his [1967]).

When language and reality are conceived in the neo-fregean manner, the task of establishing the existence of numerical objects becomes twofold. First, it must be established that numerical terms possess the syntactic features characteristic of singular terms. Second, it must be established that these terms figure in true sentences. Then, in view of the underlying neo-Fregean conception of language and reality, there can be no doubt that there are numerical objects corresponding to those expressions.

The neo-logicist proceeds to accomplish these tasks by providing a 'logical reconstruction' of ordinary arithmetical practice. He begins by introducing syntactically singular expressions that figure in true sentences. The linguistic practice thereby established is then shown to be a reconstruction of ordinary arithmetical language in the following sense. The pattern of use established for the novel singular terms and the sentences in which they figure is the very same pattern of use associated with ordinary numerals and the arithmetical sentences to which they belong. Next, appeal is made to a further component of neo-fregeanism (Wright [1999], p. 322; MacBride [2002]):

(SP4) *Meaning Supervenes on Use*: if sentences 'S_1' and 'S_2' exhibit the same pattern of use, then if 'S_1' is true then 'S_2' is also true; if expressions 'n_1' and 'n_2' exhibit the same pattern of use, then if 'n_1' refers to an item n, then 'n_2' also refers to n.

It follows that since the novel sentences introduced are true, the ordinary arithmetical sentences that share their pattern of use are also true. And since the singular terms introduced refer to certain worldly items, the corresponding numerical terms (which share the relevant pattern of use) also refer to those things.[10]

[10] Earlier versions of neo-logicism offered (HP) as an 'analysis' of the ordinary notion of number or 'analytic of' that concept (Wright [1983], pp. 106–7; Hale [1997], p. 99). For an alternative neo-logicism that insists on the advantages of the continued employment of the traditional notion of conceptual analysis see Demopoulos [forthcoming].

The neo-logicist employs the 'method of abstraction' to introduce the sentences and constituent singular terms required for a reconstruction of ordinary arithmetic (see Fine [1998] for a compendious treatment of abstraction). Singular terms ('a', 'b') possess a distinctive syntactic feature that distinguishes them from other forms of expression ('F', '$\sim$', '$\forall$'): they may intelligibly figure in identity contexts ('$a = b$'). So the task of introducing novel singular terms is, at least in significant part, the task of demonstrating that these terms may intelligibly figure in novel identity sentences. The method of abstraction is tailor-made to perform this task. It introduces singular terms by explaining the use of identity sentences (in which these terms figure) by appeal to the established use of familiar sentences (Wright [1983], pp. 29–30, [1990], pp. 154–5, [1997], pp. 275–6; Hale & Wright [2000]).[11]

Suppose we already speak a language containing expressions ('α_1' ... 'α_k') referring to the elements of a domain and a special relational predicate ('$\approx$') expressing an equivalence relation in which those elements figure. Then, according to the method of abstraction, we may extend the expressive capacities of our discourse by introducing a novel operator ('Σ'). The operator is introduced by the following stipulation:

$$\text{(AP)} \quad (\forall\alpha_\varphi)(\forall\alpha_\kappa)[(\Sigma(\alpha_\varphi) = \Sigma(\alpha_\kappa)) \leftrightarrow (\alpha_\varphi \approx \alpha_\kappa)]$$

(AP) is an 'abstraction principle'. Rather than saying *explicitly* what the operator 'Σ' is to mean, (AP) introduces 'Σ' *implicitly* (contextually) in terms of expressions with established uses. Two related components in the stipulation which (AP) embodies may be discerned. First, (AP) stipulates that 'Σ' is to perform the role of operator—on familiar expressions

[11] In fact the neo-fregean (the proponent of the underlying conception of ontology) requires that the introduced terms exhibit a complex package of proof-theoretic traits associated with singular terms in natural language. Drawing upon the inferential conception of singular termhood advanced in Dummett ([1981a], pp. 54–80) the neo-logicist has developed a complex account of the proof-theoretic criteria associated with singular terms (Hale [1979], [1987], pp. 15–41, [1994b], [1996], and Wright [1983], pp. 10–12, 53–64). There are three pertinent concerns that may be raised here. (1) It may be doubted whether Hale has succeeded in isolating the relevant traits of ordinary singular terms (Williamson [1988], pp. 487–8; Wetzel [1990]; Stirton [2000a]). (2) It may be questioned whether there is any good reason to insist that genuine numerical singular terms should exhibit all the tangled web of traits associated with natural language terms. Why should it not suffice to show (for the purpose of establishing their object-invoking character) that the introduced terms interact in a stereotypical—if rudimentary—fashion with identity, existential quantifiers and negation? (3) One may also wonder just how the occurrence of the introduced expressions in identity contexts bestows upon them the proof-theoretic characteristics of singular terms.

(‘α_j’, ‘α_k’)—whose application results in complex expressions figuring in identity contexts (‘$\Sigma(\alpha_j) = \Sigma(\alpha_k)$’). Second, (AP) also stipulates that the truth conditions of the novel identity contexts so introduced coincide with the truth conditions of another form of statement (‘$\alpha_j \approx \alpha_k$’) which we already understand. Since (AP) thereby bestows significance on identity sentences featuring the novel terms, and terms that figure in identity sentences are syntactically singular, it follows (*Syntactic Decisiveness*) that (AP) succeeds in introducing novel singular terms (‘$\Sigma(\alpha_j)$’, ‘$\Sigma(\alpha_k)$’) formed by the application of the novel operator. So, in addition to endorsing a quite general conception of language and reality ((C1)–(C4)), the neo-logicist also advances a particular account of how novel singular terms may be generated:

(MA1) *Syntactic Novelty*: the method of abstraction provides a mechanism for introducing novel expressions (‘$\Sigma(\alpha_j)$’, ‘$\Sigma(\alpha_k)$’) with the characteristic syntactic features of singular terms by the stipulation of an abstraction principle (AP).

But the neo-logicist does not claim simply that the method of abstraction provides a means of introducing novel terms. He also claims that where we possess the ability to confirm or disconfirm familiar statements, abstraction principles provide us with a means of establishing the truth or falsity of the novel identity statements in which those terms occur. For if we are already able to establish that a sentence of the form ‘$\alpha_j \approx \alpha_k$’ is true, then—in virtue of the stipulated coincidence of truth conditions—we are thereby able to establish that an identity sentence of the form ‘$\Sigma(\alpha_j) = \Sigma(\alpha_k)$’ is also true. And if the sentence is confirmed, *Referential Minimalism* and *Linguistic Priority* then entitle us to suppose that there really is an object (the referent of the terms ‘$\Sigma(\alpha_j)$’ and ‘$\Sigma(\alpha_k)$’) in the world to which the truth we have discovered bears witness.

In this way, the neo-logicist argues, the method of abstraction allows us to advance from knowledge of the truths expressed in one language (truths about α_j and α_k) to a knowledge of truths expressed in an extended language (truths about $\Sigma(\alpha_j)$ and $\Sigma(\alpha_k)$). Abstraction principles throw into relief objects of which we were previously unaware, objects ‘every bit as objective as’ (Wright [1983], p. 13; Hale [1987], p. 11) the more familiar referents of our discourse. Two further theses concerning the method of abstraction are thus endorsed:

(MA2) *Semantic Novelty*: where the original language contains insufficient resources to characterise the entities required to satisfy (AP), the novel expressions (‘$\Sigma(\alpha_j)$’, ‘$\Sigma(\alpha_k)$’) introduced by the method of abstraction will, if they refer at all, pick out objects to which the familiar expressions (‘α_j’, ‘α_k’) do not refer.

(MA3) *Referential Realism*: moreover, if the singular terms introduced by abstraction refer, then statements concerning the objects to which they refer ($\Sigma \ldots$) merit a realistic interpretation, an interpretation that may also be legitimately applied to statements concerning the referents (αs) of familiar expressions.[12]

By equipping us with abstraction principles, the neo-logicist promises to radically transform our epistemic powers. By artful stipulation of abstraction principles, he promises to show how we may advance from knowledge of the concrete to knowledge of the abstract, and, from knowledge of logic to knowledge of arithmetic.

Consider our ordinary facility in characterising and talking about directions. Directions are, apparently, abstract entities. They lack causal powers and are troublesome to locate in either space or time. It is correspondingly mysterious how concrete creatures such as we are could talk about and knowingly utter truths concerning directions. The neo-logicist offers to dispel the mystery by providing a logical reconstruction of our ordinary direction talk. The logical reconstruction is based upon a first-order language—a language of which we possess an unproblematic grasp—containing names ('a', 'b', 'c' ...) referring to straight concrete lines and a relational predicate ('... is parallel to ...') expressing an equivalence relation on them. Then novel terms ('D(a)', 'D(b)' ...) standing for novel abstract objects may be introduced, the neo-Fregean claims, by means of the following stipulation (Wright [1983], pp. 29–30, [1990], p. 155; Rosen [1993], pp. 155–8):

($D^=$) $\quad D(a) = D(b) \leftrightarrow$ a is parallel to b

The stipulation of this abstraction principle—along with a battery of cognate stipulations—establishes that the truth conditions of sentences of the form '$D(a) = D(b)$' coincide with the truth conditions of sentences about lines that we already understand, sentences of the form 'a is parallel to b'.[13] Since we are equipped, let it be assumed, with reliable methods of acquiring information about the concrete world, instances of the right hand side of

[12] This thesis is articulated in a manner that deliberately refrains from characterising statements concerning the objects introduced as realist outright. Since the publication of his earlier work on the philosophy of mathematics it has been an enduring feature of Wright's evolving views on realism that there is no single dimension of realist comparison. Instead Wright ([1992], pp. 180–2, pp. 192–3, pp. 198–9) emphasises that there are many such dimensions and that whilst statements about some abstract objects may be suitably characterised as realist along one dimension ('minimal truth') they may fail to do so along another ('wide cosmological role'). See Divers & Miller [1999] for further discussion.

[13] The following additional stipulations are also needed to provide, respectively, for the predication of, and quantification over, the novel objects introduced by ($D^=$): (i) $\phi D(x) \leftrightarrow Fx$ (where 'is parallel to' is a congruence for 'F'). (ii) $(\exists x)\, \phi\, x \leftrightarrow (\exists x)Fx$ (where 'ϕ' and 'F' satisfy (ii))

($D^{=}$) may sometimes be confirmed by surveying the domain of concrete lines. And since the stipulations laid down equip us with an understanding of the novel expressions they introduce it also follows that we may, when right-hand side instances are confirmed, derive the truth of instances of left hand side instances of ($D^{=}$) that describe the identity of Ds. So even though we are only concrete, abstraction principles place it within our power to grasp the existence and nature of abstract things. Moreover, the neo-Fregean claims, the pattern of use established for novel terms formed from the operator 'D*x*' is identical to the pattern of use associated with talk of directions. He concludes that ($D^{=}$) provides an epistemological mechanism for grasping the existence and nature of directions.

By appeal to another logical reconstruction, the neo-Fregean contends, we may advance from knowledge of logic to knowledge of arithmetic. A second-order language serves as the basis of the reconstruction. It contains expressions ('F', 'G'...) denoting concepts and a relational predicate ('... 1–1...') expressing the equivalence relation of one-to-one correspondence amongst the items fallings under concepts. Then novel terms ('Nx:Fx', 'Nx:Gx'...) standing for novel objects may be introduced, the neo-logicist claims, by stipulating the truth of a familiar looking principle:

(HP) $(\forall F)(\forall G)\,[(Nx : Fx = Nx : Gx) \leftrightarrow (F1 - 1G)]$

This abstraction may be employed to introduce into a second-order language a novel operator 'Nx:Φx'.[14] The introduction is achieved by stipulating that the truth conditions of identity statements ('Nx:Fx = Nx:Gx') featuring singular terms formed by the application of the novel operator to familiar concept words ('Nx:Fx', 'Nx:Gx') coincide with the conditions under which an equivalence relation amongst concepts may be said to obtain ('F 1–1 G'). The neo-logicist now appeals to a further doctrine concerning the extent of logic (Wright [1983], pp. 132–5, [1998a], pp. 247–8):

(2OL) *Second-Order Logic is Logic*: there is a recognisable class of logical inferences whose validity turns upon the occurrence of second-order notions, those of *property* and *relation*.

[14] The attempt to introduce concepts by the method of abstraction may be fruitfully compared to the attempt to introduce logical constants (Gentzen-style) by introduction and elimination rules (Hale & Wright [2000a], pp. 117–8, p. 146, p. 148; cf. Prawitz [1965]). Consider Gentzen's remark ([1934], p. 189): 'an introduction rule gives, so to say, a definition of the constant in question [whilst] an elimination rule is only a consequence of the corresponding introduction rule, which may be expressed somewhat as follows: at an inference by an elimination rule, we are allowed to "use" only what the principal sign of the major premise "means" according to the introduction rule for this sign.' Conceived Gentzen-style, the right-to-left and left-to-right readings of (HP) respectively correspond to introduction and elimination rules for the numerical functor 'Nx : Φx'. This way of thinking about (HP) has been prominent in Tennant's logicist programme ([1987], pp. 275–300, [1997b], pp. 322–6).

In particular, the neo-logicist claims, there is a class of valid inferences in which the second-order relation of 1–1 correspondence occurs, and a logical grasp of these patterns informs us that certain instances of the right-hand side of (HP) are true. In such cases we are warranted by the stipulation which (HP) effects to derive the truth of the corresponding left-hand side sentences describing the identity of the object referred to by the expressions 'Nx:Fx' and 'Nx:Gx'.

