ON THE FLIPSIDE: REFINEMENTS OF POLYTOPAL SUBDIVISIONS AND SECONDARY POLYTOPES

MAXINE CALLE

ABSTRACT. Given all the ways to triangulate the vertex set V of a polytope, we can construct the secondary polytope $\Sigma(V)$. The combinatorial structure of the secondary polytope neatly corresponds to regular subdivisions of V. We outline previous work done in this area relating to refinements of subdivisions, splits, and flips. We demonstrate this relationship via some examples of secondary polytopes.

1. Introduction and Background

The main object of our study, a *(convex) polytope* P can be defined as the convex hull

$$P = conv(V) = \{x \in \mathbb{R}^d \mid x = \sum_{i=1}^{n} \lambda_i v_i,\ \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0 \text{ for } i = 1, \ldots, n\},$$

for a finite point set $V = \{v_1, \ldots, v_n\}$ in Euclidean $\mathbb{R}^d$. We assume that $P \subseteq \mathbb{R}^d$ is full-dimensional, and since our study is limited to convex polytopes, we drop the prefix. A *face* of P is its intersection with some hyperplane that does not intersect the relative interior of P, and the dimension of the face is the dimension of its affine hull. Recall that the faces of dimension $0, 1, d-1, d$ are called vertices, edges, ridges, and facets, respectively.

The task of analyzing and (if possible) visualizing such objects is no easy one, and so it is natural to try to find ways to decompose polytopes into smaller, easier-to-handle pieces. Being particularly fond of triangles, we might subdivide P into simplices (recalling that a k-simplex, denoted $\triangle_k$, is defined as the convex hull of $(k+1)$ affinely independent points). Such a decomposition is known as a *triangulation*, and a particular method of construction gives rise to *regular* (or *coherent*) triangulations. Although we could triangulate arbitrary point sets, for our purposes we take $V = vert(P)$. Then by triangulating V, we in fact triangulate the polytope P itself. The more general investigations of triangulations of point sets arise in the study of optimization, topological combinatorics, computational geometry, and algebraic topology (see [9, Chapter 1] and [12, Section 2] for more discussion).

There are many interesting questions we could ask about the triangulations of a vertex set V, and the studying the space of triangulations of a vertex set V has provided a basis for many investigations in polytope theory. In particular, from the set of all regular triangulations of V, we can construct a new polytope $\Sigma(V)$ which is known as the *secondary polytope*, where each vertex corresponds to a triangulation. This class of polytopes was developed by Gelfand, Kapranov, and Zelevinsky around 1989, stemming from their work in generalized hypergeometric functions, discriminants, and resultants related to toric varieties [3]. Following this original work came alternative constructions of secondary polytopes, including the generalized class of *fiber polytopes* investigated by Billera, Filliman, and Sturmfels [1, 2].

Given its construction, $\Sigma(V)$ has interesting combinatorial properties. The face lattice of $\Sigma(V)$, where the partial ordering of faces is determined by inclusion, is isomorphic to the refinement poset of regular subdivisions of V. So the lower-dimensional faces of $\Sigma(V)$ correspond to the more refined subdivisions of V, with the correspondence between vertices and triangulations on the lowest level.

To better understand the structure of the secondary polytope we look at particular subdivisions of V. There is an edge between two vertices in $\Sigma(V)$ precisely when there is a *geometric bistellar flip*, or just *flip* for short, between the two corresponding triagulations of V. The idea of "links" between triangulations was introduced by Gelfand, Kapranov, and Zelevinsky in their original work, and the ensuing exploration has been taken up by many, particularly Santos [12]. From here we can create the *flip-graph*, whose vertices are triangulations and edges are the flips between them. A somewhat surprising result from [9] is that although the graph of $\Sigma(V)$ will always be contained in this flip-graph, the two are not necessarily isomorphic.

Alternatively, we can describe some facets of $\Sigma(V)$ by *splits* of V, as investigated by Herrmann in [4]. These splits correspond to the coarsest subdivisions of V, at the upper level of the refinement poset. The structural connection between faces of the secondary polytope and the subdivisions of V is a rich and interesting area of study.

