Wittgenstein and Symbolic Mathematics

Sören Stenlund
Department of Philosophy
Uppsala University

Abstract

The notion of symbolic mathematics has its roots in the invention of the algebraic symbolism in the 17th century. Franciscus Vieta (1540 –1603), who made decisive contributions to this development, uses the word 'symbol' (lat. *symbolum*) in the sense that is relevant here. Symbolic mathematics is to be contrasted with mathematics as an ontological science, for instance as the science of quantity and magnitude, which was the prevailing view in ancient Greek mathematics and in the renaissance version of the Aristotelian and Euclidian heritage. The algebraic symbolism and techniques were decisive for the invention of the differential and integral calculus and of much of modern mathematics, but the ontological conception has still survived, in great tension with the symbolic conception. It has survived in particular through the influence of formal logic and modern mathematical logic. The purpose of this paper is to show that Wittgenstein's view of mathematics has much in common with the symbolic (non-ontological) view of mathematics. In particular, it is argued that Wittgenstein's symbolic conception of mathematics is the appropriate background for understanding Wittgenstein's critique of mathematical logic and its philosophical impact.

Keywords: symbolic, operation, Vieta, Aristotles, Euclid, Klein, Hilbert, Hertz, ontology, logic, prose, calculus, rigor.

The foundational status of formal logic.

Some of the most controversial statements made by Wittgenstein are remarks 46 and 48 in Part V of the RFM. In the latter, Wittgenstein says:

> 'Mathematical logic' has completely deformed the thinking of mathematicians and of philosophers, by setting up a superficial interpretation of the forms of our everyday language as an analysis of the structures of facts. Of course in this it has only continued to build on the Aristotelian logic. [1]

[1] In PR §§87-98 Wittgenstein offers more specific examples of what he means by 'superficial interpretation'.

And a few lines earlier, he has already stated why he takes mathematical logic to give a 'superficial interpretation' of the forms of our everyday language. He writes:

> The curse of the invasion of mathematics by mathematical logic is that now any proposition can be represented in a mathematical symbolism, and this makes us feel obliged to understand it. Although of course this method of writing is nothing but the translation of vague ordinary prose.[2]

Many logicians (and philosophers) tend to react to these remarks as though they were only the emotional reactions of someone who dislikes mathematical logic. But Wittgenstein hardly disliked mathematical logic *as mathematics*. *Principia Mathematica* as well as the propositional calculus and the predicate calculus are mathematical symbolisms; they can be seen as pure mathematical calculi on the same level as many other mathematical or algebraic calculi. What Wittgenstein is calling into question here is a certain use of mathematical logic. He is questioning *the foundational status* that mathematical logic acquired in the discussions of the foundations of mathematics from the turn of the twentieth century onward. This foundational position of formal logic, even with respect to language in general, is what Quine has in mind when he says in the introduction to his book *Mathematical Logic*, regarding the 'logical particles', that "They are basic to all discourse". (p. 2). We find another explicit statement of the foundational status of mathematical logic in Gödel's article on Russell's mathematical logic. Gödel says there that mathematical logic "is a science prior to all others, which contains the ideas and principles underlying all sciences".[3] The systems of mathematical logic are assigned the role of *universal* logical frameworks. But this is no longer a thesis that is motivated and argued for; it is rather an *attitude* about which there is tacit agreement among philosophical logicians.

With regard to the foundational status of formal logic, I think that Wittgenstein is right to suggest that modern mathematical logic continues to build on the Aristotelian logical doctrine as it is expounded in the *Organon*. The foundational status of Aristotelian logic was transferred to modern times, and in particular to modern mathematics, mainly through the influential renaissance reading of Euclid's *Elements* as an application of Aristotle's logical

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[2]By the "translation of vague ordinary prose" Wittgenstein means the reading of logical formulas that is based on the translation of "¬ A" as "not A" or "it is not the case that A", of "A & B" as "A and B", of "∃x.A(x)" as "there is an x such that A(x)", etc.

[3] Gödel (1964, p. 447).

doctrine, in particular the doctrine of demonstrative science (*episteme*) found in the *Posterior Analytics*.[4] Euclid's *Elements* was one of the most influential classical works, its authority having been compared to that of the *Bible*. Newton's *Principia* follows the format of Euclid's *Element*s, and its precepts were fundamental to Kant's *Critique of Pure Reason,* to mention only two examples. It is not until the latter half of the 19th century that Euclid's (and thereby also Aristotle's) authority regarding logic is questioned. But what modern logicians (such as Frege, for instance) called into question in Aristotle's logical doctrine were a number of specific features, such as the subject-predicate form of all propositions. The very idea that there *is* a logical doctrine with a foundational status was not questioned; rather, it was asserted anew in the name of mathematical logic.

What then is involved in the foundational status of formal logic? I would suggest the following things. First, formal logic is concerned with judgments or propositions originally connected with the *episteme* (scientific knowledge) of the Aristotelian *ontological* concept of a science. In this sense, a science and its propositions are always *about a certain independently given subject-matter*. The subject-matter of a theory does not originate with the theory and its propositions; the theory is a representation (a sort of copy) of the subject matter that already exists in physical nature. Since logic is taken to be basic to all scientific discourse, this ontological (or descriptive) conception of propositions applies to mathematics as much as to physics.

In modern mathematical logic, 'the independently given subject matter' are the objects or entities in the domain over which the bound variables range, and to which names in propositions *refer*. This is of course intimately connected to the view of all simple propositions as having the form of a function applied to one or more arguments, given the modern concept of function according to which the objects in the argument domain of a function are given logically prior to and independently of the function defined for that domain. (Remember that in Frege's ontology there are two categories: functions and objects (*Gegenstände*); numbers, for instance, are objects that numerals *denote*.)

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[4] See Mancuso /1996, Ch. 1 and 4.) Mancuso argues convincingly that "the Aristotelian epistemological framework was pervasive in the seventeenth century and very influential indeed in later centuries" (Mancuso 1996, p. 92). By the "Aristotelian epistemological framework" Mancuso means primarily what we find in Aristotle's *Posterior Analytics*.

