Internet Appendix for "Treasury Inconvenience Yields during the COVID-19 Crisis"

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A Euler Equations for Treasury Pricing

We derive the ODE system for $\{C_0(\cdot), C_\beta(\cdot)\}$ based on the dealer's Euler equation first. We then offer a detailed numerical procedure to solve the ODE system, and give the results on equilibrium OIS curves.

A.1 Euler Equation and ODE System

Guess that

$$P_{\beta t}^{\tau} = \exp\left[-\left(A(\tau) r_t + C_0(\tau)\right)\right]$$
$$P_{\beta t}^{\tau} = \exp\left[-\left(A(\tau) r_t + C_\beta(\tau)\right)\right]$$

and recall the yield is defined as $y_t^\tau = -\frac{\log P_{\tilde{\beta} t}^{\tau}}{\tau} = \frac{A(\tau) r_t + C_{\tilde{\beta}}(\tau)}{\tau}$.

Return dynamics. One can show that

$$\frac{dP_{0t}^{\tau}}{P_{0t}^{\tau}} = \left[A'(\tau) r_t + A(\tau)\kappa(r_t - \overline{r}) + C_0'(\tau) + \frac{\sigma^2}{2}A^2(\tau)\right]dt + \left\{e^{C_0(\tau)-C_\beta(\tau)} - 1\right\}dN_t - A(\tau)\sigma dZ_t, \tag{IA.1}$$

$$\frac{dP_{\beta t}^{\tau}}{P_{\beta t}^{\tau}} = \left[A'(\tau) r_t + A(\tau)\kappa(r_t - \overline{r}) + C_\beta'(\tau) + \frac{\sigma^2}{2}A^2(\tau)\right]dt + \left\{e^{C_\beta(\tau)-C_0(\tau)} - 1\right\}dN_t - A(\tau)\sigma dZ_t, \tag{IA.2}$$

where dN_t denotes the Poisson shock with intensity $\xi_{\tilde{\beta}}$.

We perform the calculation for $\frac{dP_{\beta t}^{\tau}}{P_{\beta t}^{\tau}}$; $\frac{dP_{0t}^{\tau}}{P_{0t}^{\tau}}$ follows similarly. First,

$$\begin{aligned} dP_{\beta t}^{\tau} &= \exp\left[-\left(A(\tau) r_t + C_\beta(\tau)\right)\right]\left[A'(\tau) r_t dt + A(\tau)\kappa(r_t - \overline{r})dt + C_\beta'(\tau)dt - A(\tau)\sigma dZ_t\right] \\ &\quad + \exp\left[-\left(A(\tau) r_t + C_\beta(\tau)\right)\right]\frac{1}{2}A^2(\tau)\sigma^2 dt + \left[e^{-(A(\tau)r_t + C_0(\tau))} - e^{-\left(A(\tau)r_t + C_\beta(\tau)\right)}\right]dN_t, \end{aligned}$$

which implies that

$$\frac{dP_{\beta t}^{\tau}}{P_{\beta t}^{\tau}} = A'(\tau) r_t dt + A(\tau)\kappa(r_t - \overline{r})dt + C_\beta'(\tau)dt + \frac{A^2(\tau)\sigma^2}{2}dt - A(\tau)\sigma dZ_t + \left\{e^{C_\beta(\tau)-C_0(\tau)} - 1\right\}dN_t.$$

It is easy to calculate that $\mathbb{E}_t\left[\frac{dP_{\beta t}^{\tau}}{P_{\beta t}^{\tau}}\right] = \mu_{\beta t}^{\tau}dt$ with

$$\mu_{\beta t}^{\tau} \equiv A'(\tau) r_t + A(\tau)\kappa(r_t - \overline{r}) + C_\beta'(\tau) + \frac{1}{2}A^2(\tau)\sigma^2 + \xi_\beta\left(e^{C_\beta(\tau)-C_0(\tau)} - 1\right). \tag{IA.3}$$

Euler equations. As before, let us focus on the distress state. The arbitrager is maximizing the mean-variance objective over

$$\int_0^T x_{\beta t}^{\tau}\left(\frac{dP_{\beta t}^{\tau}}{P_{\beta t}^{\tau}} - r_t dt - \Lambda_{0t}dt\right)d\tau.$$