Especially in lower dimensions, this structural relationship is well-understood. In higher dimensions, things start to get more complicated. Not much is known outside of specific examples, and even seemingly-accessible cases can present difficult puzzles. For example, the 4-cube has $87,959,448$ regular triangulations [9], meaning it has a secondary polytope with just as many vertices. Even the task of describing all regular triangulations of higher dimensional polytopes is a difficult one. Active areas of research include the connectivity and diameters of flip-graphs, and it is known that in higher dimensions there are point sets with disconnected flip-graphs and triangulations without flips at all [12, 9].

This paper seeks to outline and highlight work relating regular subdivisions and secondary polytopes. In Section 2, we introduce polytopal subdivisions and triangulations and outline the original construction of the secondary polytope. Section 3 investigates the connection with refinements of subdivisions, flips, and splits. Finally, Section 4 explores the well-known example of the n-gon and $(n-3)$-associahedron, before investigating the crosspolytope.

2. Triangulations and the Secondary Polytope

Just as there is more than one way to skin a cat, there is more than one way to "chop up" a given polytope. The particular subdivisions we are interested in decompose the polytope into smaller polytopes of the same dimension, with the idea that we could stitch them back together to get our original polytope again. As stated earlier, we take V to be the vertex set of some polytope, although the definitions can work to characterize subdivisions of more general point configurations. Recall that we assume that if V has n points in d

dimensions (corresponding to a d-dimensional polytope with n vertices), the dimension of the affine hull of V is d.

Definition 2.1. A *(polytopal) subdivision* of a finite point set $V \subseteq \mathbb{R}^d$ is a finite collection of d-dimensional polytopes $\mathcal{T}$ such that

(i) For any $\sigma, \sigma' \in \mathcal{T}$, the intersection $\sigma \cap \sigma'$ is a (possibly empty) face of both,
(ii) The union $\bigcup_{\sigma \in \mathcal{T}} \sigma = conv(V)$, so $\mathcal{T}$ covers $conv(V)$.

The elements of a subdivision are called *cells*.

Note that the first two properties are the more general definition of a polytopal complex. The subdivision is a *triangulation* when all the cells are simplices, and the *trivial subdivision* is $\{conv(V)\}$. Our restriction of V as a vertex set of a polytope requires all points of V to be used as vertices in the subdivision, although this condition need not hold in the general definition.

Most of the work that has been done on secondary polytopes has used regular triangulations, which are constructed using a lift function $\omega : V \to \mathbb{R}$ that assigns a weight to each element of V. Call the convex hull of this lifted point configuration $conv(V_\omega) \subseteq \mathbb{R}^{d+1}$. A *lower face* of this polytope is a face defined by a (non-vertical) hyperplane H such that $conv(V_\omega)$ is above H. More intuitively, the lower faces are the ones we would "see" if we were standing below $conv(V_\omega)$ and looked up. A *regular* triangulation (or subdivision, more generally) is one that can be obtained by projecting this set of lower faces via the canonical projection map that "forgets" the last coordinate:

$$\pi(x_1, \ldots, x_d, x_{d+1}) \mapsto (x_1, \ldots, x_d).$$

Regular triangulations are often the prime examples of triangulations due to their (relatively) easy construction, and it is known that every point configuation has regular triangulations. Although there are many polytopes that have both regular and non-regular triangulations (see the section dedicated to "The Mother of All Examples" in [9]), for the construction of secondary polytopes we consider only regular triangulations (unless explicitly stated otherwise).

Definition 2.2. Let $\mathcal{T}$ be a triangulation of $V = \{v_1, \ldots, v_n\}$. For each $v_i \in V$, let $\mathcal{T}_i = \{\sigma \in \mathcal{T} \mid v_i \in \sigma\}$. The *Gelfand-Kapranov-Zelevinsky vector of* $\mathcal{T}$, or *GKZ-vector*, is defined as

$$v_{\mathcal{T}} = \sum_{i=1}^{n} \sum_{\sigma \in \mathcal{T}_i} vol(\sigma) e_i$$

where e_i is the i^{th} standard basis vector.

So the j^{th} component of a GKZ-vector is the sum of the volumes of simplices (in the triangulation) that v_j is a vertex of. After computing the GKZ-vector across all triangulations, we form the secondary polytope as their convex hull. By this construction, scaling V (and so changing the volumes of the simplices in its triangulation) will not affect the combinatorics of the secondary polytope.