To the foundational status of formal logic belongs also the idea that logic is a universal system that displays the form of propositions essential for their being true or false as well as for the form of the deductive relationships between propositions. This has been taken to mean in particular that a logically well-articulated proposition carries its meaning or logical content in and of itself as a proposition, regardless of its context of use. (This well-articulatedness of the logical content of propositions is what Wittgenstein is questioning when he says that it rests upon translation into "vague ordinary prose".)

The discussion in the philosophy of mathematics since the 1920's has taken place on the basis of this view of the nature of logic as if it were a well-established scientific fact that everyone taking part in the discussion is supposed to know. Doubts had been expressed earlier about the foundational significance of formal logic for mathematics by Poincaré and the early Brouwer, but their criticism was largely ignored and forgotten in the enthusiasm about the progress in mathematical logic that took place in the 1930's through the work of Gödel, Tarski, Church and others.

Let me conclude this introduction with a more specific example of what Wittgenstein means when he talks about 'superficial interpretation of the forms of our everyday language'. In PR § 96, he says that "Russell and Frege construe a concept as a sort of property of a thing. But it is very unnatural to construe the words 'man', 'tree', 'treatise', 'circle' as properties of a substratum." Suppose you have a projector that projects a black circle on a white screen, and suppose also that by pressing certain buttons you can change the color (but nothing else) of the circle, into red, blue etc. The colors can then be seen as properties of a substrate (properties of one and the same the circle). But suppose furthermore that by pressing other buttons you can change the shape of the projected object. By pressing a certain button the circle turns into an ellipse. Having pressed that button, it would not make sense to say that the projected circle now has the form of an ellipse. The circularity of the original circle is not an external property of a substrate like the colors, it is rather the *form* of the projected object, *the form in which it is given as what it is* (a circle and not an ellipse).

In mathematical logic, where all properties and relations are rendered as functions and thereby as external properties and relations, it is impossible to do justice to this difference between the form and a property of a substrate. A form cannot be described: it is shown or displayed, and there is a sense of generality connected with the notion of form which is not the generality expressed by a quantifier that binds a variable at the argument-place of a function. [5] This notion of form is of fundamental importance in his philosophy of mathematics because getting clear about the workings of *mathematical symbolism* is one of Wittgenstein's main concerns there. According to Wittgenstein, inaccurate symbolism is a main source of problems and confusions in the philosophy of mathematics.

A philosophical vocabulary

Against the background of the foundational status of mathematical logic, a philosophical vocabulary has been developed and has become quite well-established. Included in this vocabulary are the names of the main philosophical views and positions in the philosophy of mathematics: Platonism, realism, formalism, intuitionism, constructivism, finitism, conventionalism, verificationism, fictionalism, etc. And sometimes these positions are furnished with qualifications such as 'strict', 'strong', 'weak'. etc. It is also within the context of logical foundationalism that technical distinctions come into play, such as for instance, between 'reference', 'syntax', 'semantics', and other notions that are used to capture the nature of mathematics.

A characteristic feature of this vocabulary is that a view or position is often defined or explained in terms of other positions and the tensions between different views. A certain position in this vocabulary derives its meaning in relation to the other positions. The formalist position, for example, is often explained in terms of the views or methods of the Platonist or realist views that formalists reject. Thus for a finitist or formalist, the realist or Platonist views are false or mistaken, but the realist view would nevertheless seem to *make sense* for the formalist as a possible alternative view of mathematics.

There are also attempts to summarize Wittgenstein's philosophy of mathematics within this vocabulary. One example is the article "Wittgenstein's philosophy of mathematics" in the *Stanford Encyclopedia of Philosophy* (written by Victor Rodych). Before I say what I find

[5] In his Cambridge lectures of 1939, especially lecture XVII (Diamond 1976, p. 161), Wittgenstein develops in great detail a critique of Frege's and Russell's concept of number based on their failure to distinguish between a form and a property of a substrate.

problematic with that article, I want to state that I find it quite useful and helpful. The article tries to bring out crucial features of Wittgenstein's philosophy of mathematics, and the author connects his claims to relevant quotations from Wittgenstein's texts. He is unusually generous in providing references to other commentators on each topic, and he has a very comprehensive bibliography.

What I find most problematic about this article is that it makes Wittgenstein look like a dogmatic revisionist. There is a long list of views in the article that Wittgenstein is simply said to reject. There is no mention that Wittgenstein was expressly aware of the danger of dogmatism in the early 1930's, and worked diligently to avoid it. It was precisely due to this awareness that he insisted that his language-games be regarded as mere "objects of comparison". Moreover, Wittgenstein explicitly stated that a successful philosophical investigation of mathematics must leave mathematics as it is. (PI § 124). Already in his discussions with the Vienna Circle, he remarked

> It is a strange mistake of some mathematicians to believe that something inside mathematics might be dropped because of a critique of the foundations. Some mathematicians have the right instinct: once we have calculated something it cannot drop out and disappear![6]

What Wittgenstein has in mind here is obviously the critique of Brouwer and (the early) Hermann Weyl, who wanted to drop large parts of classical mathematics. In this particular controversy, Wittgenstein sides more with Hilbert. Wittgenstein continues this reflection by contrasting the revisionist attitude with his own critical approach:

> And in fact, what are caused to disappear by such a critique are names and allusions that occur in the calculus, hence what I wish to call 'prose'. It is very important to distinguish as strictly as possible between the calculus and this kind of prose.

I shall argue that this idea of "misleading prose" in mathematics is an important feature of Wittgenstein's philosophy of mathematics that has not been paid sufficient attention. It is particularly important in his questioning of the foundational status of mathematical logic, and in his critique of the extensional view in mathematics. I shall return to the importance of the

[6] McGuinness (1979, p. 149).

prose/calculus distinction, and argue that it connects Wittgenstein's view of mathematics with the rigorization movement in mathematics in the late nineteenth century.

One might think that Wittgenstein abandoned the idea of misleading prose and the calculus/ prose distinction when he began to talk about mathematical language-games in his later philosophy of mathematics. But that is not true, and he never meant to say that *all* use of verbal language in mathematics is misleading. The two remarks I quoted at the beginning of this paper were written in the years 1942-1944, and thus by the late Wittgenstein. And in the last part of RFM, written in 1944, we find him referring to the "curse of prose" (RFM VII, § 41).