Given (IA.1), we have

$$\mathbb{E}_t\left[\int_0^T x_{\beta t}^{\tau}\left(\frac{dP_{\beta t}^{\tau}}{P_{\beta t}^{\tau}}-r_t dt-\Lambda_{\beta t}dt\right)d\tau\right]=\int_0^T x_{\beta t}^{\tau}\left(\mu_t^{\tau}-r_t-\Lambda_{\beta t}\right)d\tau dt,$$

$$\mathbb{V}ar_t\left[\int_0^T x_{\beta t}^{\tau}\left(\frac{dP_{\beta t}^{\tau}}{P_{\beta t}^{\tau}}-r_t dt-\Lambda_{\beta t}dt\right)d\tau\right]=\left(\int_0^T\left\{x_{\beta t}^{\tau}\left(e^{C_\beta(\tau)-C_0(\tau)}-1\right)d\tau\right\}\right)^2\xi_0 dt+\left(\int_0^T x_{\beta t}^{\tau}A\left(\tau\right)d\tau\right)^2\sigma^2 dt.$$

Therefore the dealer's FOC with respect to $x_{\beta t}^{\tau}$ is (omitting dt from now on)

$$\mu_{\beta t}^{\tau}-r_t-\Lambda_{\beta t}=A\left(\tau\right)\frac{\sigma^2}{\rho_d}\int_0^T x_{\beta t}^{u}A\left(u\right)du+\left(e^{C_\beta(\tau)-C_0(\tau)}-1\right)\cdot\frac{\xi_\beta}{\rho_d}\int_0^T\left\{x_{\beta t}^{u}\left(e^{C_\beta(u)-C_0(u)}-1\right)du\right\}.$$

Expanding $\mu_{\beta t}^{\tau}$ in (IA.3) and collecting terms we have

$$\begin{aligned}&\left(A'\left(\tau\right)-1\right)r_t-\Lambda_{\beta t}+A\left(\tau\right)\kappa\left(r_t-\overline{r}\right)+C_\beta'\left(\tau\right)+\frac{1}{2}A^2\left(\tau\right)\sigma^2 r_t+\xi_\beta\left(e^{C_\beta(u)-C_0(u)}-1\right)\\ =&A\left(\tau\right)\cdot\frac{\sigma^2}{\rho_d}\left[\int_0^T x_{\beta t}^{u}A\left(u\right)du\right]+\left(e^{C_\beta(u)-C_0(u)}-1\right)\cdot\frac{\xi_\beta}{\rho_d}\int_0^T\left\{x_{t\beta}^{u}\left(e^{C_\beta(u)-C_0(u)}-1\right)du\right\}.\end{aligned}$$

Because this equation holds for all r_t, we must have $A\left(\tau\right)\kappa+A'\left(\tau\right)=1$; with initial condition $A\left(0\right)=0$ we have

$$A\left(\tau\right)=\frac{1-e^{-\kappa\tau}}{\kappa}. \tag{IA.4}$$

Moving on to the function $C_0\left(\cdot\right)$, and collecting terms, we have

$$\begin{aligned}&C_\beta'\left(\tau\right)+\xi_\beta\left(e^{C_\beta(u)-C_0(u)}-1\right)\left(1-\frac{1}{\rho_d}\int_0^T\left\{x_{\beta t}^{u}\left(e^{C_\beta(u)-C_0(u)}-1\right)du\right\}\right)\\ =&A\left(\tau\right)\cdot\frac{\sigma^2}{\rho_d}\int_0^T x_{\beta t}^{u}A\left(u\right)du+\Lambda_\beta+A\left(\tau\right)\kappa\overline{r}-\frac{1}{2}A^2\left(\tau\right)\sigma^2.\end{aligned} \tag{IA.5}$$

This is the equation when $\tilde{\beta}_t=\beta$.