For illustrative purposes, consider the statement that the entities falling under the concept *being non-self identical* figure in a relation of 1–1 correspondence:

(1) $(x : x \neq x)\ 1 - 1\ (x : x \neq x)$

Our second-order logical knowledge informs us that (1) is true. Since everything is self-identical nothing falls under the concept *being non-self-identical*. So, trivially, the things that fall under that concept and the things that fall under that same concept are one-one correlated. But (1) is also an instance of the right-hand side of (HP). Hence, a corresponding instance of the left-hand side of (HP) may be derived from (1):

(2) $(Nx : x \neq x) = (Nx : x \neq x)$

Since (2) is true and contains expressions exhibiting the characteristic syntactic features of singular terms, it follows that there is an object $(Nx : x \neq x)$ in reality corresponding to this way of speaking. We are therefore entitled to generalise existentially on (2) to yield the result:

(3) $(\exists y)\ (y = Nx : x \neq x)$

So we may advance from our knowledge of a logical truth (1), and our grasp of what is merely a stipulation (HP), to knowledge of the existence of an abstract object (3). And more generally, the neo-logicist claims, we may advance simply on the basis of (HP) and logic to knowledge of the truths of *Frege arithmetic*—the system that results from (HP) and logic—and the abstract objects it describes.

The neo-logicist then claims that *Frege arithmetic* exhibits a pattern of use that is identical to the pattern of use associated with ordinary pure arithmetic. The neo-logicist appeals to Frege's theorem to establish this result. For Frege's Theorem demonstrates that the two systems are so much alike that the fundamental laws of arithmetic (Peano's Postulates) can be interpreted and proved in *Frege arithmetic*. Furthermore, the neo-logicist claims, there are additional definitions that may be adjoined to (HP) to generate an extended system—*applied Frege arithmetic*—that shares a pattern of use with ordinary applied arithmetic (Wright [1999], pp. 330–2). Since it has already been established that we may grasp the truths expressed and objects described by *Frege arithmetic*, the neo-logicist concludes that (HP) provides an epistemological mechanism for grasping the existence and nature of objects described by familiar arithmetic.

4 Reductionism

Neo-logicism is a bundle of relatively independent doctrines. One or other constituent doctrine may succeed even when others falter. In particular, it may be that neo-logicism as a whole proves untenable even though neo-fregeanism (the underlying conception of ontology) prevails. If we are to benefit from an engagement with neo-logicism, it is therefore imperative that we distinguish between the many varied objections to the position and isolate the constituent doctrine (or doctrines) relevant to each. It is (in part) because this task is already a considerable theoretical undertaking that the study of neo-logicism (regardless of its eventual success) has proved to be significant.

It is also vital to bear in mind that the neo-logicist has provided a genuinely innovative model of how reference may be disclosed and knowledge of abstract objects achieved. Criticisms of neo-logicism cannot simply presume the correctness of some entrenched model of knowledge and reference and then criticise the new model for failing to match it. To avoid begging the question in favour of, for example, a causal theory (or any other prior epistemology), a critic must isolate the element, or elements, of the neo-logicist account that falter in their intended semantic or epistemic purpose.[15]

The intuitive 'reductionist' response to neo-logicism that comes readily to mind upon first encountering the position provides an instructive example. According to this objection, the stipulation of an abstraction principle does establish the equivalence of sentences of a novel form with sentences of a familiar shape. Moreover, it may sometimes be established that the sentences so introduced are true by confirming the relevant familiar sentences with which they coincide in truth conditions. Nevertheless, reductionism complains, the stipulation of abstraction principles fails to enable us to talk about numbers. For by displaying the equivalence of novel sentences which appear

[15] Of course, the neo-logicist may also argue directly against causal theories (Wright [1983], pp. 84–103, [1988] pp. 442–4; Hale [1987], pp. 78–101, pp. 170–4, pp. 180–93, pp. 264–5). He may argue that there is no proper motivation for an entirely general causal theory of knowledge or reference that applies both to empirical and to non-empirical subject matters. Or he may argue *ad hominem* against, for example, a causal theory of knowledge on the grounds that it cannot account for our knowledge of the future or general empirical truths. In a similar spirit, the neo-logicist may seek to make capital out of the (almost) inevitable concession of his opponent that some form of non-empirical knowledge is possible. Suppose a nominalist opponent is willing to grant that there is an explanation available of our tendency to form true beliefs about a particular species of necessary truth (logical consequence) without which we could not reason. Then the neo-logicist can respond by claiming that a similar explanation may be provided of our tendency to form true beliefs about another species of necessary truth (mathematics) (Hale [1994c]).

to make reference to such entities with familiar sentences that make no such reference, abstraction principles provide a mechanism for 'explaining away' or 'reducing' apparent reference to numerical objects. But despite its superficial integrity, this objection really fragments into (at least) three contrasting complaints that take issue with different components of neo-logicism.

'Syntactic reductionism' grants that the expressions introduced syntactically appear to be singular but also notes that the familiar sentences (relative to which the novel sentences are introduced) do not contain expressions purporting to refer to any novel entities. Since the sentences introduced are stipulated to be equivalent to these familiar sentences, it follows that the syntactic appearance to which the introduced expressions purport to convey reference cannot be anything other than semantically misleading (contrary to *Syntactic Decisiveness*). By contrast, 'semantic reductionism' and 'anti-realist reductionism' accept that expressions introduced by the method of abstraction are genuinely referential but differ according to the referential status that may legitimately be assigned to those terms. Semantic reductionism agrees with neo-logicism that the reference of the introduced terms merits a realistic interpretation but denies that these terms need pick out any novel objects (contrary to *Semantic Novelty*). In agreement with neo-logicism, 'anti-realist reductionism' allows that these terms pick out previously undisclosed items but denies that claims about such entities should be interpreted realistically (contrary to *Referential Realism*). So rather than presenting a unified response, the various objections conflict with one another, marking different points of agreement and disagreement with neo-logicism. Since syntactic reductionism casts doubt directly on the conception of ontology that underpins neo-logicism, we will begin by investigating this most fundamental of objections.[16]

The method of abstraction establishes that an equivalence relation obtains between familiar and novel sentences. If the syntactic reductionist argument is to succeed, then the relation established must be a congruence (in the following sense) with respect to semantic properties: if two sentences figure in this relation then if one sentence exhibits (or lacks) a semantic property, the other does, too. By assumption, familiar sentences lack the semantic property of referring to novel objects. So if it is also assumed that the relation established is a congruence with respect to semantic properties, it follows that

[16] Syntactic reductionism is sometimes labelled 'austere' (Wright [1983], p. 86). For further consideration of **semantic reductionism**, see Dummett ([1991], pp. 126, 165), Rosen ([1993], pp. 168–9), Hale ([1994a], pp. 201–5), and Hale & Wright ([2001b], pp. 352–66). Different species of **anti-realist reductionism** (which have fallen under the titles 'intermediate' and 'tolerant') have been advanced by Dummett ([1981a], pp. 498–500, [1981b], pp. 424–7, pp. 452–7, [1982], pp. 240–2, [1991], pp. 189–99, pp. 234–5, p. 239 and [1998], pp. 384–6). For discussion, see Wright ([1983], pp. 64–84), Hale ([1987], pp. 150–93), Miller ([1991]), Hale ([1994a], pp. 200–7), and Wright ([1998b], pp. 268–71).

the novel sentences introduced via the relevant equivalence also lack the semantic property of referring to novel objects. Therefore, there are two ways open to the neo-logicist to respond to syntactic reductionism. Either he may reject the assumption that—contrary to syntactic appearances—the familiar sentences already effect reference to the 'novel' entities. Or he may reject the assumption that the stipulated equivalence is congruent with respect to the semantic property of reference.[17]

'First generation' neo-logicism advocates the rejection of the former assumption (Wright [1983], pp. 31–2; Hale [1987], pp. 158–62). Since the familiar sentences contain no terms apparently referring to novel objects, the syntactic reductionist claims that the introduced sentences featuring terms apparently so referring must be grammatically misleading. In response, the neo-logicist argued that the reductionist way of looking at things was arbitrary and could very well be turned 'on its head'. Since the introduced sentences contain terms referring to the objects in question (numbers), the familiar sentences that do not contain terms referring to such objects must be misleading. Even though these familiar sentences contain no *isolatable* element that refers to the relevant objects, they must in virtue of their equivalence with the sentences introduced have *tacitly* so referred all along. So rather that extending the subject matter of our discourse the method of abstraction merely serves to render explicit the nature of the referents that we already implicitly thought and talked about.

Regrettably, the first-generation response to syntactic reductionism abrogates a fundamental principle constraining reference (Wright [1988], pp. 456–9, [1990], pp. 164–70; Dummett [1991], pp. 168–70, pp. 194–5). In order for a sentence to be used by a speaker to effect reference to an object, the sentence in question must express an *identifying thought* of that object. An identifying thought characterises an object as being of a certain kind and thereby enables the speaker to identify the object in question. So a speaker who understands and endorses a sentence that effects a reference must know what kind of object he thereby picks out. It follows, contrary to the proposal, that familiar sentences cannot even tacitly effect a reference to objects of a kind of which a speaker competent with sentences of that sort has no inkling. A sentence that purports to convey reference to a given kind of object cannot express the same identifying thought as a sentence that fails to articulate the resources required for identifying an object of that kind.

[17] The neo-logicist also argues that syntactic reductionism leads to 'unpalatable consequences' or is otherwise self-defeating (Wright [1983], pp. 32–35, p. 40, pp. 89–90, p. 173, and Rosen [1993], p. 164). I will address these arguments in forthcoming work on properties. Heck ([2000]) argues that syntactic reductionism cannot account for the interaction between numerical terms and plurality quantifiers.

As a result, 'second-generation' neo-logicism takes the alternative tack of dropping the assumption that the stipulated equivalence is a congruence with respect to reference (Wright [1988], pp. 459–60, pp. 469–73, [1990], pp. 164–7; Hale [1994a], pp. 192–7). The neo-logicist begins by contrasting reference with ontological commitment. Whereas a speaker can only refer to an item if they have can identify it, they may be ontologically committed to the existence of an object even when they entirely lack a conception of how to isolate the relevant kind of object from the environmental backdrop. A sentence is ontologically committed to those entities the existence of which it entails, and a speaker may fail or be unable to grasp all the entailments of a sentence they endorse (Wright [1988], p. 473, [1990], p. 165). Next the neo-logicist observes that the method of abstraction does not stipulate that novel sentences and the familiar ones by which they are introduced express the same identifying thought. The method only stipulates that they coincide in truth-value and thereby fixes their necessary equivalence. Since necessary equivalent sentences may refer to different things, it follows that the equivalence established by stipulation between novel and familiar sentences fails to be a congruence with respect to the semantic property of reference. So, contrary to syntactic reductionism, it does not follow from the fact that novel sentences are stipulated to be equivalent to familiar ones that the sentences introduced are incapable of referring to objects of which we were previously unaware.[18]

A suspicion is, however, likely to linger that even second-generation neo-logicism fails to address the syntactic reductionist concern. Let it be granted that there are *some* necessarily equivalent sentences that effect reference to distinct kinds of objects. But it remains to be established that the sentences stipulated to be necessarily equivalent by an abstraction principle fall amongst the class of such sentences. For example, according to the neo-logicist account, right-hand-side instances of (HP) are ontologically committed to numbers even though no reference to numbers is achieved there. However, there is only reason to suppose that such commitments are present if appropriate left-hand-side instances of (HP) do achieve reference to numbers, and it may be questioned whether so much is achieved. The method of abstraction specifies the syntactic roles of novel terms and fixes the truth conditions of the *whole* sentences in which they occur. But it neglects to specify referents for the individual terms it introduces. It fails to fix a semantic contribution for each to make. One might therefore (non-arbitrarily) wonder

[18] It may seem that the neo-fregean response simply abuses the label 'ontological commitment'. For example, Boolos reserves the term for whatever our normal use of a sentence forces us to admit (Boolos [1985], pp. 77–8). By contrast, the commitments which the neo-fregean discerns are only brought to light when a suitable abstraction is laid down. But this only shows that the label may be used in different ways, not that the neo-fregean employment of it—tied to the notion of necessary equivalence—is illegitimate.

whether the expressions introduced (on the left-hand side of an abstraction) are genuine referring expressions rather than mere syntactic simulacra. Consequently, one may also doubt whether the familiar sentences that feature on the right-hand side of abstractions are ontologically committed to the entities to which the necessarily equivalent left-hand-side sentences are intended to refer.

In order to appreciate how the neo-logicist may improve upon the explicit responses he has offered to syntactic reductionist doubts, it is important to recognise that the neo-fregean conception of ontology is itself intended to be (in part) the consequence of a semantic principle.[19] This principle dictates that there can be no requirement that the semantic contribution of a referring term be explicitly fixed in the manner syntactic reductionism demands. It is insufficiently emphasised and rarely enough mentioned that it is the later Wittgenstein, rather than Frege, whose (proximate) influence results in the adoption of this principle (Wright [1983], xxi, pp. 41–7, p. 129).[20]

Wittgenstein, in his attack on the Augustinian conception of language, noted that ostensive definition (where a name is explicitly correlated with its intended referent) could never serve as a basis for acquiring a first language (Wittgenstein [1953], §§26–38). For in order to appreciate the significance of an ostensive definition, Wittgenstein argued, the trainee must already possess a considerable linguistic understanding. If they lack this understanding then they will also lack the conceptual wherewithal to single out the feature of the environment under scrutiny and appreciate the grammatical character of the expression introduced. Language acquisition must therefore—prior to an explicit appreciation of the reference of ingredient expressions—rely upon training in the use of whole sentences. It is the acquisition of the ability to collectively employ these sentences in suitable configurations that eventually endows the trainee with the conceptual background required for that appreciation.

[19] The neo-logicist does not officially respond to syntactic reductionist doubts in the manner proposed in the text, but I am recommending that he do so. According to the official neo-logicist line, the fact that the significance of introduced expressions is determined contextually should not blind us to the contribution made by these expressions to the *content* of the novel sentences (Wright [1998b], pp. 270–1). But this reply to the syntactic reductionist is hostage to i) an articulation of the relevant notion of content, and ii) an account of why the contribution effected by the introduced expressions to the content is genuinely objectual in character.

[20] See Tait ([1986]) for a related position based upon Wittgenstein's critique of Augustine. It should be noted that Frege's own metaphysical position may be far more akin to the Wittgensteinian position ascribed to the neo-logicist than is often recognised. See Ricketts ([1986]), Weiner ([1990], pp. 133–218), and Reck ([1997]).