There is one general remark in the *Stanford Encyclopedia* article which I want to mention, and it is the following:

> The core idea of Wittgenstein's formalism [in the philosophy of mathematics] from 1929 through 1944 is that mathematics is essentially syntactical, devoid of reference and semantics.

I think that there is *some* truth, a trifle, in this remark, but it is misleading since this use of the words 'formalism', 'syntactical', 'reference' and 'semantical' belong to the philosophical vocabulary which, as I noted earlier, is strongly conditioned by the foundational status of mathematical logic. (More about the problems with this use of philosophical terminology later.) As a matter of fact, I don't think that it is possible to articulate a fair account of Wittgenstein's philosophy of mathematics within this vocabulary. Wittgenstein's conception of mathematics is incompatible with the foundational status of mathematical logic. I think that this is the key to understanding Wittgenstein's remarks cited at the beginning of the article.

The symbolic point of view and the history of mathematics

Wittgenstein's conception of mathematics, already from the beginning in the *Tractatus*, has much in common with what has been called *symbolic mathematics*. I think that this is true of the early and middle as well as the late Wittgenstein. I am inclined to say that for Wittgenstein, the most authentic form of mathematics in modern times is symbolic mathematics.[7] The symbolic view of mathematics offers us a perspective from which the unity of Wittgenstein's philosophy of mathematics becomes apparent.

A clear manifestation of Wittgenstein's symbolic point of view is his claim that mathematical propositions are not 'real' propositions. According to Wittgenstein, they don't have a descriptive content; they do not describe real states of affairs. Already in the *Tractatus,* mathematical propositions were called "pseudo-propositions". And around the beginning of the 1940's, he expressed a symbolic, non-ontological conception of mathematics as follows:

> Let us remember that in mathematics we are convinced of *grammatical* propositions; so the expression, the result, of our being convinced is that we *accept a rule*.
>
> Nothing is more likely than that the verbal expression of the result of a mathematical proof is calculated to delude us with a myth.
>
> I am trying to say something like this: even if the proved mathematical proposition seems to point to a reality outside itself, still it is only the expression of acceptance of a new measure (of reality).
>
> Thus we take the constructability (provability) of this symbol (that is, of the mathematical proposition) as a sign that we are to transform symbols in such and such a way. [8]

This is something that cannot be made much sense of within a philosophical vocabulary based on the foundational status of mathematical logic, where all possible propositions *are* propositions in the ontological sense. They have a descriptive content and are *about* something in a 'referential sense'. There is no place in the vocabulary for propositions in another sense. The closest one can come to propositions in some non-ontological sense are propositions in the "nominalistic" sense that are merely about concrete signs, so-called 'syntactical propositions'. Thus within the philosophical vocabulary, one is forced to see Wittgenstein's philosophy of mathematics as some superficial kind of formalism in which mathematics has been deprived of all meaning (it is "purely syntactical, without semantics", according to the *Stanford Encyclopedia* article).

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[7] Here I am using the word 'authentic' in more or less the same sense as when we say that chemistry is an authentic natural science today, which alchemy is not.

[8] RFM III, §§26-27.

But as Wittgenstein said in the last quotation, if he is depriving mathematics of something in his critique of foundations, it is the (often misleading) prose that accompanies the mathematical calculi, and he is doing so in order to clarify the sense that mathematical notions have within the calculi.

So what is symbolic mathematics? – To answer this question, a historical perspective on the development of mathematics is necessary. The view of modern mathematics as symbolic mathematics has been suppressed and misrepresented through the self-understanding of modern mathematics that is influenced by at least three different tendencies. The first is the influence of the Aristotelian and Euclidian heritage already mentioned; the second is the influence of mathematical logic on the discussion about the foundations of mathematics in the twentieth century. But there is also a third tendency, more common among many professional mathematicians: the picture of mathematics as a "science of mathematical objects and states of affairs", bearing strong similarities with physics. The "abstract reality" that is the subject matter of mathematics in this view is similar to the nature of the natural sciences in that "it is as it is by itself"; the subject matter is something *given in advance*, just as there are processes and phenomena studied in physics that are given in advance of the emergence of science of physics. As to the character of the objects and states of affairs in this 'mathematical nature', one tends to rely on analogies with laws of nature in physics. An equation such as $5 + 7 = 12$ is seen as a law of nature about what happens (what number of objects you get) when you put together 5 objects with 7 other objects into one collection.[9] As the equation $E = mc^2$ expresses an eternal truth about physical nature, so do the true equations of algebra express eternal truth about 'mathematical nature'. They are taken to be eternal, not in the sense of timeless, but as having permanence, as being invariant over time. There is no coming into existence and passing away in this abstract 'mathematical nature'. Everything that is possible is already actualized and the mathematician is a discoverer, not an inventor.

One feature of mathematics that is neglected or marginalized in this view is the human aspect, that is, mathematics as a human activity. The absence of historical sensitivity is therefore also

[9] This view seems to be presupposed when Hilary Putnam discusses scientific principles undergoing change and speculates as follows: "But are we not in the same position with respect to a sentence like 'In the year 2010 scientists discovered that 7 electrons and 5 electrons sometimes make 13 electrons'? Or with respect to 'In the year 2010 scientists discovered that there are exceptions to 5 + 7 = 12 in quantum mechanics?' If this is right, and I think it is, […]" Putnam (1990, p. 254).

a common characteristic of the prevalent images of the nature mathematics. The history of mathematics is seen as *our heritage* in Grattan-Guinness' sense.[10] Grattan-Guinness contrasts "heritage" with history proper: the latter is concerned with "what actually happened in the past", whereas the former has to do with mathematics in the past only as different stages in the development of mathematics *towards modern contemporary mathematics*. This means, among other things, that there is an emphasis on modern mathematical notions, views and results when writing about our mathematical heritage. According to Grattan-Guinness, both kinds of historical study are legitimate as long as they are not conflated. But the conflation of history and heritage is largely what occurred in the history of ancient mathematics. Heritage has been presented as though it is history. This is no doubt connected with the fact that many classical texts in the history of mathematics have been written by professional mathematicians who have wanted to start out from an eternal conception of what *real* and *genuine* mathematics is. And, from this perspective, what is real and genuine mathematics if not our modern mathematics? As a consequence, ancient mathematical texts are read and understood from the points of view of modern mathematics. What cannot be made sense of and find support in modern mathematics tends to be ignored, dismissed or falsified.