For the normal state $\beta_t=0$, we have

$$\frac{dP_{0t}^{\tau}}{P_{0t}^{\tau}}=\mu_{0t}^{\tau}dt+\left\{e^{C_0(\tau)-C_\beta(\tau)}-1\right\}d\hat{N}_t-A\left(\tau\right)\sigma dZ_t,$$

where

$$\mu_{0t}^{\tau}\equiv A'\left(\tau\right)r_t+A\left(\tau\right)\kappa\left(r_t-\overline{r}\right)+C_0'\left(\tau\right)+\frac{1}{2}A^2\left(\tau\right)\sigma^2+\xi_0\left(e^{C_0(\tau)-C_\beta(\tau)}-1\right).$$

And the ODE for $C_0(\cdot)$ based on the dealer's Euler equation at the stress state can be derived analogously:

$$\begin{aligned}
&C_0'(\tau) + \xi_0 \left(e^{C_0(\tau)-C_\beta(\tau)} - 1\right) \left(1 - \frac{1}{\rho_d} \int_0^T \left\{ x_{t\beta}^u \left(e^{C_0(\tau)-C_\beta(\tau)} - 1\right) du \right\}\right) \\
=&A(\tau) \cdot \frac{\sigma^2}{\rho_d} \int_0^T x_{0t}^u A(u)\, du + \Lambda_\beta + A(\tau)\kappa\bar{r} - \frac{1}{2} A^2(\tau)\sigma^2. \qquad \text{(IA.6)}
\end{aligned}$$

We hence have arrived at the ODE system (IA.5) and (IA.6) for $\{C_0(\cdot), C_\beta(\cdot)\}$, with boundary conditions

$$C_0(0) = C_\beta(0) = 0, C_0'(0) = \Lambda_0 = \lambda B_0, C_\beta'(0) = \Lambda_\beta = \lambda B_\beta.$$

A.2 Numerical Methods

This section outlines the numerical procedure in solving for (IA.5) and (IA.6). Denote

$$\begin{aligned}
D_0(\tau) &\equiv A(\tau) \cdot \frac{\sigma^2}{\rho_d} \int_0^T x_{t0}^u A(u)\, du + \Lambda_0 + A(\tau)\kappa\bar{r} - \frac{1}{2} A^2(\tau)\sigma^2, \\
D_\beta(\tau) &\equiv A(\tau) \cdot \frac{\sigma^2}{\rho_d} \int_0^T x_{t\beta}^u A(u)\, du + \Lambda_\beta + A(\tau)\kappa\bar{r} - \frac{1}{2} A^2(\tau)\sigma^2,
\end{aligned}$$

which have been solved in closed-form given x_{t0}^u and $A(\tau) = \frac{1-e^{-\kappa\tau}}{\kappa}$ (recall Eq. (IA.4)).

The numerical procedure is as follows.

1. Start with $K_0^{(0)} = K_\beta^{(0)} = 1$.

2. With $\left(K_0^{(n)}, K_\beta^{(n)}\right)$, define

$$\hat{D}_0(\tau) \equiv D_0(\tau) + K_0^{(n)}, \hat{D}_\beta(\tau) \equiv D_\beta(\tau) + K_\beta^{(n)}.$$

   We obtain the solution by solving the ODE system (IA.7)

$$\begin{cases} C_0'(\tau) + K_0^{(n)} e^{C_0(\tau)-C_\beta(\tau)} & = \hat{D}_0(\tau) \\ C_\beta'(\tau) + K_0^{(n)} e^{C_\beta(\tau)-C_0(\tau)} & = \hat{D}_\beta(\tau) \end{cases} \qquad \text{(IA.7)}$$

   with the initial conditions $C_0(0) = C_\beta(0) = C_0'(0) = C_\beta'(0) = 0$ by following these steps.

(a) From the first equation in (IA.7), we have

$$e^{C_0(\tau)-C_\beta(\tau)} = \frac{\hat{D}_0(\tau) - C_0'(\tau)}{K_0^{(n)}}. \tag{IA.8}$$

Taking the derivative with respect to τ on both sides of the first equation in (IA.7) and plugging in $e^{C_0(\tau)-C_\beta(\tau)}$, we get

$$C_0''(\tau) + \left(\hat{D}_0(\tau) - C_0'(\tau)\right)\left(C_0'(\tau) - C_\beta'(\tau)\right) = \hat{D}_0'(\tau).$$

(b) Now, using $C_\beta'(\tau) = \hat{D}_\beta(\tau) - K_\beta^{(n)} e^{C_\beta(\tau)-C_0(\tau)}$ from the second equation and plugging in (IA.8), we get,

$$C_0''(\tau) + \left(\hat{D}_0(\tau) - C_0'(\tau)\right)\left(C_0'(\tau) - \hat{D}_\beta(\tau)\right) + K_\beta^{(n)} K_0^{(n)} = \hat{D}_0'(\tau).$$