Definition 2.3. The *secondary polytope* of V is defined as

$$\Sigma(V) = conv(\{v_{\mathcal{T}} \mid \mathcal{T} \text{ triangulation of } V\}) \subset \mathbb{R}^n.$$

By definition, $\Sigma(V)$ lives in $\mathbb{R}^n$, however it will not be full-dimensional, given the affine relations among its vertices. For example, all the GKZ-vectors will have the same coordinate sum, so lie in a common hyperplane. The $d+1$ total affine relations between the GKZ-vectors means that the secondary polytope can live in only $n-(d+1)$ dimensions, and the main result of Gelfand, Kapranov, and Zelevinsky establishes this fact.

Theorem 2.4 ([3]). *For $V \subset \mathbb{R}^d$ a configuration of n points, the secondary polytope $\Sigma(V)$ has the following properties:*

(i) $\dim(\Sigma(V)) = n - d - 1$,
(ii) *Vertices of $\Sigma(V)$ are in bijection with the regular triangulations of V.*

Proof. See [3] for the original proof, and [1] or [9] for alternative proofs. □

One of the alternative constructions of the secondary polytope given by Billera, Filliman, and Sturmfels [1], describes it as a special case of a *fiber polytope*. This more general class of polytopes gives rise to the secondary polytope as the affine projection $\pi : \triangle_n \to conv(V)$ that bijects the vertices of the n-simplex to V. This is all to say that despite its algebraic origin, this polytope is of independent geometric and combinatorial interest.

3. Refinements, Flips, and Splits

With this constructive understanding of secondary polytopes, we examine the structure of the set of all regular subdivisions of V from a different perspective. Our main tools in this endeavor are *refinements* and the resulting *refinement poset*. Again, although the general definition works for non-regular subdivisions and their refinements, we restrict ourselves to considering regular polyhedral subdivisions. Consequently, in this context the refinement poset deals strictly with regular subdivisions and refinements.

Definition 3.1. Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are two subdivisions of V. We say $\mathcal{T}_2$ *refines* $\mathcal{T}_1$, denoted by $\mathcal{T}_2 \preceq \mathcal{T}_1$, if for all $\sigma_1 \in \mathcal{T}_1$ there exists $\sigma_2 \in \mathcal{T}_2$ such that $\sigma_2 \subseteq \sigma_1$.

A refinement means roughly that some cells of the subdivision are subdivided further. We say a subdivision is *coarsest* if it refines the trivial subdivision and no other, whereas triangulations are called the *finest* subdivisions. A subdivision whose only refinements are triangulations is called an *almost-triangulation*, as exemplified in Figure 1 below. Every almost-triangulation has exactly two proper refinements, both triangulations [9].

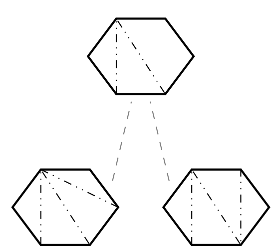

Figure 1. An almost-triangulation of a hexgon and its two refinements.

The refinement relation partially orders the set of all subdivisions of V. This refinement poset, denoted $Subdivs(V)$, can get very large very quickly, since higher dimensional polytopes will have many possible subdivisions each of which will have many possible refinements. Building off of Figure 1, Figure 2 shows a subset of the refinement poset for the hexagon, with a coarsest subdivision at the top and its triangulations at the bottom, with the almost-triangulations in between. Even though there are relatively few vertices for this polygon, the refinement poset is rather sizeable, with 14 triangulations, 21 almost-triangulations, and 9 coarsest subdivisions.

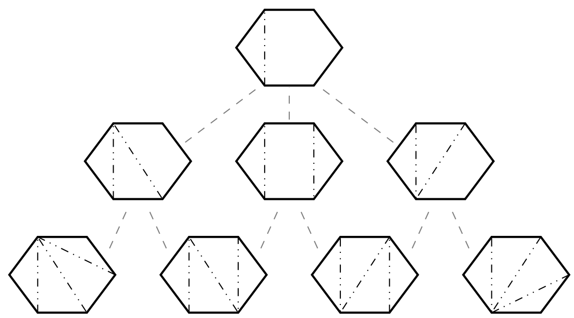

Figure 2. A subset of the refinement poset for the hexagon.