It is this last thought that exerts such a powerful and liberating influence upon the neo-logicist.[21] Once it is granted that individual citations of reference are only intelligible against such a background, it comes to seem that so long as sufficient background training in whole sentences is available then even numerical expressions may be introduced so as to have a reference. The method of abstraction is expressly designed by the neo-logicist to provide this training (by fixing familiar truth conditions for the whole numerical sentences introduced). More generally, if it is the use of sentences that ultimately serves as the medium of our engagement with the world, then it is ultimately in linguistic terms (rather than, for example, by appeal to causal or spatio-temporal concepts) that ontological notions such as 'object' and 'property' should be defined. Neo-fregeanism is designed to be the expression of just that idea.

We here touch here upon some of the most obscure, underdeveloped but nevertheless challenging aspects of neo-logicism. If Wittgenstein's critique of Augustine really has the metaphysical import the neo-logicist claims, then it is not only syntactic reductionism that must be jettisoned. In fact *any* of the prevalent conceptions of analytic ontology that purport to cleanly separate linguistic and ontological issues must be abandoned (see, for example, Mellor [1993] and Armstrong [1997]). The question is, of course, whether Wittgenstein's critique really is cogent or powerful enough to generate such metaphysical consequences.

There are clearly a number of junctures where legitimate doubts may be raised. To begin with, it may be doubted whether it is training in whole sentences that circumvents the massive under-determination inherent in the process of language acquisition. It may, for example, be claimed that it is hardwired grammars, shared quality spaces built into perceptual systems, or, more generally, innate ideas that overcome the debilitating effects of under-determination. And even if it is granted that training in whole sentences must come first in the order of linguistic *acquisition*, further argument is clearly required to show that sentences are primary in the order of semantic *explanation* (thereby rendering sentences the primary medium of reference with the external world).

Let it be granted, however, that the referential function of an expression is ultimately bestowed by the pattern of use to which its host sentences contribute.

[21] The note of liberation is also struck (albeit for different dialectical purposes) by Wittgenstein: 'Now one can ostensively define a proper name, the name of a colour, the name of a material, *a numeral*, the name of a point of the compass and so on. The definition of the number two, 'That is called "two"—pointing to two nuts—is perfectly exact' (Wittgenstein [1953], §28, my italics). Also consider: 'Grammar tells what kind of object anything is' (§373); 'Like everything metaphysical the harmony between thought and reality is to be found in the grammar of language' (Wittgenstein [1967], §55); 'The connection between "language and reality" is made by means of the clarification of words, which belongs to the learning of language, so that language remains closed within itself, and autonomous' (Wittgenstein [1969], §112).

There are, nevertheless, sentences the use of which fails to enjoin the referential function of any word. Consider (to take only the simplest of examples) the use of the one word sentence 'slab' in Wittgenstein's elementary language game ([1953], §2). It follows that there are two additional theoretical hurdles—even supposing other doubts to have been quelled—which the neo-logicist must negotiate. First, the neo-logicist must distinguish the sentences whose constituent expressions may properly be deemed referential from sentences (like 'slab') that may not. This will mean (given the explanatory resources left available to the neo-logicist) identifying the pattern (or patterns) of use to which a sentence must contribute in order for any of its constituent expressions to be referential. Second, it must be shown that the novel sentences introduced by the method of abstraction exhibit a reference-imposing pattern of use. If this is not shown, then it remains a possibility that the sentences introduced by (HP) are no more than sophisticated variants of 'slab'-like locutions that embody no referential function. Until these two hurdles have been negotiated, it remains an open possibility that really the surface syntax of sentences introduced by (HP) and other abstractions deceives us and—as the syntactic reductionist would have it—these sentences do not feature any novel singular terms.

5 Rejectionism

Rejectionism takes issue with the method of abstraction. There are weak and strong versions of the complaint. Local rejectionism finds fault with specific existential commitments engendered by the stipulation of abstraction principles. For example, the local rejectionist may grant that there are numbers of some concepts. Nevertheless, he may *reject* (HP) on the grounds that it assigns numbers to all concepts. By contrast, global rejectionism finds fault with the existential commitments of abstraction principles per se. Global rejectionism is underwritten by the assumption that linguistic stipulation simply cannot guarantee the existence of anything non-linguistic. So if the method of abstraction does (apparently) succeed in introducing novel objects, then the method of abstraction cannot be licit. Apply this general thought to (HP). (HP) does entail the existence of mathematical objects. In fact, (HP) is equivalent to the outright existential assertion (Boolos [1986], pp. 174–5, [1987], pp. 186–8, [1997], p. 308):

(Numbers) $(\forall F)(\exists! y)(\forall G)\ [(y = Nx : Gx) \leftrightarrow (F 1-1\ G)]$

Therefore (HP) cannot be a stipulation. Insofar as (HP) is conceived as a stipulation, the equivalence it endorses must be *rejected*. All that can be legitimately stipulated is the conditional claim that if numbers exist, then (HP) characterises them (Field [1984], p. 169; Boolos [1997], pp. 306–7). So unless he is possessed of some prior and independent assurance that numbers exist, the neo-logicist cannot responsibly affirm (HP).

The neo-logicist may respond to local rejectionism either by restricting (HP) to avoid the objectionable commitments, or else by denying them to be objectionable. The concern with 'anti-zero' provides a case example. The neo-logicist reconstruction of arithmetic is based upon the assignment of a number (zero) to the concept non-self-identical. If it is legitimate to assign a number to this concept, it ought also to be legitimate to assign a number to its complement (identity). Call this number 'anti-zero'. Since everything falls under the concept of identity, it follows that anti-zero is the number of all the things there are. Boolos offers the following argument (Boolos [1987], p. 184, p. 197, [1990], p. 216, [1995], p. 291, [1997], pp. 313–4). The commitment of (HP) to anti-zero is incompatible with Zermelo-Fraenkel set theory. Sets are amongst the items that fall under the concept *identity*, but ZF denies that there is a number of all the sets there are. So if (HP) is true, then ZF set theory is false (indeed *a priori* false if (HP) is *a priori* true). But this set theory is 'our best established theory of number'. It is would be incredible to suppose that the stipulation of an abstraction principle could render this theory false. Consequently, (HP) must be rejected.

Boolos' argument is not immediately compelling. ZF identifies (given appropriate definitions of cardinal number) the sets that are numbers but since ZF concerns itself only with the set-theoretic universe, ZF remains silent about numbers that are not sets. By contrast, (HP) works outside the restricted universe of set theory. It assigns numbers directly to concepts as values of the function denoted by 'Nx:Φx' regardless of whether those properties have sets for extensions or any extensions at all. As a result, (HP) makes commitment to numbers belonging to concepts whose extensions lack a number in ZF set theory. But it is only under the assumption that ZF and (HP) offer competing analyses of the same univocal notion of cardinality that these different commitments need be seen as conflicting. This assumption does not appear mandatory. Is it not possible that cardinality fractures under analysis into distinct but compatible notions?

At any rate, the neo-logicist chooses to accept Boolos' argument and responds instead by modifying (HP) to avoid a commitment to anti-zero (Wright [1999], pp. 313–5). The neo-logicist argues that a restriction of the range of the initial second-order quantifiers in (HP) to *sortal* concepts is in any case wanted. Roughly speaking, a concept F is sortal if there are determinate identity criteria for objects that fall under F (Wright [1983], pp. 2–4). It is only if a concept is sortal that it can be intelligibly assigned a number. (The guiding neo-logicist intuition here: you can't number the objects presented to you unless you can identify and distinguish between the objects given). Since the concept of identity is not itself sortal—the concept is too general to provide any discriminating identity criteria—the restriction of (HP) to sortal concepts rules out anti-zero (see Wright [1999], pp. 314–5 for further argument).

The restriction may be contested (perhaps non-sortal concepts *do* have numbers) but it also fails to make the problem Boolos identifies disappear. For (HP) remains committed to the numbers of the sortal concepts *cardinal* and *ordinal*, numbers that ZF set theory denies exist. The neo-logicist therefore proposes a further restriction on (HP) (Wright [1999], pp. 315–6; see Rumfitt [2001a] for an alternative modification). The neo-logicist is animated by the thought that *indefinitely extensible* concepts cannot be assigned a number. Dummett elucidates the notion in the following terms: 'an indefinitely extensible concept is one such that, if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterise a larger totality all of whose members fall under it' (Dummett [1993b], p. 441).[22] Example: let C be a collection of cardinal numbers; let C* be the union of the result of replacing each $k \in \mathrm{C}$ with a set of size k. The cardinal of the powerset of C* is larger than any cardinal in C (by Cantor's theorem). Hence, the concept *cardinal* is indefinitely extensible. The neo-logicist therefore proposes that concepts can only be assigned a number if the objects that fall under them form a definite (i.e. not indefinitely extensible) collection. So the (principled) restriction of (HP) to definite concepts rules out a commitment to the number of cardinals and ordinals and avoids a clash with ZF set theory.

This neo-logicist response, however, raises as many questions about indefinite extensibility as it answers about (HP). The notion of indefinite extensibility has proved to be a captivating one, but the notion has also resisted precise characterisation. Clark [forthcoming] argues for the claim that the notions of definite and indefinite can only be rendered precise in set-theoretic terms, thereby revealing the neo-logicist reconstruction of arithmetic to be parasitic upon a prior grasp of set theory. More generally, neo-logicism claims to provide a *logical* reconstruction of arithmetic, a reconstruction that only uses logical definitions and proofs. But if indefinite extensibility resists precise characterisation, the neo-logicist can hardly assume the notion is logically definable. One may even doubt whether the notion is ultimately intelligible or merely chimerical.

Global rejectionism takes offence at the very idea that an abstraction principle made true by stipulation might incorporate existential commitments (Field [1984], pp. 168–9, [1993], pp. 286–7; Boolos [1987], p. 199, [1990],

[22] The notion originates with Russell's Vicious Circle Principle (Russell [1906], p. 144) but has figured most prominently in the writings of Dummett ([1963], [1991], pp. 313–21, [1993]). Clark ([1993a], [1998]), Oliver ([1998]), Shapiro ([1998]) and ([2003]) provide critical discussion.

p. 211, p. 214, [1997], pp. 304–8). The neo-logicist offers two fast-track arguments for dismissing global rejectionism:

(1) It is wrong to characterise (HP) as an existential assertion. It is merely a *conditional* claim that does no more than fix the truth conditions of novel statements (Wright [1990], pp. 162–3; Hale & Wright [2000], pp. 145–6).
(2) Since it treats the success of an abstraction in introducing novel objects as a *reductio ad absurdum* of the method of so introducing objects, global reductionism simply begs the question against the neo-logicist (Wright [1990], pp. 167–8).

Neither argument is likely to prove persuasive. It is true that (HP) is conditional in form, and only in conjunction with second-order logic does (HP) entail the existence of numbers. However, it is only in conjunction with logic that *any* statement is existence entailing. So this reflection fails to distinguish (HP) from other existentially committing claims. The neo-logicist replies that it is the *way* in which logic is needed to deduce existential consequences from (HP)—by finding a logical truth to serve as an instance of its right-hand side—that manifests the existential neutrality of (HP). But his opponent is unlikely to see the relevance of the way in which existential consequences are derived and it is difficult to see how the debate might be fruitfully adjudicated from there on. It is also difficult to see how the charge of question-begging might move a neutral observer. For the rejectionist may make the very same charge: by assuming that the stipulation of an abstraction can legislate regarding existence, the neo-logicist begs the question against rejectionism. If we are open-minded and wish for a principled reason to choose between the competing positions, additional arguments will have to be forthcoming.

The neo-logicist offers, however, two further arguments against global rejectionism. The first argument claims that rejectionism fails to provide an adequate account of the concept *number* (Wright [1983], p. 148–52, [1990], pp. 158–60; Hale & Wright [2000], pp. 143–5). The stipulation of (HP) supplies a sufficient condition for the existence of numbers: that there be equinumerous concepts. Consequently, the neo-logicist can intelligibly make out conditions under which numbers exist and conditions under which numbers do not exist. However, by insisting that it is only the conditional 'if numbers exist then (HP)' that may be legitimately stipulated, the rejectionist fails to supply a sufficient condition for the existence of number. His treatment of the concept *number* is consequently lacking. The rejectionist is left with no conception of 'a ground on which we could, in practice or in principle, rely to determine whether or not [numbers exist]'.

The argument may be questioned in a variety of ways. It neglects the possibility that the rejectionist may supply sufficient conditions for the existence of numbers in the following manner: numbers exist if number theory performs an indispensable role in science. The argument also appears to place too high a threshold on conceptual adequacy. It is a familiar enough thought that philosophically interesting concepts cannot be supplied with necessary and sufficient conditions for their application (where the conditions are specified discursively without employing the target concept). Furthermore, if the entities we seek to describe with these concepts are irreducible and metaphysically fundamental, then it is inevitable that there can be no sufficient conditions for the existence of these objects specifiable independently of the relevant concepts. Example: assume substance dualism. Then there are no material conditions that are sufficient for the existence of a mind. Does this mean that there can be no creditable thought concerning the existence of minds (other than perhaps our own) because we have no conception of a ground for their application that does not employ mental concepts?[23]

It is the second argument offered by the neo-logicist that promises the potentially load-bearing response to global rejectionism (Wright [1997], pp. 275–8, [1999], pp. 308–12; Hale [1997], pp. 103–5). The rejectionist reads an abstraction as making the substantial claim that along with the items we usually talk about there are additional novel objects correlated with them. It is because these claims appear to be substantial or 'inflationary' that the rejectionist denies they can be open to stipulation. But the neo-logicist suggests that abstraction principles be interpreted differently. Consider (HP). This principle serves to introduce the concept *number* in such a way that there is 'no gap' between the obtaining of an equivalence relation between concepts and the corresponding identity between numerical objects; the obtaining of the relevant equivalence relation between concepts 'constitutes' the identity of their numbers. So rather than making substantial or inflationary claims, abstraction principles merely effect 'reconceptualisations' or 'recarvings' of the same state of affairs. (HP) shows us that the very same state of affairs that concerns concepts may be reparsed as a state of affairs concerning numbers.[24]

This argument lies open to the following counter. Let it be granted that abstraction principles display how the same state of affairs may be parsed in different ways. But it does not follow, as the neo-logicist claims, that abstraction principles are any the less substantial or inflationary for that concession. For it is still a substantial claim that two ranges of entirely distinct

[23] Discussion of these issues resulted in a debate of several instalments between Hale, Wright and Field. See MacBride ([1999], pp. 443–5) for an overview of the debate and references.

