A CASE FOR QUOTIENTING: EQUIVALENCE AND POSTMODAL MATHEMATICAL STRUCTURALISM

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1. Introduction

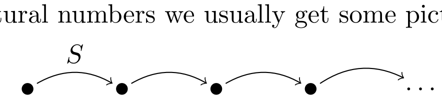
Structuralism has been quite a popular philosophy, particularly among philosophers of science and mathematics, but often structuralists are accused of being vague as to what, exactly, their metaphysical thesis is. The structuralist position is usually articulated by some statement along the lines of "*Patterns are primary, and the entities that appear in the pattern are secondary.*" Pushed a little harder, a structuralist might say that entities depend, metaphysically somehow, on the structures they appear in.

Many arguments for structuralism are advanced on epistemic grounds. We can only talk about properties like charge and mass in relation to each other, we might claim, and we only observe the pattern wherein they interact: *all we know about is structure.* Similarly, we might say that we do not know anything about the number 2 that its independent of its relations to other numbers, and further we are not really concerned with seeking such knowledge. As Michael Resnik [4, p.529] states,

> *For some time mathematicians have emphasized that mathematics is concerned with structures involving mathematical objects and not with the "internal" nature of the objects themselves...we are not given mathematical objects in isolation but rather in structures.*

Although we do not 'observe' mathematical entities in the same way we observe a massive or charged object, there is still this idea that all that really matters about the numbers is the structure that relates them to each other.

This is the claim of the mathematical structuralist: mathematical entities are just positions in structures. The structure of the natural numbers, for instance, can be characterized as an infinite series related by a successor relation. With such a conception of the natural numbers we usually get some picture like the following.



It does not matter to a structuralist which individuals we might put in place of these little black dots, nor which relation S is chosen over them. So long as the pattern is intact, we have the natural number structure, because all that really matters about the natural numbers is their structure.

But what does it mean to say the pattern is 'all there is' and what is it about reality that makes it so? What kind of ontology is the mathematical structuralist committed to, and what kind of metaphysical tools do they employ to articulate their thesis? Ted Sider [6] extensively pushes these types of questions, and seemingly frustrates the structuralist at every turn. Building off of Sider's work, we will seek to outline exactly what kind of problems a structuralist is facing, particularly in the context of articulating theoretical equivalence. Section 2 distinguishes between the three types of mathematical structuralism, given by Shapiro [5], each of which has its own distinctive ontological background. In Section 3, we trace Sider's argument for a transition from modal to postmodal metaphysics [6, Chapter 1], wherein he claims that a large failing of modal concepts is their inability to meaningfully articulate a mathematical structuralist thesis. We then examine theoretical equivalence in Section 4, and how it functions within the context of mathematical structuralism. In the following sections, we expand upon the debate between two extreme approaches to theoretical equivalence [6, Chapter 5]. We conclude that if forced to choose between the fundamentality-based approach or quotienting, the latter option is the mathematical structuralist's best shot.

2. Mathematical Structuralism

In order to get a good handle on the dilemmas the mathematical structuralist might face, we will first expand upon the general position. Although there is a lot of room for various interesting epistemic puzzles and discussions, our primary concern will be metaphysics. We mostly refer to Resnik [4] and Shapiro [5], who have done a lot of work in developing mathematics as a 'science of patterns.' Sider gives a brief discussion in the context of rejecting individuals [6, Section 4.7].

A classic question for philosophers of mathematics is figuring out what exactly numbers are. Answers to this puzzle are as classic as the philsophers who gave them, from Plato to Aristotle to Frege. The structuralist position is most often compared to a platonist one, wherein numbers are given some special immaterial existence outside of space and time. The natural numbers exist independently of the mathematician. However the traditional platonist and the structuralist might not always agree. The platonist might hold that the *essence* of the number 2 can be stated without invoking the number 6, but a structuralist would maintain that the essence of 2 is precisely its relations to the other numbers within the structure. The numbers cannot be independent of each other since they are not independent of the structure in which they are positioned. This idea is similarly articulated by Resnik [4, p.530]:

> *In mathematics... we do not have objects with an 'internal' composition arranged in structures, we have only structures. The objects of mathematics... have no identity or features outside of a structure.*

What are the ontological consequences of such a view? There are two issues here that demand our attention: we might ask about the ontological status of whole structures, like the natural number structure, and the status of places within those structures, the natural numbers themselves.