Lemma 3.2 ([9]). *Let $Subdivs(V)$ be a refinement poset. Then*

(i) *There is a unique maximal element, the trivial subdivision $\{conv(V)\}$,*
(ii) *A subdivision is a minimal element of $Subdivs(V)$ if and only if it is a triangulation.*

Proof. For (i), every cell of every subdivision is contained in the trivial subdivision, so every subdivision refines the trivial one.
For (ii), let $\mathcal{T}$ be a triangulation of V and $\mathcal{T}' \preceq \mathcal{T}$. So for any cell $\sigma' \in \mathcal{T}'$, there is some $\sigma \in \mathcal{T}$ such that $\sigma' \subseteq \sigma$. Since the vertices of σ are affinely independent, σ' is a face of σ. Then by property (i) of Definition 2.1, $\sigma' \in \mathcal{T}$. So for all $\sigma' \in \mathcal{T}'$, $\sigma' \in \mathcal{T}$ so $\mathcal{T} \preceq \mathcal{T}'$. Then by the definition of a poset, $\mathcal{T} = \mathcal{T}'$. For the converse, if we select a lift function ω that is sufficiently generic, then the subdivision produced by that function is a triangulation (see [9, Lemma 2.3.15]). Now, let $\mathcal{T}$ be some subdivision of V. For each $\sigma \in \mathcal{T}$, let $\omega : \sigma \to \mathbb{R}$ be sufficiently generic. Then the subdivision of σ induced by ω is a triangulation and so by taking the union across all $\sigma \in \mathcal{T}$, we get a subdivision of V whose cells are simplices. So every subdivision can be refined to a triangulation. □

Given that each vertex of $\Sigma(V)$ corresponds to a minimal element in $Subdivs(V)$, there clearly is some link between the two objects. A somewhat-indirect proof establishes the beautiful structural relation between the secondary polytope and the polytopal subdivisions of the primary polytope.

Theorem 3.3 ([9]). *The refinement poset $Subdivs(V)$ is ismorphic to the face lattice of the secondary polytope (removing the empty face from the face lattice).*

Although this theorem may seem obvious once we sit with it long enough, the proof is indirect and involves constructing a polyhedral fan related to regular subdivisions of V and showing that it is in fact the normal fan of the secondary polytope. We refer the interested reader to Section 5.2 in [9].

Building on Theorem 2.4, we see from Theorem 3.3 that the edges of $\Sigma(V)$ are in bijection with the almost-triangulations of V, and the facets with the coarsest subdivisions. Given this neat correspondence between $\Sigma(V)$ and $Subdivs(V)$, we might ask how we can describe the faces of $\Sigma(V)$ with respect to subdivisions of V. Two operations that subdivide V, known as *flips* and *splits*, can help us to better understand the edges and facets of $\Sigma(V)$, respectively.

Geometric bistellar flips are minimal changes between triangulations, and have been used in both theoretical and applied studies [12]. We can understand flips as elementary changes between refinements, where the elementary change occurs within a minimally affinely dependent subset (or *circuit*) of V [9, Section 4.4]. An equivalent definition characterizes flips in terms of almost-triangulations.

Definition 3.4. An almost-triangulation $\mathcal{T}$ is a *(geometric bistellar) flip* between two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ if these are the only two triangulations refining $\mathcal{T}$. We say $\mathcal{T}_1$ and $\mathcal{T}_2$ are *connected by a flip* supported on $\mathcal{T}$.

We can reinterpret Figure 1 in terms of flips:

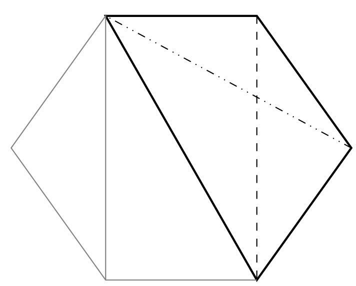

Figure 3. An almost-triangulation of a hexagon as a flip between two triangulations. The minimal change between the two triangulations is illustrated by the broken lines within the minimally affinely dependent set on the right side of the hexagon.

The flip relation between triangulations can be interpreted as the adjacency relation on $Subdivs(V)$, and it is clear then that flips correspond to the almost-triangulations of V. This result allows us to construct a graph for the triangulations of V.

Definition 3.5. The *flip-graph* of V, denoted $\mathcal{G}_{\mathcal{T}}(V)$, is the graph whose nodes are all the triangulations of V and whose edges are flips between them.