[24] Hale ([1997]) offers to clarify the notions of *state of affairs* and *reconceptualisation* in play. See Potter & Smiley ([2001]) for a critical discussion, and Hale ([2001]) for a further response.

items (e.g. numbers and concepts) can be necessarily correlated in the way in which the stipulation demands. Even if only a single state of affairs is involved, it remains a substantial claim that the same state of affairs may be parsed in two distinct ways. The neo-logicist is not entitled to stipulate that there is 'no gap' between the obtaining of an equivalence relation and the instantiation of a novel concept. How could it ever be *stipulated* that if there is one kind of object then there is another distinct but necessarily correlated kind?

There is no hiding from it but that the neo-logicist must tackle deeper, underlying metaphysical issues if he is to effectively deal with the global rejectionist. The rejectionist assumes a bifurcated conception of language and reality that makes it unintelligible how a linguistic stipulation could legislate with regard to existence. So the neo-logicist must proffer a more monistic conception that renders this sort of legislation intelligible. It is tempting to suppose that neo-fregeanism (the underlying conception of ontology that certainly offers a high degree of accord between language and reality) will provide the required outlook. But this would be a mistake. The rejectionist can well agree that if a singular term t_1 figures in a true sentence S, then there is an object to which t_1 refers. In fact it is (in part) because the rejectionist accepts neo-fregeanism that he dismisses the method of abstraction. The rejectionist reasons in the following manner. If the method of abstraction were effective then true novel sentences could be introduced that—by the dictates of neo-fregeanism—referred to novel objects. However, it offends against the rejectionist's bifurcated view that the existence of an object could be revealed simply by stipulating that a novel sentence is equivalent to a familiar (true) sentence. He concludes that the method of abstraction cannot be effective; the stipulations must be rejected although he may continue to endorse the thesis that singular terms in true sentences refer.[25]

What really is at issue between the global rejectionist and the neo-logicist? It appears to be the sense in which states of affairs are structured. The neo-logicist assumes that when an abstraction principle is stipulated, any states of affairs that exhibits the parsing characterised by one side of the abstraction will also exhibit a distinct parsing characterised by the other side. There is no leap of faith to be made when we lay down an abstraction principle to characterise the relevant necessary connections that obtain between the different objects revealed by these parsings. The presence of the different

[25] This is not the only way of developing the global rejectionist strategy but it has the merit of helping to sieve out the underlying commitments of neo-logicism. Shapiro & Weir ([2000], pp. 179–188) and Potter & Smiley ([2001], pp. 336–8) endorse a species of rejectionism that simultaneously takes issue with the method of abstraction *and* neo-fregeanism. They deny that abstraction principles may be legitimately stipulated in the context of classical logic but argue that they may be so stipulated in the context of a free logic. They then point out that neo-fregeanism fails in a free logical setting: none of the existential consequences upon which the neo-logicist relies are derivable in a free logic.

arrays is guaranteed by the structure of the sentences stipulated to be equivalent by the abstraction principle. How could this possibly be if the states of affairs that make up the world did not have their structure imposed upon them by our sentences?[26]

The disagreement between the global rejectionist and the neo-logicist thus appears to come down to this. The rejectionist assumes that the structure of states of affairs is *crystalline*—fixed quite independently of language. By contrast, the neo-logicist assumes that states of affairs lack an independent structure, that states of affairs are somehow *plastic* and have structure imposed upon them by language. As a consequence it is unintelligible for the rejectionist that the method of abstraction might be ensured to disclose additional structure in a state of affairs. By contrast, from the neo-logicist point of view it is inevitable (supposing the abstraction meets certain formal requirements to be discussed in Section 8) that the method of abstraction will succeed in delineating novel structure.[27]

Seen, however, in the light of the concluding Wittgensteinian reflections of the previous section (Section 5) the dispute between the rejectionist and the neo-logicist may run far deeper than the envisaged disagreement about the metaphysical status of structure. Rather the disagreement may turn upon whether there is *any* metaphysically robust account to be given of how an area of discourse 'fits' reality. The rejectionist and his envisaged opponent offer competing realist and anti-realist accounts of how this is achieved—by discourse either mirroring or imposing the structure of states of affairs. But the neo-logicist need not be saddled with either a realist or an anti-realist account. Rather, he may reject the dichotomy between mirroring and imposing structure and invoke instead a form of quietism. This form of quietism denies that there is any intelligible question to be raised about whether there might be some ulterior failure of 'fit' between language and reality. Instead, we can only

[26] It is worth recalling that Dummett attributed just such a position to Frege: 'the world does not come to us articulated in any way; it is we who, by the use of our language (or by grasping the thoughts expressed in that language) impose a structure on it' (Dummett [1981a], p. 504).

[27] The metaphysical conception does appear to surface on occasion (Wright [1983], p. 48, [1992], pp. 181–2). There is something close to an explicit endorsement: 'if we endorse the syntactic priority thesis, we abandon the view, characteristic of the *Tractatus*, that the structure of a state of affairs is somehow determined independently of the syntactic structure of any statement of it' (Wright [1983], p. 129). Unfortunately, this remark does not quite articulate the conception in question. It fails to distinguish the conception in question from the syntactic priority thesis (neo-fregeanism). Moreover, it does not endorse the thesis that the structure of a state of affairs is determined by (potentially) many different statements of it. The remark is also problematic given Wittgenstein's own endorsement of the context principle ('Only propositions have sense: only in the context of a proposition does a name have meaning': Wittgenstein [1961a], 3.3) and apparent acceptance of the syntactic priority thesis (Ishiguro [1969], McGuinness [1981]). There are, however, earlier remarks made by Wittgenstein that suggest more clearly the sort of doctrine the neo-logicist wishes to replace in the *Notebooks* (Wittgenstein [1961b], 4 Nov. 1914).

submit to the norms of our discourse and record whether according to them the use of language determines the presence of objects.

The neo-logicist has not relied upon this underlying quietist conception in the defence of his position, a conception that exhibits more than a passing resemblance to the (murky) doctrine of internal realism (Carnap [1950]; Putnam [1987], pp. 16–20; see also Wright [1992], [1998c] for a corresponding account of truth). Nor has he relied upon an anti-realist conception of structure either. Nevertheless, if the reasoning of this section carry any force then it is, in the final analysis, with the tenability of quietism or anti-realism that neo-logicism must stand or fall.

6 The 'Julius Caesar' problem

Frege (recall) rejected (HP) as an adequate definitional basis for introducing numbers. He rejected (HP) on the grounds that it failed to specify *which* objects are numbers. To justify the employment of (HP) (and abstraction principles in general), the neo-logicist must somehow solve or dissolve the so-called 'Caesar' problem. The neo-logicist expresses the 'Caesar' problem into the following terms. The intended purpose of (HP) is to introduce a novel sort of object, number. If (HP) is to fulfil this role, then (HP) must succeed in introducing a *sortal concept*, a concept under which things of the sort number fall. Any genuinely sortal concept is associated with two distinct criteria:

Criterion of application: a criterion that discriminates between those objects to which the concept applies and those to which it does not.
Criterion of identity: a criterion that discriminates between identical and distinct objects to which the concept applies.

(HP) provides a criterion of identity for numbers: it tells us whether numbers are identical or distinct (supposing that they are already described in numerical terms); they are identical when their associated concepts are 1–1 correspondent, and distinct when their concepts are otherwise related. But (HP) does not supply us with a criterion of application: it does not tell us whether Caesar does or does not fall under the concept number. Since (HP) does not supply a criterion of application for numbers, (HP) fails to introduce a sortal concept of number.

The neo-logicist first offered to solve the 'Caesar' problem by demonstrating that (HP) does in fact supply a criterion of application (Wright [1983], pp. 107–17, p. 122; Hale [1994a], pp. 197–200; Wright [1998a], pp. 249–50; Hale & Wright [2001b], pp. 367–70). The argument proceeds by appeal to another principle (presumably *a priori*) that governs sorts in general:

Sortal Inclusion Principle (SIP): a sort of objects F is included within a sort of objects G only if the content of a range of identity statements about Gs—those linking terms denoting Gs that are candidates to be Fs—is the same as that asserting satisfaction of the criterion of identity for the corresponding Fs.[28]

Suppose, for example, that there is a range of singular terms ($f_1 \ldots f_n$) that denote Fs and another range ($g_1 \ldots g_n$) that denote Gs. Provided that some true identity '$f_1 = f_2$' has the same content as another true statement '$g_1 = g_2$', then—according to (SIP)—an (at least) necessary condition is satisfied for some Fs to be Gs.

Next it is noted that the content of identity statements about persons is different from the content of identity statements about numbers. The former statements (plausibly) concern facts about physical and psychological continuity and connectedness. By contrast, the criteria of identity supplied by (HP) determine that the latter statements concern facts about 1–1 correspondences between concepts. (SIP) then dictates that the sort *person* cannot be included within the sort *number*. So, contrary to Frege's concern, it cannot be that Caesar is amongst the objects introduced by (HP). More generally, it follows that the concept *number* (when introduced via (HP)) can only apply to objects whose identity and distinctness concerns facts about 1–1 correspondences between concepts. A criterion of application is thereby 'extracted' from the criterion of identity that (HP) provides and the 'Caesar' problem solved.

This argument has been criticised both on the grounds that it is too strong and on the grounds that it is too weak. It appears to be too strong because it rules out the possibility of apparently intelligible identifications between different sorts of objects (Dummett [1991], pp. 161–2; Hale [1994a], pp. 197–200; Sullivan & Potter [1997], p. 139; Hale & Wright [2001b], pp. 371–80). For example, mathematicians may sometimes 'identify' the integers and natural numbers with the complex numbers. Since different criteria of identity are associated with these different species of number it follows by (SIP) that the sortal concepts *integer* and *natural number* cannot apply to complex numbers and consequently no natural number or integer can be identified with a complex number. Counter-arguments of this form are, however, unlikely to be conclusive, for it is open to the neo-logicist to deny

[28] Hale and Wright have offered differing versions of (SIP) where the notion of 'content' is variously replaced by 'adequate explanation of truth condition' and 'what determines an identity statement as true or false'. On some occasions the 'only if' is also tentatively replaced by an 'if and only if'. It would take us too far a field to consider the differences these formulations might make to the import of (SIP) (Wright [1983], pp. 114; Hale [1994a], p. 198; Hale & Wright [2001b], pp. 368–9). The objections considered are intended to be generic and apply to all these different variations.

(reasonably) that the identifications in question need be taken literally and claim instead that they indicate a modelling or role-occupying relationship between sorts. So, to return to our example, the mathematician may be construed not as identifying the natural numbers and integers with the complex numbers but as employing the complex numbers to serve as a *model* of the integer and natural number structures and thereby prove results about the latter that might otherwise have been difficult to achieve. This sort of interpretation also makes ready sense of the fact that mathematicians may *choose* to 'identify' different sorts of numbers on different occasions; facts of identity are hardly a matter of choice, whereas there is latitude for discretion in which sort of number we may use to model another.

Let us therefore turn to the criticism that the neo-logicist argument is too weak because (arguably) it fails to supply a criterion of application. This criticism admits of more or less concessive variations. According to the concessive form, the proposed solution to the 'Caesar' problem does show that the sortal concept introduced by (HP) cannot apply to Caesar. Nevertheless, it fails to show that the concept in question cannot apply to other different sorts of abstract items. Consider, for example, classes whose identities are also settled by facts concerning 1–1 correspondences and can be shown to be isomorphic to the natural number sequence (Benacerraf [1965], pp. 272–85; Parsons [1965], pp. 194–5; Wright [1983], pp. 121–2). In response, the neo-logicist points out that the incapacity of (HP) to generate a criterion of application which distinguishes between isomorphic copies of the natural numbers can only be considered a failing if it is in general possible for criteria of application to discriminate in a more fine-grained way. However, Quine's arguments for the inevitable inscrutability of reference throughout language suggests that our terms can never so discriminate. Consequently, no special failing of the (purportedly) sortal concept introduced by (HP) has been identified.[29]

According to the less concessive form of criticism, the proposed solution does not succeed even in showing that the concept introduced cannot apply to Caesar.[30] Here is one way of developing such a concern. What could the occurrence of the term 'content' in (SIP) mean? If it means 'sense', then the principle is clearly false since there may be different senses that share a reference. If, on the other hand, it means 'reference', then (SIP) is clearly true. For (trivially) two sortals cannot overlap unless identity statements involving

[29] See Wright ([1983], pp. 123–7) and for discussion, McGinn ([1984]), Hazen ([1985]), Hale ([1987], pp. 194–244), and Williamson ([1988], pp. 488–90). Important contributions to this field of debate are supplied by Field ([1974]), Hodes ([1984]), Brandom ([1996]) and McGee ([1997]).

[30] A battery of related criticisms are proposed and discussed in Dummett ([1991], p. 161), Hale ([1994a], pp. 197–9), Rosen ([1993], pp. 171–4), O'Leary-Hawthorne & McDonough ([1998], pp. 351–3), and Hale & Wright ([2001b], pp. 383–5).

them say of the same objects that they are identical. However, it does not follow from (SIP) so interpreted that different concepts cannot overlap. Simply because identity statements can be settled in different ways it does not follow that they are not statements concerning the same objects. One and the same referent may just fall under different modes of presentation (personal and numerical).