There is historiographical discussion about this attitude. The historian of mathematics Sabatai Unguru, for instance, writes: "to read ancient mathematical texts with modern mathematics in mind is the safest method for misunderstanding the character of ancient mathematics […] "[11] Unguru's article "On the Need to Rewrite the History of Greek Mathematics", published in1975, ignited a heated debate. [12] Well-known mathematicians such as André Weil, B.L. Van der Waerden, and Hans Freudenthal took part in the controversy, claiming that the Greeks already possessed the basic elements modern algebra in the form of 'geometric algebra'. The Greeks were somehow in possession of the 'algebraic content', but they had not yet found the most advantageous form for expressing that content.

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[10] Grattan- Guinness (2004).

[11] Quoted from Kastanis and Thomaidis "The term 'Geomtrical Algebra', target of a contemporary epistemological debate."

[12] Unguru (1075). See Kastanis, N. and Thomaidis, Y. "The term "Geometrical Algebra", target of a contemporary epistemological debate", for a survey of this debate. http://users.auth.gr/~nioka/Files/GEOMALGE.pdf

With the heritage view of ancient Greek mathematics, the conceptual transformation that takes place in the seventeenth century is not seen. However, the invention of the algebraic symbolism and the symbolic point of view in the seventeenth century was a profound conceptual transformation in mathematics.

Vieta's Analytical Art

The notion of symbolic mathematics has its origin in the seventeenth century with Fransiscus Vieta's work *Isagoge or Introduction to the Analytical Art*. A penetrating account of the origin and nature of symbolic mathematics is found in the phenomenologist Jacob Klein's book *Greek Mathematical Thought and the Origin of Algebra*.[13] As Klein explains, it is difficult to understand symbolic mathematics without a historically sensitive perspective on the *transformation* of ancient Greek mathematics that took place in the seventeenth century. What Klein means by a 'historically sensitive' perspective is primarily that we cannot read classical mathematical texts with modern mathematics in mind. He expresses this feature of his historical approach as follows:

> […] most of the standard histories attempt to grasp Greek mathematics itself with the aid of modern symbolism, as if the latter were an altogether external "form" which may be tailored to any desirable "content". And even in the case of investigations intent upon a genuine understanding of Greek science, one finds that the enquiry starts out from a conceptual level which is, from the very beginning, and precisely with respect to the fundamental concepts, determined by modern modes of thought. To disengage ourselves as far as possible from these modes must be the first concern of our enterprise.[14]

That we have to do with a *transformation* and not just with what is usually called a historical development is important. Klein argues convincingly that it is only against the background of an understanding of crucial features of Greek mathematics that this transformation of mathematics becomes clearly visible. This is so because the influence of the viewpoint of modern mathematics in well-established accounts of Greek mathematics has blocked our understanding of Greek mathematical thinking. We tend to see the modern concept of number

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[13] Klein (1968). See also Hopkins (2011).

[14] Klein (1968, p. 5)

or magnitude, for instance, as the result of an almost continuous historical development of the Greek notions of number and magnitude up to the modern ones. But this idea of a ‘continuous development’ is highly questionable.

The modern symbolic concept of number is not a determinate *numbers of things of* a multitude, as the ancient Greek *arithmos.* Jacob Klein argues that the modern symbolic concept of number does not appear through a continuous development and extension of the ancient Greek concept. The latter belongs to ancient Greek non-symbolic, *ontological* science. The modern symbolic concept of number is an essentially different concept, or rather, introduces a new conceptual dimension, which was possible to articulate only in conjunction with the invention of the algebraic symbolism in the seventeenth century. According to Klein, the invention of the algebraic symbolism was an essential transformation in the sense that it provided *new techniques and operational practices as the basis for new conceptions*. But an effect of this conceptual transformation was also that the original Greek understanding of numbers was lost, which is why Klein claims that his ‘intentional’ historical investigation of Greek mathematics is necessary.

The Greek mathematician who seems to have come closest to the conceptual transformation in which the algebraic symbolism originated was *Diophantus of Alexandria* (third century AD, sometimes referred to as "the father of algebra"). He was the author of a series of books called *Arithmetica* that deal with solving (what we call) algebraic equations. But according to Klein, it is Franciscus Vieta, 1540 –1603, who develops the logical and mathematical consequences of Diophantus’ work, and who deserves to be called the ‘inventor’ of modern mathematics. An important step in Vieta’s work was his innovative use of letters as parameters in equations. And it is Vieta who introduces the word ‘symbol’ (lat. *symbolum*), and talks about the symbolic concept of number.[15]

Diophantus worked with an arithmetical calculus of determinate numbers. Vieta advances Diophantus’ problems by introducing a new *general analytic* or an *analytic art*, using not only determinate number but ‘species’ or ‘forms’ of number – a calculus of species or forms. In the articulation of these symbolic forms of number, the use of letters as variables is essential.

[15] Klein claims that the term “symbolum,” used for letter signs as well as for connective signs, originated with Vieta himself (Klein 1968, p. 276).

The words 'species' and 'form' alludes to the '*eidos*' of Greek philosophy, but it is important to understand how its sense is transformed in Vieta's 'analytic art'. This is how Klein explains the difference:

> […] the "being" of the species in Vieta, i.e. the "being" of the objects of "general analytic," is to be understood neither as independent in the Pythagorean or Platonic sense nor as attained "by abstraction" […] in the Aristotelian sense, but as *symbolic*. *The species are in themselves symbolic formations* – […]. *They are, therefore, comprehensible only within the language of symbolic formalism.* […] Therewith the most important tool of mathematical natural science, the "*formula*," first becomes possible, but above all, a new way of "understanding," inaccessible to ancient *episteme* is thus opened up.[16]

This new understanding is the symbolic point of view. The core of the conceptual transformation achieved by Vieta is the replacement of the concern with determinate numbers (of ancient non-symbolic arithmetic) with the identification of a number represented by its form. In a certain sense, one might say that in the symbolic conception, form becomes the content. But 'form' in this sense is not 'syntactical form' in the modern meta-mathematical sense. 'Form' in Vieta's sense is displayed in the operational practices.