It seems intuitive that the graph of $\Sigma(V)$ would be isomorphic to the flip-graph of V, and indeed this statement was generally believed to hold, but a surprising result of de Loera, Rambau, and Santos in shows it to be false in general [9, Section 5.3]. There can be *non-regular* flips between triangulations that will not appear as edges of the secondary polytope. The secondary polytope is only defined in terms of *regular* triangulations, so will not "see" the non-regular flips that are edges in the flip-graph. This implies that the graph of the secondary polytope $\Sigma(V)$ is *not always* equal to the flip-graph of V, although it will appear as a spanning subgraph.

Theorem 3.6 ([9]). *The graph of the secondary polytope $\Sigma(V)$ is contained in the flip-graph of V.*

The flip-graph will contain everything in the lower two levels of $Subdivs(V)$, and so as a subposet of $Subdivs(V)$ (isomorphic to the face lattice of a polytope), it follows from Balinski's Theorem that this graph must be connected.

Corollary 3.6.1. *The flip-graph of* $V = \{v_1, \dots, v_n\} \subseteq \mathbb{R}^d$ *is* $(n-d-1)$*-connected, so every triangulation of* V *has at least* $n-d-1$ *flips.*

As opposed to looking at the lower levels of $Subdivs(V)$ and lower-dimensional faces of $\Sigma(V)$, we could investigate the facets of the secondary polytope. This question was taken up by Herrmann and Joswig in [5], using *splits* to describe coarsest subdivisions.

Definition 3.7 ([5])**.** A *split* of V is a subdivision S which has exactly 2 maximal cells, denoted S_+ and S_-. The *split-hyperplane* H_S is the span of $S_+ \cap S_-$.

Since S will not introduce new vertices, H_S does not intersect any edge of $conv(V)$ at its relative interior. Similarly, any hyperplane which separates $conv(V)$ without separating an edge will define a split. All splits of V are the "simplest", regular, non-trivial subdivisions, and so reside in the top level of the refinement poset. Given a lift function, there is a specific split-generated linear inequality that defines a facet of $\Sigma(V)$.

Proposition 3.1 ([5])**.** *Let* S *be a split of* V *generated by the lift function* $\omega_S : V \to \mathbb{R}$. *The facet-defining inequality for* $\Sigma(V)$ *is given by*

$$\sum_{v_i \in V} \omega_S(v) x_i \geq (d+1) \sum_{\sigma \in S} vol(\sigma) \overline{\omega}_S(c_\sigma),$$

where $\overline{\omega}_S : conv(V) \to \mathbb{R}$ *is the extension of* ω *to* $conv(V)$ *and* c_σ *is the centroid of* σ.

Each split of V determines a facet-defining inequality for $\Sigma(V)$, however it is not necessarily the case that all coarsest subdivisions of V are splits. In fact, the condition that all subdivisions of V are refinements of splits is called *totally splittable*. The secondary polytope can be defined using the inequalities for splits if the *oriented matroid* of V meets certain criteria.

Definition 3.8. Define an equivalence relation between point configurations that have the same (up to relabeling) sets of circuits. The *oriented matroid* of V is the equivalence class of V with respect to this relation.

Theorem 3.9. *The secondary polytope* $\Sigma(V)$ *can be defined by the inequalities given by splits if and only if* V *has the same oriented matroid as a simplex, crosspolytope, polygon, prism over a simplex, or a (possibly multiple) join of these polytopes.*

The interested reader is encouraged to see [9, Section 5.3.3] or [4, 5] for a more in-depth treatment. Using splits and flips, we can further characterize the secondary polytope and explore the fascinating connection to subdivisions of a vertex set V and their refinements.

4. Examples

Fascinating as secondary polytopes are, there are not many well-understood examples, especially for higher dimensions. Prime examples will likely come from the classes of primary polytopes whose triangulations can be elegantly characterized. In their original work, Gelfand, Kapranov, and Zelevinksy give several such examples, including the characterization of secondary polytopes for convex polygons [3].

Example 4.1. The secondary polytope of the n-gon can be realized as the $(n-3)$-dimensional associahedron [3, 6]. In this case, the graph of $\Sigma(V) \cong G_{\mathcal{T}}(V)$, since every triangulation will have exactly $n-3$ flips, each of which is associated to an internal diagonal [9, Section 1.1]. Taking $n = 6$, the secondary polytope of the hexagon is the 3-dimensional associahedron, illustrated in Figure 4.