To evade this dilemma the neo-logicist may seek to interpret the notion of content in a manner that is weaker than sense but stronger than reference. But is there a principled intermediate notion available for such a purpose? The neo-logicist cannot simply gerrymander a notion which ensures that identity statements concerning persons and numbers differ in content. For it may be that Caesar and a number fail to be the same object in the sense correlative to this manufactured notion but are identical with respect to some alternative conception of content. So it needs to be established that the notion of content employed by the neo-logicist is the one relevant to enquiries concerning the status of persons and numbers. Moreover, it may be that there is no general notion of content available. Such is the heterogeneous nature of sortals it may be that there is no 'one' thing that is *the* content of an identity claim. It is unclear whether there need be anything in common between *a posteriori* identities in biology and necessary identities in mathematics or even metaphysics. It is consequently unclear whether the neo-logicist can evade the given dilemma.

Potter & Sullivan (see their [1997], pp. 143–5) raise an important, additional concern. The neo-logicist solution rests upon the assumption that (HP) provides an *exhaustive* characterisation of the nature of the objects it introduces. With this assumption in place it follows (by (SIP)) that no number can be Caesar. This is because the nature of persons far outruns that which can be articulated in terms of 1–1 correspondence. But, as Sullivan & Potter point out, it cannot in general be assumed that the means whereby an expression or expressions are defined serve to exhaustively characterise the objects, if any, thereby picked out. If a class of objects exists independently of us, then it must be allowed that the underlying nature of these objects may evade our initial characterisation of them. It is a symptom of this general thought that—in an area where realism is usually taken for granted—we readily grant that the real essence of a natural kind may far outrun the nominal essence employed to initially define and introduce a term for that kind. So, if we are realist about the class of objects introduced, the fact that they are introduced by (HP) cannot by itself demonstrate that they are nothing but objects whose associated concepts figure in relations of 1–1 correspondence.

In response to these various difficulties, the neo-logicist has proposed a novel solution to the 'Caesar' problem that replaces the earlier one. This solution rests upon a distinctive *epistemological* feature of putative trans-categorical identities: 'there is simply no provision for evidence for or against such identities' (Hale & Wright [2001b], p. 394). It is difficult not only to rule out the possibility that Caesar might be both a person and an object whose identity criteria are given by (HP). It is just as difficult, the neo-logicist claims, to settle whether Caesar belongs to any other arbitrary category (just consider any other absurd trans-categorical identification). Grant that realism really does enjoin us to take seriously the possibility of trans-categorical identities between numbers and persons. It follows that we are also enjoined to countenance the possibility of arbitrary trans-categorical identities whose (apparently) discrete subject matters merit a realistic interpretation. But such identities are, in general, evidence transcendent. The neo-logicist concludes that the inability of (HP) to supply evidence for or against just one of them can hardly be taken to signal a particular defect.

The novel neo-logicist solution to the 'Caesar' problem thus takes the form of a dilemma (Hale & Wright [2001b], pp. 385–96). According to the ontology favoured by the neo-logicist, the world is comprised of a range of categories, where categories are distinguished by the different criteria of identity associated with the objects that fall under them. Questions of intra-categorical identity are settled by the distinctive criterion of identity of the relevant category. Now, either it is granted that there are trans-categorical identities or it is not. If it is granted, then—given that the identities at issue are evidence transcendent—it follows that all questions of trans-categorical identity and distinctness are left undecided. Consequently, there can be no legitimate demand that (HP) decide whether Caesar falls amongst the objects it introduces. If it is not granted that such identities obtain then (*a fortiori*) questions of trans-categorical identity are settled in the negative simply on the grounds that the objects concerned belong to different categories. Since Caesar and the objects introduced by (HP) belong to different categories (associated with distinctively different personal and numerical criteria of identity), it follows that Caesar is distinct from any of the objects (HP) introduces. In sum: either the 'Caesar' problem requires no solution or else the 'Caesar' problem is solved.

This latest neo-logicist argument clearly requires development in several different directions before its force can be accurately assessed. For instance, the notion of category the argument employs requires principled and precise articulation (do sets and numbers belong to different categories? do different species of numbers belong to different categories?). Similarly, it needs to be shown that all trans-categorical identities share the same evidential status. But in advance of such developments it is illuminating to step back and reconsider the character of the 'Caesar' problem itself. The question we need to ask is: which component neo-logicist doctrine or doctrines does the 'Caesar' problem cast in doubt?

The 'Caesar' problem expresses a generic doubt about the method of abstraction (the capacity of (HP) to introduce novel objects). But it admits of precisification in (at least) two distinct ways.[31] It may be taken to express a doubt concerning the capacity of (HP) to introduce genuine singular terms. Or it may be taken to raise the question whether all the terms fabricated by (HP) refer to novel objects (e.g. $Nx : x \neq x$) rather than familiar ones (e.g. Caesar). The solutions (or dissolution) of the 'Caesar' problem proposed by the neo-logicist address this second concern. They seek either to demonstrate that the introduced terms do have novel reference, as *Semantic Novelty* demands, by showing that the relevant class of trans-categorical identities are false. Or, alternatively, they seek to show that the demands of *Semantic Novelty* are misplaced and there is no need to determine a unique referent for any term (HP) introduces. However, if the first concern is not addressed these putative solutions to the 'Caesar' problem are otiose. Supposing that (HP) fails to introduce any singular terms, it follows that the relevant class of trans-categorical identities are *meaningless* rather than false and that the terms (HP) introduces are not even apt to refer.

Here is one means whereby this version of the 'Caesar' problem might be developed. If the stipulation of truth conditions for sentences of the form '$Nx : Fx = Nx : Gx$' is to introduce genuine singular terms, then it must be ensured that these statements are genuinely logically complex. They must be understood as saying of (the relevant) Nx:Fx that it satisfies the predicate '$\ldots = Nx : Gx$'. Consequently, (HP) will only succeed in fixing truth conditions for genuine identity claims if it also fixes a meaning for such predicates as '$\ldots = Nx : Gx$' and '$Nx : Fx = \ldots$'. To achieve this, (HP) must also fix truth conditions for the sentences in which these predicates occur. So, in general, (HP) must fix truth conditions for sentences of the form '$Nx{:}Fx = q$' (where 'q' is any singular term whatsoever). But (HP) only fixes truth conditions for sentences of the form '$Nx{:}Fx = Nx{:}Gx$' where dual occurrences of the numerical term flank the identity sign. The problem is not simply that (HP) fails to settle a truth-value for identity statements where 'q' is replaced by other terms (like 'Caesar'). The problem is that (HP) fails even to ready such sentences for the receipt of a truth-value; (HP) fails to assign them truth conditions in the first place. (HP) therefore fails to provide a basis for supposing that '$Nx : Fx = Nx : Gx$' is genuinely logically complex, embedding genuine singular terms.

[31] Heck [1997c] argues that Frege's original formulation of the 'Caesar' problem was provoked by a further range of issues related to the character of the proof of the infinity of the natural number series. See Hale & Wright ([2001b], pp. 345–51) for critical discussion.

This problem highlights the need for further clarification of the role of syntax in the method of abstraction. It suggests that—even for the neo-logicist—the assignment of an expression to a given syntactic category does not suffice to generate semantic significance for that expression. The supply of truth conditions for the *wffs* in which the expression grammatically figures is also required. It is only when these two requirements harmonise (matching up the relevant class of *wffs* with truth conditions) that the expressions in question may be properly deemed referential in function. The 'Caesar' problem considered results from the failure of (HP) to supply enough truth conditions for the range of *wffs* in which the introduced expressions occur.

There are at least two strategies whereby the neo-logicist may seek to resolve this problem. The neo-logicist may claim that there is some sort of semantic projection from the sentences for which (HP) directly supplies truth condition to truth conditions for other sorts of contexts. But then it is incumbent upon the neo-logicist to provide some principled account of the basis and extent of the projection involved and demonstrate that (HP) figures in the relevant projective relation to these other contexts. Alternatively, the neo-logicist may weaken the requirement that truth conditions be everywhere supplied for the range of *wffs* in which an expression occurs. In that case it need not be counted a failing of (HP) that it does not provide truth conditions for all sentences of the form '$Nx : Fx = q$'. But this means that the neo-logicist must shoulder the theoretical burden of distinguishing between the cases where the failure to supply truth conditions for a range of *wffs* indicates a deficiency in the expressions introduced and cases where it does not. Then the neo-logicist must show that the expressions introduced by (HP) are of the latter type. Until these syntactical issues are addressed it remains open that the neo-logicist may have laboured in vain to solve the 'Caesar' problem (see MacBride [2003] for further development of this theme).

Related issues are generated by what may be appropriately dubbed the 'counter-Caesar' problem, a pressing difficulty that has received scant attention from the neo-logicist (different versions of the difficulty are highlighted in Benacerraf [1981], p. 20; Heck [1997*a*], p. 596, [forthcoming]; Black [2000]). The familiar 'Caesar' problem concerns our ability to determine whether the terms in different theories (*Frege arithmetic*, Roman history) refer to *distinct* objects (zero, Caesar). The 'counter-Caesar' problem concerns our ability to establish that terms in different theories refer to the *same* objects. The problem is pressing because the neo-logicist claims to offer a reconstruction of ordinary arithmetic. The reconstruction is intended to show how we may arrive by logical means at a grasp of the same objects (zero, one, two...) that we referred to all along by the use of the familiar numerals ('0', '1', '2'...).

The neo-logicist attempts to address the problem by appealing to *Meaning Supervenes On Use* (Section 3) (Wright [1999], p. 322). The reconstruction (*Frege arithmetic*) generates the same pattern of 'use' as ordinary arithmetic (Peano arithmetic). So corresponding terms drawn from the different theories will refer to the same objects (1, $Nx : x = 0$). But this reply is hostage to the provision of a meaning-determining concept of use according to which (i) the theories whose subject matters the neo-logicist wishes to identify exhibit the same pattern of use, and (ii) the theories whose subject matters we wish to distinguish exhibit different patterns of use. Whether a notion of use that possesses these features can be theoretically motivated remains to be established. Independently of the general difficulties that beset a use theory of meaning, it will be no easy task to establish this. *Frege arithmetic* includes a range of expressions for numbers that do not invariably have correlates in ordinary arithmetic (zero, infinite numbers, the number of numbers . . .). So in at least one sense of the term, *Frege arithmetic* generates a different pattern of 'use' to ordinary arithmetic.

A distinction may be drawn between *hermeneutic* and *re-constructive* forms of logicism (cf. Burgess & Rosen [1997], pp. 6–7 for a corresponding distinction amongst nominalist theories). Hermeneutic forms aim to show that what we 'had in mind' all along, when we reasoned arithmetically, is *a priori*. By contrast, re-constructive forms show no concern for the epistemological status of ordinary arithmetic. Instead they aim to elucidate *a priori* knowledge of a subject matter that suffices 'operationally' for ordinary arithmetical purposes. Clearly, the re-constructive approach obviates the need to address the 'counter-Caesar' problem; it sees no operational utility in ensuring that *Frege arithmetic* and ordinary arithmetic characterise the same subject matter. Neo-logicism is historically unusual amongst logicisms for its hermeneutic focus.[32] If the 'counter-Caesar' problem cannot be resolved, then the theoretical option remains open of switching to a re-constructive stance (see MacBride [2000], [2002] for further discussion).

7 Second-order logic

According to neo-logicism, second-order logic provides (in part) a foundation for arithmetical knowledge. For by reflecting upon the (deductive) second-order logical consequences of (HP), a subject ignorant of mathematics may acquire knowledge of arithmetic's fundamental laws. If second-order logic is to perform this foundational role then a grasp of second-order logical

[32] Frege appears to have been a re-constructive rather than a hermeneutic logicist (Benacerraf [1981], Weiner [1984], and Dummett [1991], pp. 176–9). Hempel also appears to have adopted a re-constructive stance in his own logical positivist development of logicism (Hempel [1945], p. 387).

consequence must not require prior acquaintance with mathematics. And, if the neo-logicist is not simply to replace one mystery with another, a grasp of second-order logical truth and consequence must be more epistemologically tractable than a grasp of mathematical truth. But Quine has famously argued that second-order logic is really set theory in disguise (Quine [1970], pp. 66–9). More recently, Shapiro has made an extended case for the view that there is no firm boundary between logic and mathematics (Shapiro [1991], [1999]). These arguments must be addressed if the neo-logicist programme is to realise its epistemological pretensions.

Continuity between the first- and second-order definitions of such notions as truth-in-an-interpretation, validity and consequence provide one indication that second-order logic is logic (Boolos [1975], pp. 41–3). Another indication is provided by the capacity of second-order logic to characterise such (apparently) logical concepts as *ancestral* and capture the formal validity of the natural language arguments in which those concepts figure (Boolos [1975], p. 48–9; Wright [1983], pp. 133–4). But these pointers do nothing to rule out the possibility that second-order logic might be both logical *and* mathematical. Moreover, the fact (if it is one) that second-order patterns of argument exhibit the features (formality, generality...) distinctive of logical inferences does not establish the epistemological tractability of second-order logic (ask yourself: how do we know that arguments exhibiting these features are valid?). Since second-order logic might exhibit these features and be no more epistemologically tractable than mathematics itself, the issue of whether second-order logic is logic becomes (ironically) orthogonal to neo-logicist epistemological concerns.

Three kinds of consideration have been advanced in favour of the view that second-order logic is mathematics (or akin to it).[33]

Semantics: in order to provide a semantic theory for second-order logic, a considerable body of mathematics must be called upon (Shapiro [1991], pp. 45–7, pp. 80–8, pp. 134–69).
Expressive power: the capacity of second-order languages to express a great deal of mathematics reveals the underlying mathematical character of second-order logic (Shapiro [1991], pp. 97–133, p. 194).
Ontological commitment: second-order logic involves the 'outright assumption of sets the way [set-theory] does' (Quine [1969], p. 258, [1970], p. 68).