It is important to understand that the new arithmetic-algebraic system of Vieta (as well as the *mathesis universalis* of Simon Stevin, Rene Descartes and John Wallis, who continued and completed Vieta's work) *are not new theories of arithmetic or new sciences of number* (in the Aristotelian sense of 'theory' and 'science'). They are primarily new *arts*, new practices, new methods and techniques for dealing with problems: problems not only in 'pure mathematics' but also in physics and astronomy. Vieta ends his work *Isagoge by* saying "Analytic art appropriates to itself by right the proud *problem of problems*, which is: *to leave no problem unsolved.*" Klein points out that Vieta, " in concentrating his reflections on procedure […] no longer differentiates between "theorems" and "problems", […] because he sees all theorems as problems". (Klein, p., 166)

In the fourth chapter of the *Isagoge,* Vieta lays down the rules for the operation with species or forms, as well as rules for the transformation of equations. *These rules create*, as Klein puts it, *the systematic context which "defines" the object to which they apply*. Klein calls this system of rules "the first modern *axiom system",* alluding to Hilbert's axiomatic method.

[16] Klein (1968, p. 175).

There is no talk here about logical demonstrations from postulates or first principles as in Aristotle and Euclid. The analytic art of Vieta is not seen primarily as a representation of a body of truths, or a body of knowledge of some subject-matter, but in the first instance as a system of methods and techniques for solving problems, which later develops into the analytic geometry of Descartes and the infinitesimal calculus of Leibnitz.

Wittgenstein's symbolic point of view

We can see striking similarities between Vieta's analytic art and essential features of Wittgenstein's views of mathematics, such as Wittgenstein's frequent emphasis on mathematics *as activity and as calculus*. (In the PR he makes the somewhat excessive claims that, "in mathematics, the signs themselves *do* mathematics, they don't describe it", and "You can't write mathematics [as you can write history]. You can only do it." PR, p. 186.) When Wittgenstein is working on the notion of proof in mathematics, his examples are often arithmetical or algebraic calculations, which many logicians and mathematicians would not want to call real proofs at all, since they require that a proof should be dressed up in prose. Wittgenstein is interested only in the mathematical core of proofs. Regarding Euclid's proofs, Wittgenstein would say that the core of the proof is often what is *shown* in the diagram, and that the verbal proof-rhetoric surrounding it can be disregarded.

I don't want to say that Wittgenstein was inspired by 17th-century mathematicians. He had surely become aware of the deep difference between ancient Greek mathematical thinking and modern mathematics through his reading of Spengler in the early 1930's. But the symbolic point of view is present already in the *Tractarian* conception of arithmetic and logic.[17] I think that the main source of inspiration for Wittgenstein were certain ideas issuing from the struggle for rigor in mathematics and theoretical physics in the latter half of the 19th century, in which the symbolic point of view was prominent. He was particularly influenced in this respect by Heinrich Hertz. Hertz' work on the mathematics of classical mechanics, in which Hertz showed how to deal with conceptual problems connected, for instance, with the notion of force of classical mechanics, was an important and influential contribution to the symbolic

[17] E.g. "My fundamental thought is that the 'logical constants' do not represent. That the logic of the facts does not represent." (TLP 4.0312).

point of view. I believe that Ernst Cassirer was right when he said: “Heinrich Hertz is the first modern scientist to have effected a decisive turn from the copy theory of physical knowledge to a purely symbolic theory.”[18] Hertz saw the scientific theory as the application of a symbolic system, which is an autonomous system, in its formal aspects, independent of the empirical phenomena it is used to explain.[19]

This feature of Hertz’ work deeply influenced Wittgenstein, not only the author of the *Tractatus,* but also the later Wittgenstein.[20] In the *Big Typescript,* Wittgenstein writes:

> In my way of doing philosophy, its whole aim is to give an expression such a form that certain disquietudes disappear. (Hertz) [21]

In his introduction to the Ogden translation of the *Tractatus*, Russell states emphatically that the basic and most fundamental topic of the *Tractatus* is “the principles of Symbolism” or “the conditions for *accurate* Symbolism”. This was something that Wittgenstein obviously had stressed in their private conversations. But there is no sign in Russell’s introduction that he understood this point. Russell’s introduction does not even mention the harsh critique of the theory of types found in the *Tractatus* which was based precisely on Wittgenstein’s ideas about an accurate symbolism. [22]

Mathematics as activity.

The perspective of mathematics as an *activity* is essential in the logic of symbolic mathematics, and this is also an important reason why Wittgenstein’s conception of mathematics is close to the symbolic conception. I think that the most *important* reason why

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[18] Cassirer (1957, p. 20)

[19] Hertz (1956, p. 8)

[20] This is argued in more detail in Stenlund (2012b). See also Kjaergaard (2002) for an argument for the importance of Hertz’ ideas for Wittgenstein’s philosophy of science as a whole.

[21]MS 213 (in the Bergen Electronic Editions of Wittgenstein’s works).

[22] In a letter to Russell dated August 19, 1919 Wittgenstein wrote: ”The theory of types, in my view, is a theory of correct symbolism: a simple symbol must not be used to express anything complex: more generally, a symbol must have the same structure as its meaning. That’s exactly what one can’t say. You cannot prescribe to a symbol what it *may* be used to express. All that a symbol *can* express, it *may* express.” (App. III in Wittgenstein 1998, p. 130.) Here we see how Wittgenstein’s notion of correct symbolism is connected with the showing/saying distinction and the distinction between a form and a property of a substrate.

Wittgenstein found the game analogy so useful (not only in the philosophy of mathematics but for language in general) is that it brings mathematics (and language) *as activity* into the foreground.