(c) Letting $z(\tau) \equiv C_0'(\tau)$, we have a Riccati equation for $z(\tau)$

$$z'(\tau) = \left(\hat{D}_0'(\tau) + \hat{D}_\beta(\tau)\hat{D}_0(\tau) - K_\beta^{(n)} K_0^{(n)}\right) - \left(\hat{D}_\beta(\tau) + \hat{D}_0(\tau)\right) z(\tau) + z^2(\tau)$$

with a boundary condition $z(0) = 0$. Once we numerically solve for $z(\tau)$, we can calculate $C_0(\tau) = \int_0^\tau z(u)\,du$ and derive $C_\beta(\tau)$ according to Eq. (IA.8).

3. Calculate $K_0^{(n+1)}, K_\beta^{(n+1)}$ based on the new solution using (IA.9):

$$\begin{cases} K_0^{(n+1)} \equiv & \xi_0\left(1 - \frac{1}{\rho_d}\int_0^T \left\{x_{t0}^u \left(e^{C_0(u)-C_\beta(u)} - 1\right) du\right\}\right) \\ K_\beta^{(n+1)} \equiv & \xi_\beta\left(1 - \frac{1}{\rho_d}\int_0^T \left\{x_{t\beta}^u \left(e^{C_\beta(u)-C_0(u)} - 1\right) du\right\}\right). \end{cases} \tag{IA.9}$$

4. If $\left\|\left(K_0^{(n+1)} - K_0^{(n)}, K_\beta^{(n+1)} - K_\beta^{(n)}\right)\right\| < \epsilon$, then terminate. Otherwise set $n = n + 1$ and go to Step 2.

A.3 Derivation of Equilibrium OIS Curves

We guess and verify that the equilibrium OIS prices take the following forms with $A(\tau) = \frac{1-e^{-\kappa\tau}}{\kappa}$:

$$P_{0t}^{\text{OIS},\tau} = \exp\left[-\left(A(\tau) r_t + C_0^{\text{OIS}}(\tau)\right)\right]$$
$$P_{\beta t}^{\text{OIS},\tau} = \exp\left[-\left(A(\tau) r_t + C_\beta^{\text{OIS}}(\tau)\right)\right],$$

where $\left\{C_0^{\text{OIS}}(\cdot), C_\beta^{\text{OIS}}(\cdot)\right\}$ are to be determined endogenously. Note that the Treasury-OIS spread at tenor τ, denoted by Δy^τ, then can be calculated as

$$\Delta y^\tau = \frac{\ln P^\tau - \ln P^{\text{OIS},\tau}}{\tau} = \frac{C^{\text{OIS}}(\tau) - C(\tau)}{\tau}.$$

Based on similar derivations as in Section A.1, we arrive at

$$C_0^{\text{OIS}\prime}(\tau) + \xi_0\left(\exp\left(C_0^{\text{OIS}}(\tau) - C_\beta^{\text{OIS}}(\tau)\right) - 1\right)\left(1 - \frac{1}{\rho_d}\int_0^T x_{t0}^u\left(\exp\left(C_0^{\text{OIS}}(\tau) - C_\beta^{\text{OIS}}(\tau)\right) - 1\right)du\right)$$
$$= A(\tau)\kappa\bar{r} - \frac{A^2(\tau)}{2}\sigma^2 + \frac{A(\tau)\sigma^2}{\rho_d}\int_0^T x_{t0}^u A(u)\,du \tag{IA.10}$$

$$C_\beta^{\text{OIS}\prime}(\tau) + \xi_\beta\left(\exp\left(C_\beta^{\text{OIS}}(\tau) - C_0^{\text{OIS}}(\tau)\right) - 1\right)\left(1 - \frac{1}{\rho_d}\int_0^T x_{t\beta}^u\left(\exp\left(C_\beta^{\text{OIS}}(\tau) - C_0^{\text{OIS}}(\tau)\right) - 1\right)du\right)$$
$$= A(\tau)\kappa\bar{r} - \frac{A^2(\tau)}{2}\sigma^2 + \frac{A(\tau)\sigma^2}{\rho_d}\int_0^T x_{t\beta}^u A(u)\,du \tag{IA.11}$$

with initial conditions $C_0^{\text{OIS}}(0) = C_0^{\text{OIS}\prime}(0) = C_\beta^{\text{OIS}}(0) = C_\beta^{\text{OIS}\prime}(0) = 0$.