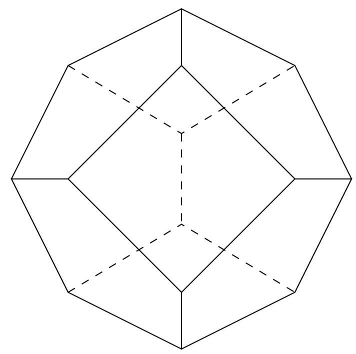

Figure 4. The 3-dimensional associahedron.

Since the full equality of Theorem 3.3 holds in this case, we can describe each face of the associahedron using elements from the refinement poset. Similarly, by Theorem 3.9, every facet of the associahedron is described by an inequality given by some split of the n-gon. Figure 5 below shows the upper-left facet of the associahedron from Figure 4 described by some subdivisions of the hexagon (some of which the observant reader might recognize from Figure 2).

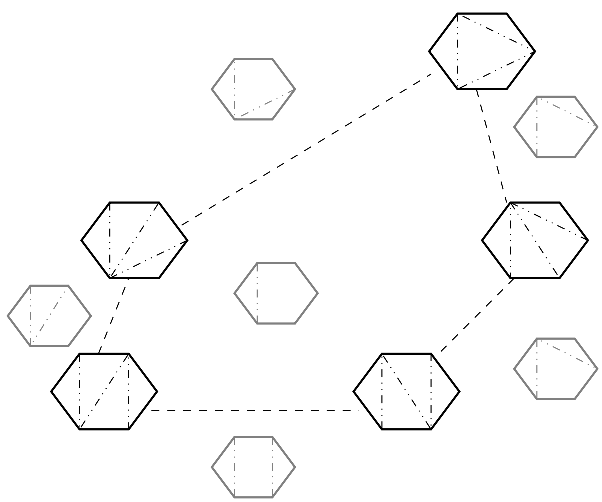

Figure 5. Facet of the associahedron labeled with subdivisions of the hexagon.

This duo is quite famous, at least in the realm of secondary polytopes. The associahedron was initially studied by Stasheff in relation to the theory of loop spaces, and so they are also referred to as *Stasheff polytopes*. The question of whether this complex could be realized floated around for some years, before a construction given by Lee [6], in addition to the work done by Gelfand, Kapranov, and Zelevinsky.

Example 4.2. Our second example is one that was not discussed in any sources we could find, but still lends itself quite nicely to the theory: the d-crosspolytope.

Definition 4.3. The *crosspolytope* is defined by

$$\diamond_d = conv(\{\pm e_1, \ldots, \pm e_d\}) \subset \mathbb{R}^d.$$

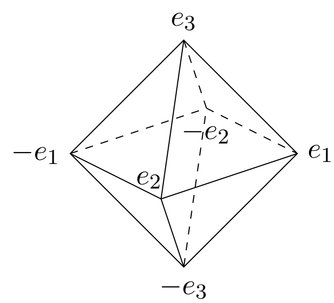

Figure 6. The 3-dimensional crosspolytope $\diamond_3$ with labeled vertices.

The crosspolytope is a simplicial polytope, meaning that each face is a simplex. Without loss of generality, we can index the vertices by ordering $v_i = e_i$ for $1 \leq i \leq d$ and $v_j = -e_{j-d}$ for $d+1 \leq j \leq 2d$. Note that there are $d2^{d-1}$ possible simplices on the vertices of $\diamond_d$. For each of these simplices, there is exactly one index $i \in \{1, \ldots, d\}$ such that e_i and $-e_i$ are vertices. So the vertex set of each simplex must contain e_i and $-e_i$ for some i and either e_j or $-e_j$ (but not both) for all $j \neq i$. We can use an indicator variable of sorts to tell us which of e_j or $-e_j$ is included, by taking $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{i-1}, \varepsilon_{i+1}, \ldots, \varepsilon_d) \in [-1, 1]^{d-1}$. Considering all such ε for a fixed i gives a complete set of simplices in a triangulation. Ranging across the indices i gives the set of all triangulations of $\diamond_d$.

Proposition 4.1. *All triangulations of the standard crosspolytope have the form*

$$\mathcal{T}_i = \left\{ conv(\{e_i, -e_i\} \cup \{(\varepsilon_j e_j \mid j \neq i\}) \mid \varepsilon \in [-1, 1]^{d-1} \right\}.$$

The crosspolytope has exactly d triangulations, corresponding to the d possible selections of which $\pm e_i$ to fix in the triangulation.