In the formal logical tradition as well as among many professional mathematicians, the tendency is the opposite: a mathematical system is seen as a representation of a body of truths rather than as a calculus. The activity element is suppressed and concealed, as though it were something inessential. The emphasis is rather on results, stated in prose. As a consequence, a language or a symbolism tends to be seen merely as a static sign-system or a notation in an almost purely typographical sense.

I think that this careless attitude towards mathematics as an activity was one source of the debate in the Unguru controversy mentioned earlier. The algebraic symbolism is seen as a mere static notation rather than as a symbolism *in which the symbols are constituted by the use of signs.* The static view of a symbolism is quite manifest in Van der Waerden's argument in the Unguru debate:

> Unguru, like many non-mathematicians, grossly overestimates the importance of symbolism in mathematics. These people see our papers full of formulae, and they think that these formulae are an essential part of mathematical thinking. We, working mathematicians, know that in many cases the formulae are not at all essential, only convenient.[23]

It seems to me that a static view of mathematical symbolism like Van der Waerden's often goes hand in hand with an extensional view of mathematics (as Wittgenstein speaks of the extensional view in PR § 99). Wittgenstein's disinclination towards the extensional view is tied to his symbolic conception of mathematics. The extensional view is perhaps the most characteristic feature of the modern ontological conception of mathematics, i.e. as the science of a mathematical reality in which nearly all is and will always be hidden from us. Notice the separation, or dualism, between form and content that is presupposed in Van der Waerden's remark. He speaks as though 'working mathematicians' could have the mathematical content in their mind without having it articulated in some notation or symbolism.

Now compare and contrast Van der Waerden's statement with the following statement by Hertz:

[23] Van der Waerden (1975, p. 205).

> We cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than what was originally put into them.[24]

It is the view of a symbolism as a mere combinatorial structure of sequences of concrete signs that is the background of the established notion of 'syntax' in modern mathematical logic and meta-mathematics. With that notion of 'syntactical form', the view of mathematics called 'formalism' becomes a hopelessly superficial view that hardly anyone has seriously held.[25]

The notion of 'syntax' originated in the traditional linguistic conception of language, according to which a language is a system – not primarily an activity! – determined by grammar and lexicon. Syntax was the doctrine of correct sentence-construction of a (natural) language. But when the notion of syntax was transferred to the so-called 'formal languages', with the distinction between object-language and meta-language, syntax becomes more 'formal' in the sense that the rules of syntax have been given a mathematical content: the syntactical objects have the structure of arbitrary finite sequences (in a purely mathematical sense) generated by recursive or inductive procedures. There is no limit to the length of well-formed formulae of the object-language, but formulae are still claimed to be concrete objects. This does not really make sense![26] With this notion of syntactical form with roots in meta-mathematics, a *dualism* between form and content is created. There arises a need to supplement the syntax with a 'semantics' that endows the syntactical objects with content or meaning. This is the origin of Tarski's 'logical semantics' or 'model theory'.

Wittgenstein's conception of *logical* syntax in the *Tractatus*, which later develops into (philosophical) grammar, is very different *in that it is related to activity, to the use of signs*. It has to do not only with the external combinatorial forms of the signs as finite sequences, but with *the forms of use of signs*, i.e. *the forms of use that constitute the symbols*. The logical syntax of the *Tractatus* needs no supplementation with something like a 'semantics', because

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[24] Quoted from Bell, (1937, p. 31).

[25] Formalism in this sense will also be a kind of finitism, and it is'finitism' in this sense Wittgenstein has in mind when he says,"Finitism and behaviorism are quite similar trends. Both say, but surely, all we have here is.... Both deny the existence of something, both with a view to escaping from a confusion." (RFM II, § 61.) Wittgenstein was no more a formalist or finitist in this sense than he was a behaviorist.

[26] The source of this confusion in the notion of syntax of formal languages is a similar confusion in Hilbert's finitism. See Stenlund (2012a).

it is *already* concerned with the sense of signs: to get clear about the forms of use of signs is to get clear about their sense (or logical content).

Wittgenstein's emphasized the operational aspect of a symbolism already in the *Tractatus*, where we find, for example, the following significant observation:

> In order to recognize the symbol in the sign we must consider the significant use. (TLP, 3.326).[27]
>
> A sign does not determine a logical form unless it is taken together with its logico-syntactical employment. (TLP, 3.327).

This difference between sign and symbol becomes quite clear in the game-of-chess analogy. A sign corresponds to a chess piece as physical or perceptual object with a certain shape, size, colour, etc., while a symbol corresponds to a chess piece as 'a piece you play with', determined by the conventions which govern its moves and its relations to other pieces in the game.

In Wittgenstein's view of mathematics since the 1930's, as I understand it, it is the operational aspect of a symbol, its function in the calculus, its role in the manipulation and transformation of expressions, which constitutes it as a symbol. A symbolism is not just a system of notation in some typographical or combinatorial sense.

Accordingly, Wittgenstein's view means that new mathematical concepts (such as, for instance, the concept of a number) necessarily emerged *together with* the new algebraic symbolism and the place-value notation for numbers. They could come into existence as new precise notions only when the operational practices of the new symbolism were in place. Or rather, the invention of new notions *was* the invention of the new operational practices of the algebraic symbolism. It is not as if the new notions were invented in advance "in the minds of mathematicians" (as Van der Waerden's remark suggests) and were then given expression and application using the new algebraic notation. That is the sort of mentalism that Wittgenstein persistently calls into question. And it is essentially the same mentalistic tendency in Frege that is being questioned when Wittgenstein writes: "In attacking the formalist conception of

[27] The original German version of this remark is "Um das Symbol am Zeichen zu erkennen, muss man auf den sinnvollen Gebrauch achten." I prefer Ogden's translation of this remark.

arithmetic, Frege says more or less this: these petty explanations of the signs are idle once we *understand* the signs. Understanding would be something like seeing a picture from which all the rules followed, or a picture that makes them all clear. But Frege does not seem to see that such a picture would itself be another sign, or a calculus to explain the written one to us."(PG, p. 40)