Regarding the first question, Shapiro distinguishes three major schools of thought regarding ontology. The first, ontological eliminitive structuralism (sometimes called "hardheaded" or *post rem* structuralism), thinks that there is no more to structures than the systems that exemplify them. What exists are systems, collections of objects with certain relations, which have structural features in common. Benacerraf is placed in this category, in accordance with his rejection of the thesis that numbers are objects [1]. Note that this position requires that we postulate that there are enough (abstract) objects to ensure that mathematics is non-vacuous, that is, at least a continuum many. For any structure we might want to have, for real analysis or Euclidean geometry, we must have a big enough ontology to ensure that a system can exemplify the legitimate structure.

Further, Shapiro argues that this robust background ontology will not be understood in structuralist terms. For suppose that we gave the background ontology of mathematics using the set-theoretic hierarchy. Then surely we would have enough objects to ensure our favorite mathematical fields were not threatened with vacuity. Then set theory would not be the theory of a particular structure, but about the background ontology. If we try to characterize set theory as the study of a particular structure, then we are stuck having to supply another background ontology. This new ontology cannot be taken as positions of some other structure, otherwise we are required again to produce a background ontology for *that* structure. And so it goes. Since the regress of systems and structure must stop somewhere, the final ontology of the hardhead will not be structural.

We might switch from talking about structures to just possible structures. The second school of thought, rigorously explored by Hellman [3], modalizes eliminitive structuralism. Hellman builds up number theory and real analysis from the supposition of the *logical possibility* of the existence of an infinity of atoms (either countable or uncountable, depending on how far we want to push it). From this logical possibility, he has the possibility of an appropriately non-vacuous structure,

$$\diamond \exists X \exists f\,(X \text{ under the relation } f \text{ exemplifies the natural number structure})\,.$$

While he seems to also implicitly assume the possibility of this 'relation variable' f that is meant to replace the ordering relation $<$, we get the logical possibility of a system exemplifying the natural numbers without needing to assume anything actual. Shapiro argues that modal eliminitive structuralists must account for a circularity between logical possibility and set theory, and Hellman's response is to take logical notions as primitive and so irreducible to set theory. In this account too, we have 'structuralism without structures' as the ultimate ontology is not structural.

The third view seems to be the only one that seeks a distinctly structuralist metaphysics. These *ante rem* structuralists (sometimes called 'mystical' or 'non-eliminitivist' structuralists), like Resnik and Shapiro, are committed to realism in ontology. Shapiro [5, p.89] describes this view as follows:

> *In mathematics, anyway, the places of mathematical structures are as bona fide as any objects are. So, in a sense, each structure exemplifies itself. Its places, construed as objects, exemplify the structure.*

On this view, structures have real but abstract and immaterial existence, regardless of whether they are exemplified in a nonstructural realm. We can see here too how the *ante rem* structuralist can answer the demand for the ontological status of mathematical objects: there really are these mathematical objects, these positions in structures, that exist in some abstract sense. Mathematical objects are tied to the structures that constitute them, so we commit to some sort of relativity in ontology.

Perhaps we might object that this view is committing to structures that bear an explanatory burden, as is sometimes bestowed upon *ante rem* universals. That is, a given system is a model of a structure because it exemplifies the structure. Shapiro responds to Michael Hand on this point, claiming instead that the system exemplifies the structure because it is a model of that structure. Shapiro favors this view over the other two as being the "most perspicuous and least artificial of the three" [5, p.90], and corresponds most to how mathematical theories are developed.

Sider raises two concerns regarding this view. He first questions whether *ante rem* structuralism really solves the problems typically faced by mathematical structuralists. "Consider the third position in the natural-number structure," he says, "Is it Julius Caesar? Is it a set? Is it identical to any of the positions in the rational-number structure? These questions seem perfectly well-formed, and out to have answers, if we take the talk of structures and positions in them at face value" [6, p.89]. These types of problems are serious classical puzzles for philosophers of mathematics, particularly for realists in ontology. The first was raised by Frege in *The Foundations of Arithmetic*, the second (and variations on it) by Benacerraf in his classic paper [1], and the third more generally in the context of cross-structural identification.

Shapiro's answer to this demand is to say that there is no answer to be discovered, since he says it makes no sense to ask about identity between a *place* in the natural number structure and some other object. Similarly, it only makes sense to expect determinant answers to questions regarding numerical relations, questions internal to the natural number structure. Asking whether 1 is an element of 4, he claims "Is similar to asking whether 1 is braver than 4, or funnier" [5, p.79]. Certainly, anything can 'be' 3, in the sense that it can occupy that place in a system exemplifying the natural number structure. The claim is that these types of questions do not have determinate answers, nor do they need them.

Whether or not we take Shapiro to adequately address Sider's point, we will turn our attention mostly to the cross-structural concerns. It seems wise to identify $2_{\mathbb{N}}$ with $2_{\mathbb{Q}}$, and Shapiro's commitment to ontological relativity might get him into a pickle here. If mathematical objects are tied to their structures, how can we identify them across different structures?