A.4 Proof for Section 4.1.3

We give the proof for the claim in the Remark of Section 4.1.3. The first result on heterogeneous risk bearing capacity $\{\rho_d^i\}$ is straightforward. To show the second result with quadratic cost, consider the Euler equation with respect to $x_t^i(\tau)$ at the state of $\tilde{\beta} = \beta$ (note that $b_{\beta t}^i(u) = x_{\beta t}^i(u) + q_{\beta t}^{i,d}(u)$ is the dealer i's balance sheet):

$$\rho_d^i\left(\mu_{\beta t}^{\tau}-r_t\right)-\rho_d^i\lambda^i\int_0^T b_{\beta t}^i\left(u\right)du = A\left(\tau\right)\sigma^2\int_0^T x_{\beta t}^i\left(u\right)A\left(u\right)du$$
$$+\left(e^{C_0(\tau)-C_\beta(\tau)}-1\right)\xi_\beta\int_0^T\left\{x_{\beta t}^i\left(u\right)\left(e^{C_\beta(u)-C_0(u)}-1\right)du\right\}.$$

Aggregating over i, using a) $\rho_d^i\lambda^i=\rho_d\lambda$, b) $\int_i \rho_d^i di=\rho_d$, and c) $\int_i x_{\beta t}^i\left(u\right)di = X_{\beta t}\left(u\right)$, we have

$$\rho_d\left(\mu_{\beta t}^{\tau}-r_t\right)-\rho_d\lambda\int_i\int_0^T B_t^i\left(u\right)du di = A\left(\tau\right)\sigma^2\int_0^T X_{\beta t}\left(u\right)A\left(u\right)du$$
$$+\left(e^{C_0(\tau)-C_\beta(\tau)}-1\right)\xi_\beta\int_0^T\left\{X_{\beta t}\left(u\right)\left(e^{C_\beta(u)-C_0(u)}-1\right)du\right\}.$$

This equation is identical to Eq. (16) since $\int_i\int_0^T B_t^i\left(u\right)du di = \int_0^T B_t\left(u\right)du = B_t$. Finally, it is straightforward to check that the equilibrium individual holdings are

$$x_t^i\left(\tau\right)=X_t\left(\tau\right)\frac{\rho_d^i}{\rho_d},\ \text{and}\ q_t^{i,d}\left(\tau\right)=Q_t\left(\tau\right)\frac{\rho_d^i}{\rho_d}.$$