The neo-logicist dismisses semantical considerations on the grounds that 'it is perfectly clear that no familiarity with the notion of class is prerequisite for

[33] Quine ([1970], p. 67) also argues against second-order logic on the grounds that the items in the range of second-order quantifiers lack adequate identity criteria. This concern may be obviated (for present purposes) by restricting the range of the relevant quantifiers to extensionally individuated items (such as Fregean concepts) (Shapiro [1991], pp. 16–7, p. 194).

an understanding of higher-order quantification occurring in natural language' (Wright [1983], pp. 132–3). One can appreciate the cogency of the inference from 'John is a parent of Philip', 'Mary is a parent of Philip' to 'John and Mary have something in common' without knowing anything about classes. A semantic reconstruction of ordinary discourse ought to respect this fact and not allow extraneous mathematical concepts to intrude upon a theoretical description of our second-order logical understanding.

This argument is far from conclusive. To begin with it appears too strong. For it might equally be doubted whether ordinary language users are familiar with the concepts of any higher-order entities (whether they be classes, the neo-logicist-favoured Fregean concepts or some other sort of item). Indeed, it might be reasonably claimed, ordinary language users understand second-order locutions in some non-objectual manner. Moreover, an opponent might reject the assumption that the task of semantic theory is to provide a theoretical description of speaker understanding. It may also be doubted whether the example provided is a genuinely second-order inference. If the inference is second-order logical then its conclusion ('John and Mary have something in common') is a logical truth. For, according to second-order logic, any two individuals have a property in common (i.e. $(\exists X)(Xj \;\&\; Xm)$). But it is debatable (at the very least) whether the conclusion would be deemed logical in a natural language setting (check your own intuitions).

There are, however, nearby arguments more suited to neo-logicist purposes. First-order logic also has a semantic theory, which will have to be specified with the aid of set-theoretic machinery. It hardly follows that first-order logic is just set theory in disguise! So why should second-order logic be deemed mathematical because it has a set-theoretical semantics? In response it may be argued that the semantics of second-order logic appears far more *intimately* bound up with mathematics. The extension of second-order logical truth turns upon independent propositions in set theory (for example, the continuum hypothesis). By contrast, independent set-theoretic propositions have no bearing on the extension of first-order logical truth.

Considerations of this sort are no more than suggestive. To begin with it should not be forgotten that the extension of first-order logical truth appears to stand in no less a significant relation to another set-theoretic principle, the axiom of infinity. More importantly, there is a distinction to be drawn between the tools one employs to investigate a given subject matter and the nature of the subject matter itself. One cannot immediately conclude from the fact that one has to employ tools of such and such a sort that the subject matter itself concerns items of that sort. The fact that sophisticated set-theoretic machinery is required to characterise second-order logic does not establish that second-order logic *is* sophisticated set theory. Nevertheless, these arguments do suggest this. It would be perplexing indeed if the structural liaisons (to use a neutral term) between second-order logic and set theory (absent in the case of first-order logic) failed to reflect something significant about the underlying nature of higher-order languages.

The neo-logicist might grant this last point and seek to undermine his opponent in a different way. The ability to grasp a given range of inference patterns need not rely upon the prior grasp of a semantic theory. After all, we commonly engage in argument routines for which there is no extant semantic theory, and continue to argue effectively even when ignorant of existing ones. More generally, the legitimate employment of logical inferences to transfer warrant from one statement to another cannot rely upon a prior grasp of even a rudimentary semantic theory. For a grasp of that theory will also presuppose a facility with logic that cannot—on pain of regress or circularity—rely upon the understanding of further semantic theory. So even if the appropriate semantic theory for second-order logic is set-theoretic, it does not follow that someone who employs a specific range of second-order inference patterns must have a prior acquaintance with set theory.

We here touch upon fundamental issues concerning the acquisition and employment of rules of inference. Two distinct issues require to be separated:

A. Does the thinker who employs a range of inference patterns require **explicit knowledge of a semantic theory** to transfer warrants from premises to conclusions?

B. Does the thinker who employs a range of inference patterns need an **explicit grasp of the logical rules** that licence these arguments?

It is certainly true that a theorist whose aim is to self-consciously elucidate the character of the consequence relation in question will have to explicitly articulate the logical rules in question and develop a semantic theory for them. But, by contrast, it appears that a thinker who reasons with—rather than theorises about—these rules need have no explicit knowledge of a semantic theory before she can legitimately employ them to transfer warrants from premises to conclusions. Arguably is it not even true in general that she needs any explicit grasp of the rules that license these arguments (Boghossian [forthcoming]; Wright [forthcoming b]). Once again, in order to avoid circularity or regress, it appears that the thinker must be allowed to acquire and employ rules of inference without presupposing a prior conception of the rules in play.

The opponent of neo-logicism has, however, ready responses available. First, it does not follow that the abilities displayed by speakers in the deployment of these specific inferences suffice for the sort of *systematic* understanding of second-order logical inference patterns that may be required for a grasp of sophisticated second-order arguments. It may be that it is only through the development of a set-theoretic semantics for second-order logic that anyone can arrive at a sufficiently self-conscious understanding. Second, second-order logic is incomplete. Consequently, we cannot recursively enumerate the second-order logical truths. So it is only through the development of a set-theoretic semantics that anyone can arrive at a *comprehensive* grasp of second-order logic (cf. Quine [1970], p. 42; Wagner [1987]).

These responses make evident a tactical error on the part of the neo-logicist. He need make no claim (such as (2OL)) concerning the general character of second-order logic. For the neo-logicist, reconstruction of arithmetic does not rely upon a comprehensive grasp of second-order logic. The reconstruction draws only upon the recursively enumerable fragment relevant to the derivation of Frege's theorem. Moreover, the neo-logicist may allow that a systematic understanding of second-order logic requires the exercise of mathematical concepts whilst nevertheless maintaining that a mathematical novice might follow the proof of Frege's theorem even when unable to explicitly formulate or theorise about the specific rules employed. In a similar spirit, the neo-logicist may grant that the semantics of second-order logic in general is bound up with set theory. But then the neo-logicist may also deny that the semantics of the fragment upon which he relies is objectionably mathematical. Of course, it remains to be established that the second-order logical rules required for the derivation of Frege's theorem exhibit these welcome features and can be grasped independently of a fuller appreciation of second-order logic.

The neo-logicist may deal more cursorily with arguments that move from the expressive power of second-order logic to its underlying mathematical character. It is true that many mathematical notions (minimal closure, well-ordering, well-foundedness . . .) cannot be defined using only first-order logical resources. But it is one thing to be able to express a mathematical claim employing these notions in a second-order language. It is quite another thing to be logically obliged to endorse such claims. Until it is shown that second-order expressible mathematical claims fall amongst the class of second-order logical truths, it remains to be established that the employment of second-order inference patterns draws upon distinctively mathematical expertise.

It may be responded that there is an effective function from true claims about sets of real numbers to the class of second-order logical truths. More generally, the same holds for just about any mathematical structure (short of set theory) (Shapiro [1991], p. 82, corollary 4.9 generalised in the obvious way to any categorical structure). So second-order consequence is a very rich mathematical notion, and assuming that we have a grasp of second-order consequence is tantamount to saying that we understand truth in just about any branch of mathematics. Once again, the neo-logicist may judiciously grant the general point. He may admit that it is only through mapping out the relations between second-order logical truth and bodies of mathematical propositions that the fine-grained structure of second-order consequence may be appreciated. But the neo-logicist may still question whether an understanding of the fragment of second-order logic which subserves Frege's theorem must rely upon a prior grasp of these relations.

According to Quine, the mathematical presuppositions of second-order logic are revealed in its commitment to the 'staggering existential assumptions' of set theory. It is now commonly accepted that Quine's claims were overblown (Boolos [1975], pp. 43–8; Shapiro [1991], p. 21). Second-order quantifiers range over the power set of the sub-classes (or properties) of their associated first-order domain.[34] Since the power set of every first-order domain includes the empty set (property), it is a second-order logical truth that the empty set (or property) exists. Yet the invariable existential commitment of second-order languages to this single existent cannot be counted staggering. More significantly, second-order quantification always generates *additional* ontology (the members of the power set of whatever the first-order domain might be). Assume, for example, that the first-order quantifiers of a theory range over the natural numbers. Then the second order quantifiers will range over all the sets (or properties) of natural numbers. But, as the example shows, the existential assumptions made may still fall far short of set theory (which includes sets of sets of natural numbers and so on).

The existential assumptions of second-order logic may not be staggering. Still the legitimate question remains: how can we know that there are even as many additional items (over and above whatever objects might fall within the first-order domain) as second-order logic commits us?[35] Is not that commitment staggering enough?

Several avenues of response are open to the neo-logicist. The neo-logicist may seek to assuage concerns about the ontological commitments of second-order logic. If language and reality are as intimately related as the neo-fregean conception of ontology conceives, then the task of establishing whether a range of properties exists does not involve some unblinkered

[34] Boolos ([1984], [1985]) argues for the opposing view, that second-order quantifiers should be understood as implicitly plural locutions that bring no additional ontology in their wake. See Hossack ([2000]) for discussion of the general issues raised. Shapiro & Weir ([2000], pp. 171–9) sceptically discuss the suggestion that neo-logicism adopt a plural semantics for second-order quantifiers.

[35] See Shapiro & Weir ([2000], pp. 160–71) for further discussion and a detailed investigation of the effects of different second-order existential assumptions upon the derivation of Frege's theorem.

inspection of an abstract realm. Rather, the task becomes inseparable from establishing whether a corresponding range of predicates feature in our language. But (applying the syntactic priority thesis (SP1–3) to predicates and properties) we can only gain assurance that the relevant items exist if we know that the predicates that refer to them figure in true atomic sentences. Unfortunately, the relevant class of properties include ones that have no instances (most crucially, the property of non-self-identity), and that consequently do not figure in true atomic sentences. This means that the neo-logicist must relax the strictures of neo-fregeanism and allow that the relevant type of predicate refer to a property irrespective of its occurrence in such sentences. It is then incumbent upon the neo-logicist to provide some principled use-theoretic justification for this manoeuvre (of the kind assayed at the close of Section 4). In effect it would have to be established that (*modulo* obvious restrictions) the provision of a sense for a predicate *eo ipso* supplies a reference, a linkage between sense and reference that does not obtain in the case of names and objects.

The neo-logicist may also deny that any assurance that properties exist is required in advance of the legitimate employment of second-order logical rules. It is a predicament—familiar enough from the propositional case—that we cannot explicitly legitimate the employment of fundamental logic laws without being caught up in some sort of justificatory circle (Boghossian [2000]). Ultimately we can only come to an understanding and appreciation of logical rules by acquiescing in the practice of their application. Why should a more discursive elucidation of our logical practices be expected in this context just because ontological commitments are generated that may be absent from the propositional case? Indeed if we are willing to construe logical operators as denoting logical *functions*, then ontological commitment may be present even there.

There remains a further, distinct strategy for the neo-logicist to pursue (Hale & Wright [2001a], pp. 430–3). Our discussion so far has laboured under the assumption that the employment of quantifiers—second-order or otherwise—inevitably harbours ontological commitment. However, this assumption may be questioned (Sellars [1960]; Prior [1971], pp. 31–47; van Cleve [1994], pp. 586–9; Rayo & Yablo [2001]). According to the assumption, the semantic function of the variables bound by a quantifier is to range over an associated domain of entities. But according to an alternative conception (hereby dubbed 'neutralism') the function of a quantified variable is not to assert the existence of any entity but rather to speak *generally*. Understood in this neutral way, bound variables merely serve to generalise over whatever the semantic function of the atomic expressions they replace may be. So, contrary to Quine, there is no reason to think that second-order quantification—considered merely as a device of generality—is committing. Rather, second-order quantification is committing only if the predicate expressions which bound second-order variables replace are already entity-invoking. Consequently, there can be no special problem that confronts second-order quantification that does not confront the sort of predicative constructions already presupposed by first-order logic.

The neutralist strategy clearly stands in need of clarification and development if it is to be used to support the neo-logicist programme. First, it needs to be established that the neutral semantics suggested for second-order quantifiers (of the sort deployed in the proof of Frege's theorem) really provides an effective explanation of their behaviour. One possible development of neutralism would provide a natural deductive treatment of the higher-order quantifiers—the idea that it is the introduction and elimination rules that fix and explain their meaning (cf. Prawitz [1965], pp. 63–73). Second, it needs to be established that generalising over the semantic function of atomic predicates does not incur commitments over and above any harboured by the occurrence of atomic predicates themselves (contrary to Quine [1953], pp. 121–3). This result might be achieved by conceiving of the commitments of second-order quantification in relation not only to the predicates that belong to an actual language but also to those of any possible extension of it.

The neo-logicist position on the status of second-order logic has evolved over the course of this discussion. The original (2OL) may now be replaced with the modified principle:

(2OL*) *Epistemic Innocence*: Knowledge of the fragment of second-order logic needed to derive Frege's Theorem may be acquired a) *a priori* and b) independently of a prior grasp of mathematics.

8 Bad company objections

Neo-logicism claims that the method of abstraction may be employed to introduce the concept *number*. If the method is legitimate, then it should be possible to employ it generally. This lays neo-logicism open to *reductio*. Suppose that the general employment of the method of abstraction results in the introduction of concepts and associated objects of which we would otherwise be wary. Or, alternatively, suppose that the stipulation of abstraction principles turns out to conflict with commitments we have already made. Then it appears that the method of abstraction cannot (in general) be a legitimate means of concept introduction. Consequently, there can be no assurance that (in particular) the method of abstraction may be successfully employed to introduce the concept *number*.

Objections of this type (so-called 'Bad Company' objections) have developed through a sequence of evolutionary stages. There are two strategies the neo-logicist may adopt to deal with these objections. First, he may deny that the consequences of applying the method of abstraction in a given case are absurd. Second, he may provide a more discerning characterisation of the method that indicates why it cannot be used to generate any absurd (putative) consequences but can be used to introduce the concept *number*.