Calculus and Prose

Vieta's analytic art draws on the works of Pappus and Diophantus, which are primarily geometrical. As a consequence, Vieta presents algebraic operations geometrically, which manifests itself in his algebraic notation. Powers such as a^2, a^3 are expressed by Vieta as *a plane* and *a cube,* respectively. An algebraic expression which we would write as

$$\frac{x^3 - 3b}{cy^2}$$

would be expressed in Vieta's notation as:

$$\frac{A \text{ cubus} - B \text{ solido } 3}{C \text{ in } E \text{ quadratum}}$$

with the reading: "*A* cubed minus 3 times *B* solid divided by *C* times *E* squared". Vieta also adopted a *principle of homogeneity* according to which only magnitudes of the same dimension can be compared, for instance, be added to one another. Thus, the expression we would write $a^2 + a$ was not permitted. Descartes greatly simplified the algebraic symbolism when he showed how to represent any magnitude (including a power) as the length of a line segment, given an arbitrary stipulated unit of length. The expression $a^2 + a$ is then interpreted as a length added to a length and is permitted. By viewing all numbers as ratios, John Wallis arrived at a notion of number according to which all numbers are homogenous. Their homogeneity is identical with their dimensionlessness, which, in turn, as Klein expresses it, is identical with their symbolic character. Thus with Wallis, the analytic art becomes a pure calculus, a system of symbolic calculation.

Using Wittgenstein's distinction between calculus and prose, one could say that Vieta introduces his algebraic system using geometrical prose. The development from Vieta up to Wallis could then be seen as one of the first clarifications through a logical separation of calculus and prose. This clarification consists in finding a more adequate symbolism: one in which 'geometrical prose' is not needed.

During the 18th and the beginning of the 19th centuries, two opposed tendencies influencing the development of mathematics can be discerned. One is the ontological view of the Euclidean-Aristotelian heritage; the other tendency is the symbolic point of view arising out of the invention of the algebraic symbolism and the new powerful methods and analytic techniques that the algebraic symbolism made possible. A number of conceptual problems in mathematics emerged during this time due to the tension between these two tendencies (e.g. problems with infinitesimal magnitudes, with imaginary numbers, convergence of series, etc.). The common 18th century mathematicians' view of mathematics as *the science of quantity* shows the continued influence of the Euclidean-Aristotelian heritage – even if one had a more general notion of quantity that could bridge the ancient Greek gap between discrete and continuous quantity.

By the latter half of the 19th century, we had arrived at what Moritz Epple calls "the end of the paradigm of the science of quantity"[28]. There were several parallel developments that initiated the abandonment of the paradigm, but it is clear that in many cases the symbolic point of view plays a decisive role. This is particularly true of the new endeavor towards rigor in mathematics that begins with Cauchy, Weierstrass, Gauss and others. Hertz can also be seen as belonging to the rigorization movement, along with Hilbert and Einstein, who, like Wittgenstein, were both deeply influenced by Hertz' symbolic approach.[29]

An exception here is the perspective on mathematics from the point of view of mathematical logic understood as its foundation. Here the ontological view of mathematical theories and the descriptive view of mathematical propositions still prevails. The aspect of mathematics as activity and calculus is entirely disregarded. The emphasis is on results, and mathematical prose is given the prominent position as the source of meaning in mathematics through

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[28] Epple, (2003, p. 291).

[29] See Stenlund, (2012b).

'semantics', 'theory of reference', 'ontology' etc. But this prose does not come from the application of the logical calculi on some subject matter outside the calculi, in which case it could be harmless. The problematic prose is the one that comes from the reading (or translation) of the logical particles and formulas into ordinary language ('vague ordinary prose', as Wittgenstein calls it), and which is presented as belonging to, and constitutive of, the logical systems. It is on the basis of this prose that certain mathematical results in mathematical logic are read as results about consistency, completeness, decidability, etc. And it is on the basis of this 'internal or constitutive prose' that mathematical logic is given its foundational status.

The logical distinction that Wittgenstein calls 'calculus versus prose' was an essential feature of the symbolic approach in the work on rigorization in mathematics in the latter half of the 19th century. The 'prose-sense' (which is often called 'the intuitive meaning') of imaginary numbers, for instance, was perceived as unclear and confused. In Cauchy's work on reforming analysis in his *Cours d'analyse,* the problem is solved through a logical separation between the prose and the calculus of imaginary numbers. According to Umberto Bottazzini, [30] "Cauchy took the ontological problem concerning the nature of imaginary numbers much more seriously than anyone before him. In the *Cours d'analyse,* chapter VII*,* he introduced them in a formal manner as 'symbolic expressions'." Cauchy does not solve the "ontological problem" of imaginary numbers by providing a positive answer to the question of their "ontological nature", but rather by making the ontological problem disappear along with the prose through the development of a rigorous calculus of imaginary numbers. At that time, Cauchy's approach required hard work. Bottazzini points out that "it took him no less than fifty-five pages to […] define algebraic operations on "expressions" like $\alpha + \beta\sqrt{-1}$ (α and β being real quantities) in a rigorous way and to establish their properties." In Bottazzini's judgement, "Chapter VII of the *Cours* can be considered one of the places where Cauchy displayed his concept of rigor best."

There was no sharp division between pure and applied mathematics in 18th century mathematics. Most mathematicians also worked in theoretical physics. This state of affairs was reflected in the language of mathematics, that is to say, *in the prose of mathematics*. Does

[30] Bottazzini, (2003, p. 217).

a certain word, say, 'quantity', derive its meaning from its use in physics, or does it have its meaning from its operationalization in the mathematical calculus, i.e. does its mathematical sense coincide with its sense as a symbol of the calculus? It was only after the influence of the symbolic view of mathematics that this question could be raised. An essential feature of the symbolic point of view in the rigorization endeavor was the logical separation of a symbolic system from its application on some subject matter outside the system. This is why it is sometimes necessary to deprive words of their meaning in order to discern the dividing line between calculus and prose. Even a word of ordinary language may have the role of a symbol in a calculus (such as, for instance, the words 'point' and 'line' in Hilbert's system in *Grundlagen der Geometrie*).