Theorem 4.4. *The secondary polytope of $\diamond_d$ can be realized as the $(d-1)$-simplex*

Proof. To prove the result, we simply need to prove the resulting GKZ-vectors are affinely independent. To do so, we first characterize the GKZ-vectors of the crosspolytope. Note that by the high degree of symmetry of the cells within possible triangulations of the standard crosspolytope, each simplex in the triangulation will have the same volume. As we could scale each GKZ-vector by the inverse of the volume, the j^{th} component of a GKZ-vector merely counts how many simplicies the given j^{th} vertex appears in. Each triangulation of $\diamond_d$ will have 2^{d-1} d-simplices. Fix some triangulation $\mathcal{T}_j$, whose cells include both $\pm e_j$. So the j^{th} and $2j^{th}$ components of $v_{\mathcal{T}_j}$ are 2^{d-1}. Further, each of the remain e_i for $i \neq j$ will appear in exactly half of the simplices. So all other coordinates of $v_{\mathcal{T}_j}$ are 2^{d-2}. Without loss of generality (since we only care about combinatorial properties), we can scale the GKZ-vectors by $\frac{1}{2^{d-2}}$. The resulting vectors are of the form $(1, \ldots, 1, 2, 1, \ldots, 1, 2, 1 \ldots, 1)$ with 2's in the j and $2j$ slots. Now suppose there exist

$\lambda_1, \ldots, \lambda_d$ such that $\sum_{i=1}^{d} \lambda_i = 0$ and $\sum_{i=1}^{d} \lambda_i v_{\mathcal{T}_i} = 0$. Considering an arbitrary k^{th} component of the sum, there is exactly one GKZ-vector $v'_{\mathcal{T}}$ with a 2 in that slot, and all others will have a 1 in that slot. So then $0 = \sum_{i=1}^{d} \lambda_i + \lambda' = \lambda'$. This implies that $\lambda_i = 0$ for all $i = 1, \ldots, d$. So the GKZ-vectors are affinely independent, and hence their convex hull is a simplex. □

This polytope will be a $(d-1)$-simplex living in $\mathbb{R}^{2d}$, which can easily be projected down to its natural habitat in $\mathbb{R}^{d-1}$. Having characterized all triangulations and flips between them, we can again conclude the full equality of Theorem 3.3 holds in this case. So we can realize the flip-graph as isomorphic to the graph of the d-simplex, as shown for $d = 3$ in Figure 7.

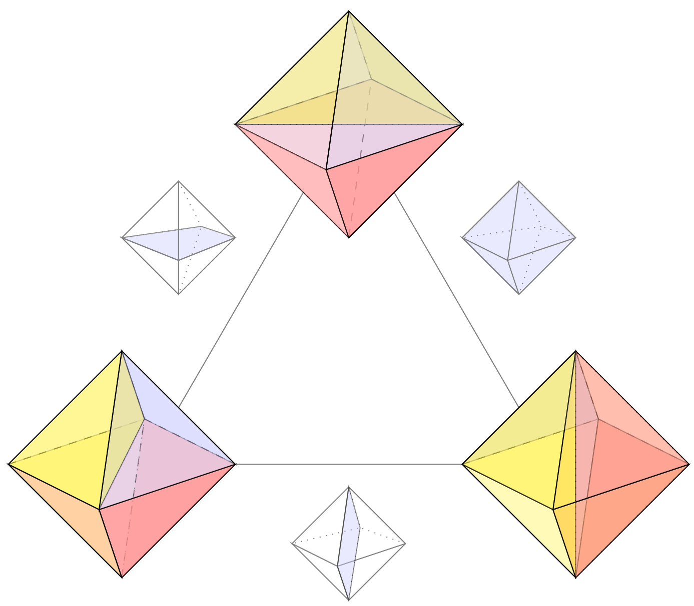

Figure 7. The flip-graph of $\diamond_3$, isomorphic to the graph of $\Sigma(\diamond_3) \cong \triangle_2$.

Further, $\Sigma(\diamond_d)$ clearly has the same oriented matroid as a simplex, so the secondary polytope can be completely described by facet-defining inequalities given by splits. Following [5], we can explicitly calculate the inequality defined by the coarsest subdivision located at the bottom of Figure 7.