The use of the first strategy is exhibited in the neo-logicist response to one of the earliest examples of a 'Bad Company' objection (Field [1984], pp. 168–9, [1993], pp. 286–7). Likening the method of abstraction to the ontological argument for the existence of God, Field proposed, by way of *reductio*, that the concept *God* be introduced by stipulating the truth of the following abstraction principle:

(G) The God of $x =$ God of y iff x and y are spatio-temporally related.

Since there are true right-hand-side instances of (G) it follows that there are also true left-hand-side instances. By applying the rule of existential generalisation it follows that God exists! Of course, it is absurd to think we might prove the existence of God by such a route (witness the travails of the ontological argument). So there must be a fault in the method of abstraction, a method that accordingly cannot be relied upon to introduce numbers. Counterarguments of this sort rapidly proliferate. A proof of substantivalism based upon the following abstraction is hardly any more tempting:

(S) The place of $x =$ the place of y iff x and y share share their spatial properties.

The neo-logicist has a ready response (Wright [1990], pp. 163–4; Hale [1994b], pp. 199–200; Hale & Wright [1994]; Bird [1997], 353–6). The meanings of the novel operators ('the God of x', 'the place of y') are impoverished. The abstraction principles that introduce them do not invest these expressions with the sorts of significance usually associated with the English expressions that share their orthographic type. So the existence of items characterised by the introduced expressions is not as absurd as it initially appears. For example, (G) fails to determine that the objects introduced have any of the familiar properties (omniscience, omnipotence etc.) of a deity. As a result (G) fails to reveal the existence of such a deity. So it need not seem absurd—as Field supposes—that there are entities of the sort (G) describes (indeed the God introduced may be none other than the familiar physical universe). Similar remarks apply to (S): it fails to introduce empty 'places' and so hardly provides a ground for affirming substantivalism.

The neo-logicist employs the second strategy to deal with an array of 'Bad Company' objections that deploy second-order abstraction principles. The most elementary (although perhaps most fundamental) of these objections simply reminds us of the historical facts (Section 2). If we introduce the concept *extension* by stipulating that the following abstraction obtains then contradiction will result (Field [1984], p. 158; Boolos [1990], p. 214; Dummett [1991], pp. 188–9; Potter & Smiley [2001], pp. 334–6):

(V) $(\forall F)(\forall G)[(\mathrm{Ext}: Fx = \mathrm{Ext}: Gx) \leftrightarrow (\forall x)(Fx \leftrightarrow Gx)]$

In response, the neo-logicist claimed that the method of abstraction should only introduce concepts with the use of *consistent* abstractions (Wright [1997], pp. 281–2). Since (HP) is equiconsistent with second-order arithmetic (Section 2) the neo-logicist may (with some considerable confidence) continue to affirm that numerical expressions may be introduced by its stipulation. But, as Heck demonstrated, this response does not cut nearly deep enough (Heck [1992]; Boolos [1993], pp. 231–3). Take any consistent second-order sentence Φ. Then there is an equivalent (and so consistent) abstraction of the form:

(H) $(\forall F)(\forall G)\ [(\#F = \#G) \leftrightarrow \Phi\ \mathrm{v}(\forall Fx \leftrightarrow Gx)]$

We are therefore free to introduce the concept # by stipulating that (H) is true. But if (H) is true, then Φ is true too. The trouble is not only that Φ may make claims that cannot plausibly be *a priori* (e.g. there are only 6 objects). The trouble is also that by stipulating different abstractions of type (H) for different values of Φ, pair-wise inconsistent results are achieved (e.g. where $\Phi_1 =$ the universe is finite and $\Phi_2 =$ the universe is infinite). How, then, can the method of abstraction be thought to furnish a legitimate means of concept introduction?

The neo-logicist responds by questioning the legitimacy of (H)-type abstractions (Wright [1997], pp. 282–6). According to Wright, 'it's natural to feel that the troublesome abstractive relations are a kind of cheat: that they merely point up the need to restrict abstraction relations which are, as it were, *real* relations, rather than artefacts of language.' However, in a neo-fregean setting where the notions of property and predicate are so intimately tied, it is difficult to see how a relevant and principled distinction could be drawn between real properties and *mere* linguistic artefacts.

There is no need to pursue this issue here, for Boolos has demonstrated that there are consistent abstraction principles of a different form that are nevertheless incompatible with (HP) (Boolos [1990], pp. 214–5, [1997], pp. 311–2; see Wright [1997], pp. 288–91 for the related 'Nuisance Principle'). Let us say that the concepts F and G *differ evenly* if the number of objects falling under F but not G or under G but not F is even (and finite). Then the concept *parity* may be introduced by the stipulation of the abstraction:

(P) $(\forall F)(\forall G)$ [(Parity (F) = Parity (G)) $\leftrightarrow$ (F & G differ evenly)]

But whereas (HP) can only be satisfied in an infinite domain, (P) can only be satisfied in a finite domain.

The neo-logicist counters by claiming that concepts may only be legitimately introduced by *conservative* abstractions (Wright [1997], pp. 293–7, [1999], pp. 318–20; cf. Field [1980], p. 12). The mere introduction of a concept should have no consequences for items that fall under distinct concepts. So if an abstraction principle merely introduces a concept, then the addition of that principle to an existing theory should not result in any new theorems about the old ontology (prior to the addition of the abstraction). (P) fails to be conservative (in this sense).[36] Add (P) to any theory (T). It follows that there are only finitely many items of any kind that belong to the ontology of (T) regardless of whether they fall under the concept *parity*. By contrast, (HP) is conservative (*modulo* the assumption of the Axiom of Choice in the meta-theory—an assumption that clearly demands greater scrutiny). The addition of (HP) to a theory (T) results in the consequence that infinitely many items belong to the ontology of (T). But (HP) requires only that there are infinitely many numbers and has no consequences for the size of the extensions of other concepts.

A further ingenious twist in the debate reveals that this restriction to conservative abstractions is still unsatisfactory. Shapiro and Weir ask us to consider abstractions of the type (Shapiro & Weir [1999], pp. 319–20):

(D) $(\forall F)(\forall G)\,[(\Sigma(F) = \Sigma(G)) \leftrightarrow ((\phi F \;\&\; \phi G) \vee (\forall x)(Fx \leftrightarrow Gx))]$

The abstractions that result from substituting 'ϕ' by 'is the size of the universe and some limit inaccessible' and 'is the size of the universe and some successor inaccessible' are conservative. Unlike (P) they do not entail that there is any upper limit on the size of the universe. But one entails that the universe is limit-inaccessible size whereas the other entails that it is successor-inaccessible size. So it is not possible for both abstractions to be true. Further constraints on the legitimate employment of abstraction principles are required. The neo-logicist suggests an additional *modesty* constraint—roughly, the idea that abstractions should only carry implications for the

[36] See Fine ([1998], pp. 626–7), Shapiro & Weir ([1999], pp. 296–8), and Wright ([1999], pp. 319–20) for further refinements on the notion of conservativeness.

number of items they introduce (the Σs) rather than undertaking a commitment to the size of the universe that does not originate in any requirement it imposes on the Σs (Wright [1999], pp. 323–30; Weir [unpublished] provides a sustained treatment of the issue).

The debate will no doubt continue. For present purposes, it is now time to step back and take stock.[37] At least four (*prima facie*) distinct ranges of issues need to be distinguished:

Logical: what are the formal conditions (consistency, conservativeness, modesty . . .) that an abstraction must meet to be an eligible vehicle of concept introduction?
Justificatory: is there some underlying justification for grouping together these conditions or are they simply put together in ad hoc response to counter-examples?
Epistemological: how can we know (and to what extent are we required to know) that an abstraction satisfies these formal conditions?
Explanatory: do abstraction principles provide an adequate explanation of the concepts they introduce?

The recurrence of counter-examples in the literature shows that the task of providing a list of formal conditions for the eligibility of an abstraction is not straightforward. Nevertheless, let us suppose that a list of conditions has been identified, an integrated body of constraints that may be motivated by the underlying brief of abstraction principles to introduce novel concepts. Is it now legitimate to employ (HP) to introduce the concept *number*?

The answer to this question will depend upon the extent to which knowledge that the relevant constraints are satisfied is required in advance of utilising the abstraction. According to neo-logicism, (HP) may be employed to introduce a mathematically ignorant subject ('Hero') to number theory. Since abstractions may be stipulated that fail to meet the formal eligibility conditions, it appears that Hero must confirm that (HP) satisfies these conditions before proceeding to derive the laws of arithmetic. If Hero cannot confirm the cogency of (HP), it is difficult to see how (HP) can provide (part of) a foundation for arithmetic. However, Hero is not in a position to demonstrate even the consistency of (HP). Gödel's second incompleteness

[37] There are evident analogies between this debate and the attempt to introduce logical constants by introduction and elimination rules. As Prior pointed out, if no constraints are imposed upon the stipulation of such rules, then the bad company connective 'tonk' may be introduced with the introduction rule for 'v' and the elimination rule for '&' (Prior [1960]). These rules allow that any arbitrary sentence whatsoever may be proved. So the attempt to introduce logical constants by defining their inferential roles appears to be bankrupt. Belnap responded by insisting that a connective may only be introduced by rules that are consistent and conservative ('tonk' fails in both regards) (Belnap [1962]). In similar spirit we may think of (V) as providing introduction and elimination rules for the spurious tonk-like functor 'Ext:Φx' and the neo-logicist responding Belnap-wise.

theorem shows that the consistency of a theory cannot be proved within the theory itself but only by appeal to a stronger system. Since Hero has no mathematical theory stronger than the system generated by (HP) and second-order logic, it follows that he cannot prove (HP) to be consistent.

Of course, those of us who already understand analysis may reassure ourselves of the cogency of (HP) by reflecting upon the equiconsistency of (HP) and analysis (Section 2). But this still leaves open the possibility that both (HP) and analysis are inconsistent, a possibility that is not ruled out by the usual (circular) means whereby the consistency of the analysis is proven (Boolos [1997], p. 313; see Wright [1999], pp. 312–3 for a more upbeat view). Moreover, Hero cannot share even in these inconclusive reflections because he is not yet in a position to appreciate analysis. The neo-logicist predicament here echoes the travails of Hilbert's programme. Hilbert thought to banish foundationalist concerns by providing a purely formal system of finitary arithmetic. But the second incompleteness theorem shows that the formalism developed by Hilbert cannot assure us of its own consistency, and the foundational status of Hilbert's formalism was thereby called into question (Gödel [1958]).

In fact, the neo-logicist appears to encounter a dilemma. Hero's justification (if he has any) for supposing (HP) to be cogent is—to put the matter crudely—either internally (cognitively) accessible or epistemologically external and beyond his cognitive ken. The warrant cannot be demonstrative and it is scarcely credible to credit Hero with immediate insight into the cogency of (HP). Therefore Hero's warrant cannot be internal and so must be external. But it is implausible to suppose that warrants for logical principles are external. The mere fact that a subject entertains and reasons from a hypothesis that happens to be consistent is not a sufficient basis for crediting the subject with warrant for the assumption that it is consistent. To suppose otherwise is to risk failing to distinguish mere conviction from knowledge and divorcing logic and mathematics from the proof-theoretic techniques that sustain them.

The neo-logicist does not address this dilemma but does offer an epistemological account that promises to navigate between the extremes of the simple forms of internalism and externalism indicated (Wright [1997], pp. 286–8, [1998], pp. 325–6; cf. Wright [forthcoming]). The neo-logicist points out that it is not only the method of abstraction but also other techniques of concept formation that are liable upon occasion to misfire. For example, the practice of introducing predicates by defining their satisfaction conditions works perfectly well in the usual run of cases. But this same method can also be used to introduce predicates that result in paradox, for example, the heterological paradox. We do not therefore doubt the legitimacy of introducing predicates by this procedure any more than we doubt the deliverance of our senses because they sometimes deceive us. Some general distinction needs to be drawn between proper and improper uses of the technique, but where the technique results in 'what appear to be perfectly innocent examples' there is no reason to doubt the cogency of the predicates defined. The neo-logicist counsels that we treat concepts introduced by abstraction in the same way. Since (HP) appears a perfectly innocent example of an abstraction principle, Hero may be credited with a species of default warrant for the cogency of (HP).

The epistemology gleaned from the neo-logicist's few remarks is evidently in a nascent stage. Still some definite doubts may be raised. To begin with, one might doubt whether like is really being compared with like. The introduction of predicates does sometimes go awry. Nevertheless, the practice of so introducing predicates is deeply entrenched and, it might be claimed, the class of mischievous examples is relatively small. By contrast, the practice of introducing concepts by abstraction is not nearly so pervasive, and the class of troublemakers constitutes a far greater proportion of the significant cases. As a consequence, one is only being epistemically responsible when one treats abstractions with suspicion. One might also doubt whether there really is a genuine epistemological contrast to be drawn between the default or defeasible *a priori* warrant for the cogency of (HP) with which the neo-logicist credits Hero and the inductive warrant for a principle concerning an abstract subject matter. After all, the only grounds that Hero appears to have for supposing (HP) to be innocent is that investigation of (HP) has failed *so far* to reveal an inconsistency (the existence or non-existence of a proof).

The epistemological issues raised by the employment of abstraction principles connect with the general issues concerning the justification of logical knowledge (Section 6). In both cases (it appears) we are beggared to provide a discursive elucidation and defence of the principles employed. Instead, we are compelled to rely upon the legitimacy of the practice in which we engage. It may be that when a more advanced state of understanding is achieved of the nature of logical justification we will have attained a perspective from which this situation may be differently described or else seems less theoretically unsatisfying. Until we have attained such a state, we may be unable to accurately assess the epistemological status of abstraction principles.