In his work on the arithmetization of analysis (which was a move away from the traditional notion of mathematics as the science of quantity), Weierstrass still uses the word 'quantity', but it is nonetheless clear that the sense of the word as he uses it is as a symbol in the arithmetical calculus. Moritz Epple (2003 p. 296) notes: " Weierstrass continued to use the notion of quantity, but expressions like "arithmetical quantity" or "number quantity" made clear what he had in mind: a logical separation of his concepts from their more intuitive counterparts in geometry and physics."[31]

As noted earlier, by this time there is a change in the symbolic point of view in that it is no longer bound up with the idea of a *mathesis universalis*, a universal mathematical system. With the invention of different geometries, different number systems and different algebraic systems, there arises a need to distinguish between pure mathematical systems and their application on something outside the systems. This change is reflected in Wittgenstein's philosophy from the beginning of the 1930th in his use of the notions of 'system', 'calculus' and 'game'. In one of his lectures on the foundations of mathematics in 1939, Wittgenstein is reported to have said:

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[31] In 1947, when Wittgenstein was working in the philosophy of psychology, he wrote in his diary: "Weierstrass introduces a string of new concepts to bring about order in the thinking about the differential calculus. And in that way on the whole, it seems to me, I must bring about order in psychological thinking through *new* concepts. (That it concerns a calculus in the first case, but not in the second, is *not* important.)". (The Bergen electronic edition of Wittgenstein's works, MS 135, p. 115-116, my translation)

> The difficulty in looking at mathematics as we do is to make one particular section – to cut pure mathematics off from its application.[32]

The difficulty with this approach is to know where to make the section. The similarity between , for instance, the two prose-sentences "A quadratic equation has two roots" and "A human being has two eyes" might suggest that they are both propositions of applied mathematics. But the former, unlike the latter, is not an application of arithmetic to a given subject matter outside the arithmetic-algebraic calculus.

At this time, a mathematical calculus or system is often compared with the game of chess. The French mathematician and philosopher Luis Couturat sketched the nature of mathematics as follows:

> A mathematician never defines magnitudes [or numbers] in themselves, as a philosopher would be tempted to do; he defines their equality, their sum and their product, and these definitions determine, or rather constitute, all the mathematical properties of magnitudes. In a yet more abstract and more formal manner he lays down *symbols* and at the same time *prescribes* the rules according to which they must be combined; these rules suffice to characterize these symbols and to give them a mathematical value. Briefly, he creates mathematical entities by means of arbitrary conventions, in the same way that the several chessmen are defined by the conventions which govern their moves and the relations between them.[33]

I find this to be quite a good formulation of the idea of the symbolic view of mathematics as it was embraced by this time, and I would say that the germ of this view of mathematics was present already in Vieta.

The Uniqueness of Wittgenstein's Symbolic Approach

It would, however, be a mistake to think that Wittgentein's symbolic conception of mathematics was in complete agreement with that of any mathematician or scientist. Wittgenstein was not a mathematician but a philosopher, and he was not the sort of

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[32] Diamond (1976), p. 150.

[33] Quoted from Bell (1937, p. 624).

philosopher who worked in the service of science. He took from Hertz and others (Boltzmann, Weierstrass, Thomae, Hilbert, Einstein) what he found useful and clarifying for his *philosophical* work.

One manifestation of Wittgenstein's aversion to scientific ambitions in philosophy, it seems to me, is the final breakdown of his extensive collaborations with Waismann on a joint effort. Waismann did not separate philosophical and scientific ambitions as sharply as Wittgenstein did. In the first of his lectures on the foundations of mathematics in 1939, Wittgenstein is reported to have said "it will be most important not to interfere with the mathematicians."[34] I believe that Wittgenstein felt that modern scientific ambitions are not always compatible with the aspiration towards clarity as an end in itself, and these aspirations of Wittgenstein's manifest themselves most clearly in his style of writing. He does not write like someone who wants to accomplish 'rational reconstructions' of problematic mathematical notions.

This is connected with the way in which Wittgenstein's method of philosophical clarification differs from *explication* in the naturalistic sense of Carnap and Quine (who says that explication is elimination of philosophical difficulties). Wittgenstein starts from a philosophical puzzle and, in a sense, works backwards towards its source. The source often turns out to be a false analogy to which we are misled by prose. He does not eliminate the difficulty by avoiding it or by finding a way around it – a roundabout way that nevertheless leads to the scientific objectives or satisfies the scientific needs. When the source of the confusion is found, the philosophical task is completed. What the mathematicians or scientists will do with that clarification (if anything) is up to them.

Let me return to Hertz' idea, to which I alluded earlier, that the signs and formulas of a mathematical symbolism have "an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them". The activity-aspect of mathematics can throw some light on this remark. The 'hidden intelligence of formulas', I would suggest, is simply the manner in which a certain symbol or formula *in its use* is connected with so many other things in the arithmetical-algebraic system which we don't survey and cannot foresee even if we master its use. This 'intelligence' seems to be hidden only because *we do not survey* the possible situations of use of the formulas. We

[34] Diamond (1976, p. 13).

do master them in practice, and we can often handle the new situations when confronted with them –even if it may require some extra effort in the new situations.

That we do not survey all possible situations of the use of mathematical symbols is true also of the 'moves of the game' at the most basic operational level, where there is complete agreement *in action* among mathematicians about how to go on. The moves and features of the calculus at this level tend to be dismissed as 'trivialities' by mathematicians. But such trivialities are often precisely the topic of many of Wittgenstein's investigations. I would say that this is why it is so difficult in many cases to understand what he is doing, what he is up to, in his writings on the foundations of mathematics. On this level of 'trivialities', the topic of 'the nature of mathematical symbolism' is a non-issue for the mathematician. No mathematician would get the idea of problematizing the following of a simple rule such as "Add 2" (i.e. iterating the operation of adding the number 2) as Wittgenstein does in his investigation of rule-following in the PI (§ 184). With respect to this difference between a mathematician's concerns and Wittgenstein's philosophical concerns, he remarks: "The philosopher only marks what the mathematician casually throws off about his activities." (PG, p. 369). A higher degree of conceptual sensitivity is a characteristic trait of Wittgenstein's symbolic approach. From the end of the 1930's onward, this sensitivity also involves mathematics as a human activity, as an anthropological phenomenon.

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