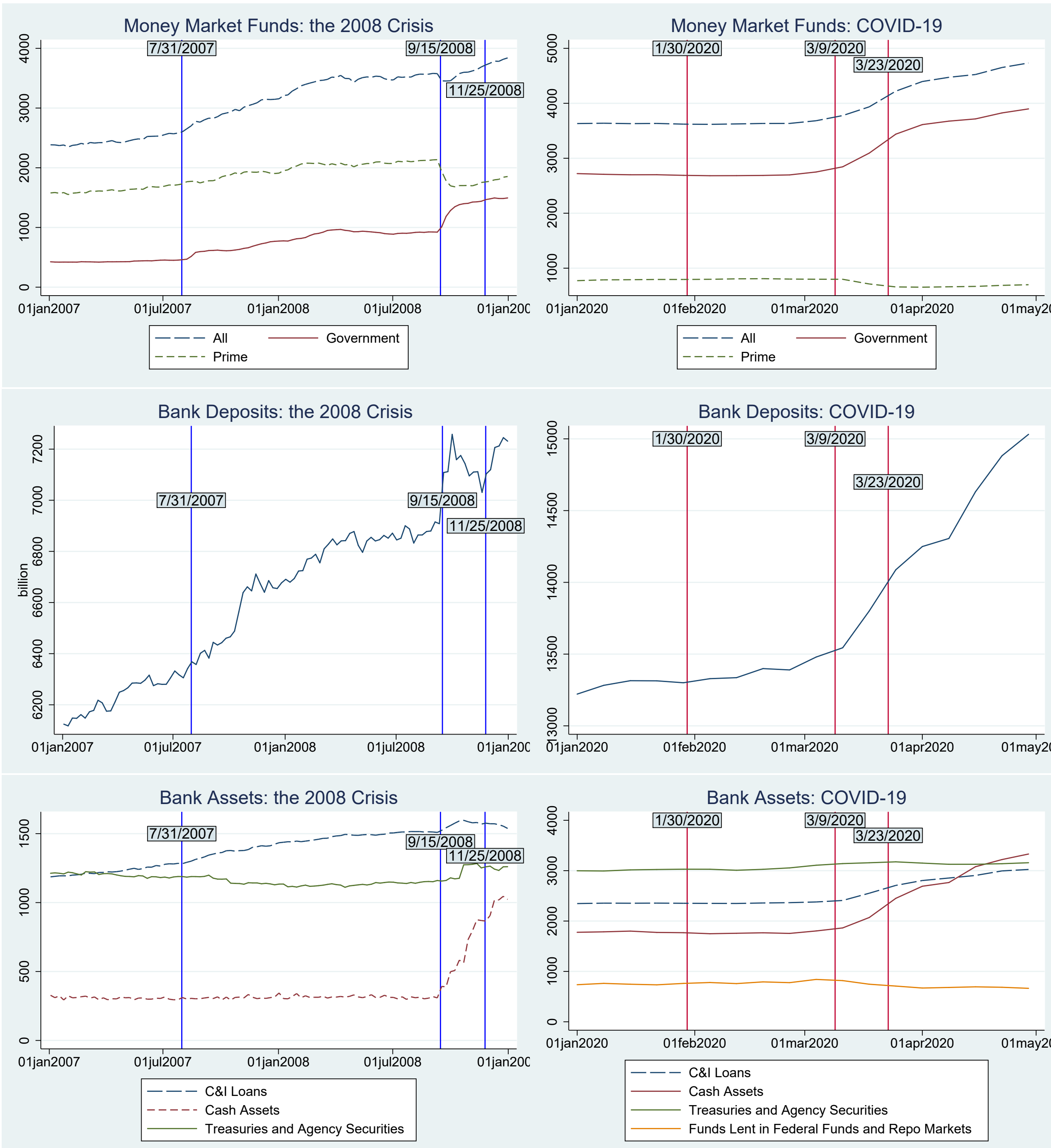
B Additional Evidence

We present two sets of additional empirical evidence, one highlighting the flight-to-cash nature of the COVID-19 shock and the other breaking down primary dealers' repo positions into different tenor buckets.

We provide two sets of additional empirical evidence. First, Figure IA.1 presents series of fund flows of MMFs and banks. The top panels show that there are negative net flows out of prime MMFs and positive net flows into government MMFs, while the middle panels show that bank deposits increased significantly, in both the COVID-19 and 200709 crises. The bottom panels show that banks allocate the incoming deposits into cash assets significantly and into long-term Treasuries and agency securities slightly. All these features are characteristic of flight-to-safety and flight-to-liquidity.