Explanatory versions of the Bad Company objection raise a different (albeit related) issue: do abstraction principles really explain what the concepts they introduce mean? Here, in broad outlines, is Dummett's development of the concern (Dummett [1991], pp. 188–9, p. 208, pp. 217–22, p. 226, pp. 232–3, [1998], pp. 369–81). (V) is demonstrably inconsistent. Yet the failing of this abstraction principle is not brute. Rather it reflects an underlying malady in the construction of (V). The malady in question is impredicativity: the fact that the extensions purportedly introduced by (V) fall within the range of the first-order quantifier that figures on its right-hand side. (HP) also suffers from impredicativity. In order to prove that there are infinitely many numbers, it needs to be presupposed that the numbers introduced on the left-hand side fall within the range of the first-order quantifiers that figure on the right-hand side of (HP).[38] Now recall the task assigned to abstraction principles. They are intended to take a novice from a familiar language to an extended language that broaches reference to a novel kind of object. But because (HP) is impredicative the novice must already understand that the quantifiers from his original language—the quantifiers on the right-hand side of (HP)—range over the entities that the abstraction principle seeks to introduce. So it appears he must already be familiar with the concept *number*. As a result the process of explaining to the novice what the concept *number* means is inevitably circular. So although (HP) may be consistent, it is just as flawed as (V) *qua* explanation of what a concept means.

This line of reasoning rests upon the assumption that a subject can only understand the right-hand side of (HP) if it is determined in advance that the range of the first-order quantifier includes numbers. However, the assumption that the range of a quantifier must be fixed in such a manner is of doubtful intelligibility (Hale [1994b], p. 209; Wright [1998a], p. 242). For to so circumscribe the domain of a given quantifier—to say *in general* what kinds of things the domain comprises—will depend upon the use of further quantifiers whose range will in turn need to be fixed. A regress then threatens. Moreover, it appears that we are capable of thinking thoughts of *unrestricted generality*, thoughts about all objects irrespective of their kind (Cartwright [1994]). For example, it is usual to think of first-order logic as detailing formulas that hold true of *every* domain. Dummett responds that there is a difference between recognising that a formula holds in any domain of whatever kind of object and recognising that it obtains in a domain of every kind of object (Dummett [1998], p. 379). However, it does not follow that first-order logic does not rest upon unrestrictedly general thought. Recognition that a formula holds of *any* domain appears to involve quantification over *every* domain—in other words, absolutely everything.

Let us grant, therefore, that one might intelligibly grasp a quantifier independently of a prior circumscription of the domain. This makes it possible for the novice to grasp the first-order quantifier on the right-hand side of (HP), whilst remaining ignorant of the fact that numbers occur within its range. The novice may then prove with the aid of (HP) that numbers exists and thereby recognise that the objects (numbers) so revealed always fell (albeit unbeknownst to him) within the range of his first-order quantifiers (Wright [1997], pp. 240–6).

[38] First-order quantifiers emerge on the right-hand side of (HP) when the notion of 1–1 correspondence is defined in logical terms: F 1–1 G iff $\exists R \forall \mathbf{x}\ ((Fx \rightarrow \exists! \mathbf{y}(Gy \ \&\ Rxy))\ \&\ (Gx \rightarrow \exists! y(Fy \ \&\ Ryx))$.

Of course, the characterisation of (HP) as a principle of unrestricted first-order generality does not put paid to all the concerns impredicativity raises for the neo-logicist reconstruction of arithmetic. The second-order logic required for the proof of Frege's theorem calls for quantification into the position of predicates that already contain bound occurrences of second-order variables. Moreover, provision must be made—in order to derive Frege's theorem—for the application of the numerical operator to predicates that already embed an occurrence of the numerical operator. This procedure results in the occurrence of numerical operators that cannot be eliminated by (HP). A further explanatory concern is thereby raised: can a novice really understand the content of these claims if (HP) says nothing about their significance?[39]

More generally, we are in a perplexing position. Impredicativity offends against our instinct for non-circular explanations and yet impredicative constructions are rife. Indeed if the Wittgensteinian critique of the Augustinian conception of language is to be taken seriously, then (in a sense) the meaning of every expression is determined impredicatively by some significant portion of the sentential contexts in which that word occurs (Section 4) (other considerations in favour of a holistic conception of language are pertinent here). Clearly we stand in need of a general schedule that details and differentiates between the various species of impredicativity. Until such a schedule is provided we may be unable to say with any authority whether the terms introduced by (HP) and second-order logic are impredicative in a stronger and more objectionable sense than the usual run of cases.

9 Conclusion

It is incontestable that any thoroughgoing defence of neo-logicism must deal with an encompassing range of some of the most fundamental questions in epistemology, metaphysics, philosophical logic and the philosophy of language. It provides an indication of the depth and originality of the underlying ideas that so much else must be understood, perhaps overturned, before we can reliably adjudicate upon the fortunes of the neo-logicist programme. We have every reason to welcome a philosophy of mathematics

[39] The neo-logicist seeks to meet this explanatory concern by assaying a ranking of uses of the numerical operator by degree of complexity. He then tries to show that a gasp of stages $1 \ldots n$ in the ranking provides for an understanding of stage $n+1$ (Wright [1983], pp. 135–6; Hale [1994a], pp. 210–3; Wright [1998a], pp. 246–55; Dummett [1998] pp. 381–4; Wright [1998b], pp. 265–8). Additional explanatory concerns are raised by Boolos ([1997], pp. 308–12), Parsons ([1997], pp. 270–1) and Demopoulos ([1998], pp. 497–500).

that demands such reflection upon habitual and ingrained assumptions, assumptions often shared amongst protagonists in a debate.

I regret to have been unable (for reasons of space) to address the specific issues raised by attempts to extend the neo-logicist programme beyond arithmetic to analysis and set theory.[40] In advance of the detailed development of such extensions, the following may be said: the success of the neo-logicist reconstruction of arithmetic need not depend on the success of any extension. It remains an open possibility—a possibility that requires an argument to foreclose—that whilst the epistemology of arithmetic may be susceptible to a neo-logicist analysis, other areas of mathematics may require a different species of epistemology. But unless the perplexities that surround the reconstruction of arithmetic are resolved, the provision of abstractions to introduce real numbers and sets will furnish little by way of philosophical illumination.

Indeed there are reasons for thinking that there is something distinctive about our knowledge of arithmetic. For, as Frege himself emphasised, arithmetic appears—by contrast to other branches of mathematics—to be entirely general in application. An understanding of arithmetic appears intimately bound up with the ordinary apparatus of individuation, an arguably constitutive feature of cognition.[41] It remains to be established what shape a completed epistemology must take to capture Frege's insight.

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Appendix 1

Axiom (V) states that the extension of one concept is identical to the extension of another if and only if those concepts are co-extensive:

[40] See Wright ([1997], pp. 298–306), ([forthcoming b]); Hale ([2000]), and Shapiro & Wright ([forthcoming]). Critical commentary is provided by Parsons ([1997]), Fine ([1998]), Shapiro & Weir ([1999]), Cook ([2002]), Shapiro ([forthcoming]) and Weir ([unpublished]).

[41] For such reasons Frege appears to have judged arithmetic to be bound up with the character of thought itself. 'We might say, indeed, almost in the well-known words: the reason's proper study is itself. In arithmetic we are not concerned with objects which we come to know as something alien from without through the realm of the senses, but with objects given directly to our reason and, as its nearest kin utterly transparent to it' (Frege [1953], §105).

(V) $(\forall F)(\forall G)\,[(\mathrm{Ext} : Fx = \mathrm{Ext} : Gx) \leftrightarrow (\forall x)(Fx \leftrightarrow Gx)]$

From (V) Frege derived (HP) and then the Peano Postulates. Tragically for Frege, this second attempt was soon to fail. For rather than being a logical truth, (V)—as Russell and Zermelo demonstrated—is contradictory. Define the term 'Russellian' as follows:

(R) an object x is Russellian if there is at least one concept F such that x is the extension of F and x is not F.

Consider the extension of the concept *being a philosopher*. The extension of that concept is not itself a philosopher and so—according to (R)—is Russellian. But what about the object—call it **r**—which is the extension of the concept *being Russellian*? Is **r** itself Russellian? On the basis of (V) it can then be demonstrated that **r** is Russellian iff **r** is not Russellian (Boolos [1998b], p. 150).

First, assume **r** is Russellian. Then, by (R), there is at least one concept F such that **r** is the extension of F ($\mathbf{r} = \mathrm{Ext{:}F}x$) and **r** is not F. **r** is also the extension of the concept *being Russellian* ($\mathbf{r} = \mathrm{Ext}$: Russellian x). So, by the transitivity of identity, the extension of F is identical to the extension of *being Russellian*. ($\mathrm{Ext{:}\,F}x = \mathrm{Ext}$: Russellian x). This last claim is an instance of the left-hand side of (V) that asserts identity amongst the extensions of concepts. By applying (V) to it, an instance of (V)'s right-hand side may be derived that asserts co-extensiveness amongst the concepts F and *being Russellian* ($(\forall x)(\mathrm{F}x \leftrightarrow$ Russellian $x)$). By assumption, **r** is Russellian. So it follows from the co-extensiveness of these concepts that **r** is F as well. Since (R) demands that **r** is not F and **r** has failed to comply with this condition, it follows that **r** is not Russellian. So if **r** is Russellian then **r** is not Russellian.

Second, assume **r** is not Russellian. Then—by (R)—there is no concept F such that **r** is the extension of F and **r** is not F. But **r** is the extension of the concept *being Russellian*. So **r** must be Russellian otherwise there will be a concept F (*being Russellian*) such that **r** is its extension and **r** is not F (i.e. not Russellian). So if **r** is not Russellian then **r** is Russellian. Thus **r** is Russellian iff **r** is not Russellian. Russell concluded that (V) is inconsistent.

(V)'s proof-theoretic failing—its capacity by simple rules of proof to give rise to Russell's Paradox—is complemented by an additional model-theoretic flaw (Boolos [1993], pp. 230–1). From the model-theoretic perspective, (V) imposes impossible demands on the size of the domain of objects that it is intended to characterise.

(V) is committed to the existence of a function from concepts to objects that assigns the same objects (extensions) to the same concepts iff those concepts are co-extensive. Contraposing on its left to right reading, (V) also says there is a function that assigns non-co-extensive concepts to distinct objects. And since (according to the model-theoretic identification of concepts with subsets) there are as many non-co-extensive concepts as there are subsets of a domain, (V) implies that there are at least as many objects in a domain as there are subsets of it. But this implication confutes Cantor's theorem. The set of all subsets of a domain is the power set of the objects that compose that domain. So if—as (V) asserts—there are at least as many objects in a domain as there are concepts of that domain, the cardinality of the set of objects in a domain is at least as great as the cardinality of its power set. But Cantor's theorem tells us that for any set X, the cardinality of X is always less than the cardinality of the power set of X. Therefore, the function to which (V) is committed cannot exist.

Appendix 2

Neo-logicism gains support from the fact that (HP) lacks the proof and model-theoretic failings of (V). Consider the following attempt to derive a Russell-style paradox from (HP) (Wright [1983], pp. 155–6, [1998a], pp. 346–7). The condition for numbers analogous to the condition laid down in (R) for extensions is:

(R*) x is Russellian* if there is at least one concept F such that x is the number belonging to the concept F and x is not F.

Take the object—call it **r***—that is the number of the concept *being Russellian**. If (HP) is flawed in the way that (V), is then it should be possible to determine that **r*** is Russellian* iff **r*** is not Russellian*.

Assume **r*** is Russellian*. Then—by (R*)—there is at least one concept F such that **r*** is the number of Fs (**r*** = Nx : Fx) and **r*** is not F. **r*** is also the number belonging to the concept *being Russellian** (**r*** = Nx : Russellian* x). So, by the transitivity of identity, the number belonging to the concept F is identical to the number belonging to the concept *being Russellian** (Nx : Fx = Nx : Russellian* x). This last claim is an instance of the left-hand side of (HP) that asserts identity amongst the numbers of concepts. By applying (HP) to it, an instance of (HP)'s right-hand side may be derived that says that the objects falling under the concepts F and *being Russellian** figure in a relation of one–one correspondence ((∀x)(Fx1 − 1 Russellian* x)).

At this point in Russell's proof (that if the extension **r** of the concept *being Russellian* is Russellian then **r** is not Russellian), an instance of (V)'s right-hand side was derived: the claim that the concepts F and *being Russellian* are co-extensive. Since by hypothesis **r** was Russellian, it followed by this co-extensiveness result that **r**—contrary to the demands expressed by (R)—fell under the concept F. It was therefore concluded that **r** was not Russellian. But no analogous reasoning is available concerning **r***. For establishing that there is a one–one correspondence between the objects that fall under the concept F and the objects that fall under the concept *being Russellian* does not also establish that the objects falling under the latter concept also fall under the former. Consequently, it cannot be inferred from the relevant instance of (HP)'s right-hand side that **r*** is F. Therefore, it cannot be shown that if **r*** is Russellian* it fails to satisfy the condition expressed by (R*) for being Russellian* (that **r*** is not F). The attempt to derive Russell's paradox from (HP) fails.

(HP) also exhibits model-theoretic differences from (V). Like (V), (HP) asserts that there exists a function from concepts to objects (Wright [1997], pp. 222–3; Boolos [1997], pp. 305–6). But, by contrast to the function (V) describes, this function, if it exists, assigns equinumerous concepts to the same objects and non-equinumerous concepts to distinct objects. Concepts divide up into classes of equinumerous concepts (where distinct concepts belong to the same class iff they are equinumerous). The function which (HP) asserts to exist therefore assigns distinct objects to the members of different classes of equinumerous concepts. So, if (HP) is true, there are at least as many objects as there are different classes of equinumerous concepts. Suppose (HP) is satisfied by a finite domain D of k elements. Then there are $k+1$ classes of equinumerous concepts under which the elements of D fall: the class of concepts with 1 instance, the class of concepts with 2 instances... the class of concepts with k instances, and, finally, the class of concepts with no instances. Since there are $k+1$ classes of non-equinumerous concepts and (HP) is satisfied by D, it follows that D must contain at least $k+1$ objects. *Ex hypothesi*, however, D contains only k objects. Therefore, (HP) cannot be satisfied by a domain containing only finitely many objects. Since where k is infinite $k=k+1$, (HP) can however be satisfied by an infinite domain. So rather than making the impossible demands that (V) imposes, (HP) requires only that the domain it characterises contain infinitely many objects.

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