This coarsest subdivision gives the split $S = \{S_+, S_-\}$ where $S_+ = conv(\{e_1, \pm e_2, \pm e_3\})$ and $S_- = \{-e_1, \pm e_2, \pm e_3\})$. The split hyperplane is $H_S = \{x \in \mathbb{R}^3 \mid x_1 = 0\}$, with the normal vector $n = e_1$. Define a weight function $\omega_S : V \to \mathbb{R}$ given by

$$\omega_S(v) = \begin{cases} |n \cdot v| & \text{if } v \in S_+; \\ 0 & \text{if } v \in S_-. \end{cases}$$

Note that this function is well-defined, since for $v \in H_S = S_+ \cap S_-$, $|n \cdot v| = 0$. With reference to Definition 3.7, the facet-defining inequality is given by

$$\sum_{v_i \in V} \omega_S(v) x_{v_i} \geq 4 \left(vol(S_+)\overline{\omega}_S(c_{S_+}) + vol(S_-)\overline{\omega}_S(c_{S_-}) \right).$$

Expanding the left side, note that $\omega(v_i) = 0$ for all $v_i \in S_-$, so

$$\sum_{v_i \in V} \omega_S(v) x_{v_i} = \sum_{v_i \in vert(S_+)} |n \cdot v_i| x_{v_i} = |e_1 \cdot e_1| x_{e_1} = x_{e_1},$$

since $|e_1 \cdot \pm e_j| = 0$ for all $j \neq 1$. Now for the right side: Extending ω to $conv(V)$, we get $\overline{\omega}_S = \omega_S$. Then $\omega_S(c_{S_-}) = 0$ and a straightforward symmetry argument gives $c_{S_+} = (\frac{1}{2}, 0, 0)$, so $\omega(c_{S_+}) = \frac{1}{2}$. As S_+ is a standard pyramid over a square of side $\sqrt{2}$, $vol(S_+) = \frac{2}{3}$. Combining these results, we have the inequality $x_{e_1} \geq \frac{4}{3}$. So the facet-defining inequality of $\Sigma(V)$ generated by this split is given by $F_S = \{x \in \mathbb{R}^V \mid cx \leq -\frac{4}{3}\}$ where $c = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. We could follow a similar process to compute the facet-defining inequalities for the other two splits to obtain the $\mathcal{H}$-description of the secondary polytope.

These two cases exemplify some of the fascinating structure underlying polytopal subdivisions. We hope to have convinced the reader that secondary polytopes are interesting combinatorial objects worthy of study. Further study might look at other examples where the set of subdivisions can be elegantly characterized, and investigate the secondary polytopes. It would also be interesting to develop a flip- or split-like method for the original polytope that would correspond to k-faces of the secondary-polytope.

References

1. L. Billera, P. Filliman, and B. Sturmfels, *Constructions and Complexity of Secondary Polytopes*, Advances in Math. **83** (1990). (155–179)
2. L. Billera and B. Sturmfels, *Fiber Polytopes*, Annals of Math., **135** (1992). (527–549)
3. I.M. Gelfand, M. Kapranov, and A. Zelevinksy, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhuser, Boston (1994). (214–251)
4. S. Herrmann, *On the Facets of the Secondary Polytope*, Journal of Combinatorial Theory, **118** (2011). (425–447)
5. S. Herrmann and M. Joswig, *Splitting Polytopes*, Mnster J. of Math. **1** (2008). (109–142)
6. C.W. Lee, *The Associahedron and Triangulations of the n-gon*, Europ. J. Combinatorics, **10** (1989). (551–560)
7. C.W. Lee, *Regular Triangulations of Convex Polytopes*, Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift, vol. 4 of DIMACS Series in Discrete Math. Theor. Comput. Sci., AMS, Providence (1991). (443–456)
8. C.W. Lee and F. Santos, *Subdivisions and Triangulations of Polytopes*, Handbook of Discrete and Computational Geometry, **3** (2017).
9. J. De Loera, J. Rambau, and F. Santos, *Triangulations: Structures for Algorithms and Applications*, Springer-Verlag, Berlin (2010).
10. G. Rota, *Indiscrete Thoughts*, Birkhuser, Boston (1997). (195–203)
11. F. Santos, *On the Refinements of a Polyhedral Subdivision*, Collect. Math., **52** (2001). (231–256)
12. F. Santos, *Geometric Bistellar Flips. The Setting, the Context, and a Construction*, International Congress of Mathematicians, **3**, Eu. Math. Soc., Zrich (2006). (931-962)
13. G. Ziegler, *Lectures on Polytopes*, Springer-Verlag, New York (1995).