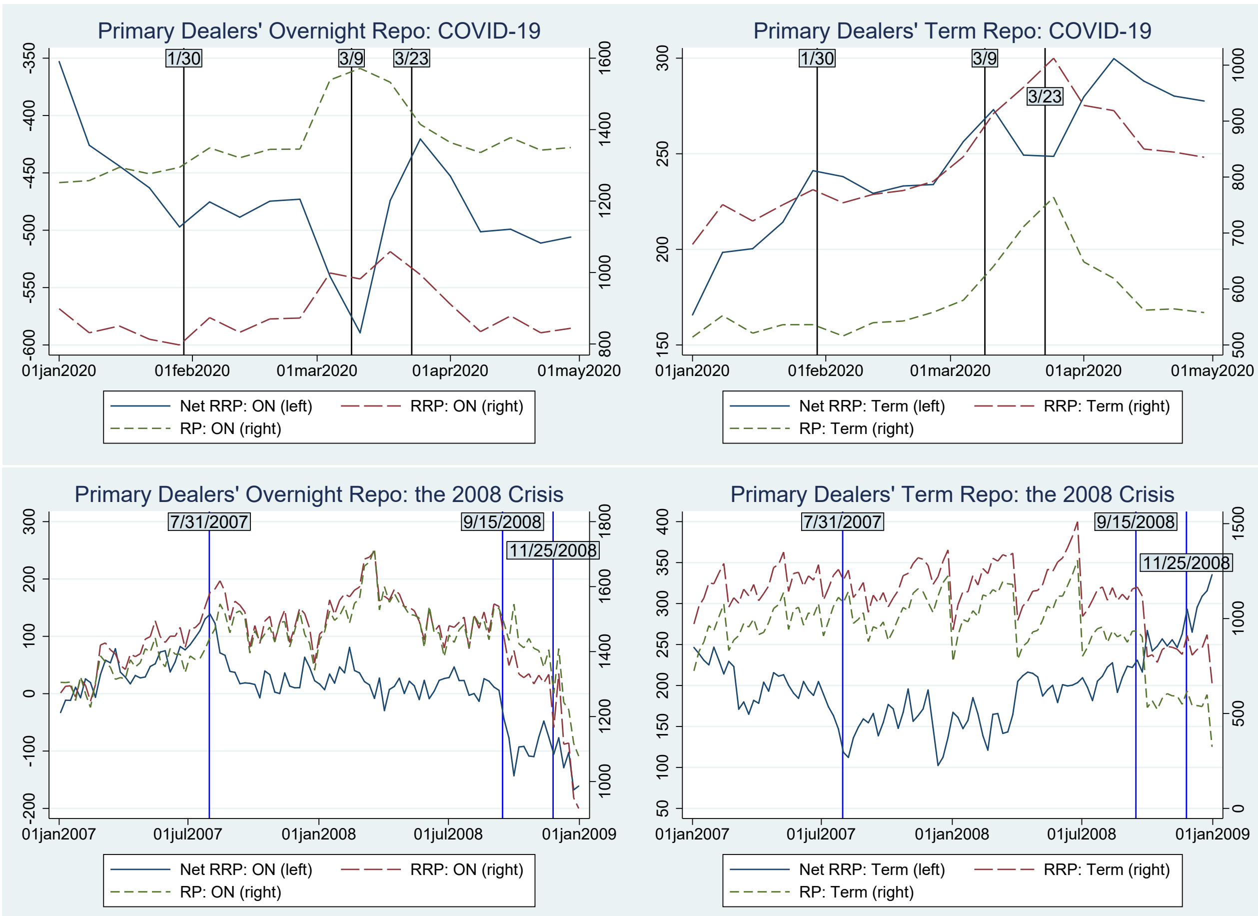
Second, Figure IA.2 provides a breakdown of primary dealers' repo and reverse repo amounts into overnight and term contracts. The net reverse repo amounts show that primary dealers conduct maturity transformations in their Treasury repo intermediation, borrowing overnight and lending term funds in both the COVID-19 and 2007–09 crises. Yet, the net borrowing amount through overnight repo is much larger in 2020 than in 2007–09, while the net lending amount through term reverse repo is similar in these two time periods. That is, primary dealers become greater net cash borrowers in the repo market post-crisis.

Figure IA.1: **Fund Flows of MMFs and Banks during the COVID-19 and 2007–09 Crises**



Notes: The top panels plot weekly series of the flows of money market funds (MMFs). The middle panels plot weekly series of the the amounts of deposits of commercial banks. The bottom panels plot weekly series of the amounts of commercial and industrial loans (C&I), cash assets, Treasury and agency securities, and funds lent in the federal funds and repo markets on the asset side of commercial banks. The units are all in billions of U.S. dollars. The sample period is from January 1, 2007, to December 31, 2008, for the left panels, and from January 1, 2020, to April 30, 2020, for the right panels.

Figure IA.2: **Breakdown of Primary Dealers' Treasury Repo Positions**



Notes: This figure plots weekly series of the primary dealers' gross repo amount, gross reverse repo amount, and net reverse repo amount for the overnight (left panels) and term (right panels) contracts. The sample period is from January 1, 2020, to April 30, 2020, for the top panels, and from January 1, 2007, to December 31, 2008, for the bottom panels. The units are all in billions of U.S. dollars.