On one end of the spectrum, Benacerraf claims that identifying positions in different structures is always meaningless, and "Statements of identity contain the presupposition that the 'entities' inquired about belong to the same general category" [1, p.65]. Interestingly, Benacerraf seems to characterize categories in terms of 'well-entrenched' predicates. In order to establish whether two entities[1] belong to the same category, there must be some possible individuating conditions, some distinguishing predicates. He argues that for any two predicates F and G, if there is no predicate C, such that C subsumes the two predicates F and G and has the appropriate individuating conditions, then identity statements crossing F and G make no sense.

So Benacerraf is committed to saying that $2_{\mathbb{N}} \neq 2_{\mathbb{Q}}$, since we could differentiate between them via a predicate like 'has a multiplicative inverse.' Perhaps such a predicate is not appropriate, since the natural number structure does not have a concept of 'multiplicative inverse.' But arguably we could come up with predicates without this flaw.[2] In any case, Benacerraf accepts the fact that we cannot have identity across distinct structures.

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[1]It is not entirely clear here what qualifies as an 'entity.' It seems that an entity can be taken to be some sort of individual, although this is not precisely indicated beyond statements like "Everything purports to be at least [an entity]. 'Entity,' 'thing,' 'object' are words having a role in the language; they are *place fillers* whose function is analogous to that of pronouns" [1, p. 65-66, emphasis added]. It is interesting to use 'place filler' for both lampposts and numbers, but claim lampposts are objects and numbers are not.

[2]Maybe some predicate that directly references the structures themselves. For example, perhaps 'occupies the position in the structure such that there is exactly one position between it and that unique position which it not a successor of anything' would work, although I'm not entirely sure.

But it seems that mathematicians often do make such identifications, finding it convenient and sometimes enlightening to do so. For example, it is much easier to think of the complex plane as the Cartesian product of the reals. Consequently, we could formulate the standard basis for the vector space $\mathbb{C}^n$ in two different ways. If we think of $\mathbb{C}^n$ as a vector space over $\mathbb{C}$, we can span the space using n orthonormal standard basis of the form $(1, 0, \ldots, 0), (0, 1, \ldots, 0)$, and so on. Alternatively, we can take $\mathbb{C}^n$ as a vector space over the reals, and so use $2n$ basis vectors of the form $(1, 0, \ldots, 0), (i, 0, \ldots, 0), (0, 1, \ldots, 0), (0, i, \ldots, 0)$, and so on. Note that in each case we specify that $\mathbb{C}^n$ is taken as a 'vector space over a field,' so likely we are implicitly utilizing something like Shapiro's ontological relativity.

Returning to the most straightforward example, it seems that often mathematicians want to claim cross-structural identity between things like $2_{\mathbb{N}}$, $2_{\mathbb{Z}}$, $2_{\mathbb{Q}}$, $2_{\mathbb{R}}$, and $2_{\mathbb{C}}$. Shapiro himself claims that we should make such identifications in some cases such as this. He says, "The point here is that cross-identifications like these are matters of *decision*, based on convenience, not matters of discovery" [5, p.81], a statement which we will come back to.

The second concern *ante rem* structuralist (as according to Sider) is that a straighforward platonist philosophy of mathematics would give us a similar ontology, that structures (and positions in them) really exist. A platonist could believe that there is a unique set of natural numbers with a unique successor relation, yet describe this as the natural number structure, and claim that individual numbers are positions in this structure. Although we saw earlier that Shapiro might object to some formulations of platonism, in order to adequately distinguish a structuralist position, we need to say further what it means that mathematical entities are just positions in structures. To do so, Sider claims, we need a distinctively structuralist metaphysics. Now we can finally get ready to enter into the realm of postmodal metaphysics. Going forward, we will primarily consider *ante rem* structuralism, since this is the only position which requires a structuralist ontology.

3. Postmodal Metaphysics

The opening chapter of Sider's *Tools of Metaphysics and Metaphysics of Science* sets us up to dive into the deep seas of metaphysics. "If this book has a single thesis," Sider says, "it is that the choice of metaphysical tools matters to first-order metaphysics, especially when it comes to 'structuralist' positions in the metaphysics of science and mathematics" [6, p.3]. He argues that modal tools are not sufficient to properly characterize many metaphysical doctrines, including structuralism.

We can use the tools of possibility and necessity to articulate a modal structuralist thesis. Nodes cannot vary independently of the patterns, claims the modal structuralist (or patterns cannot vary independently of the nodes). There are no two possible worlds that share the same facts about the nodes but different facts about the pattern. For example, if we believe that individuals cannot vary independently of qualitative facts, there can be no world in which Sider has swapped places with Barack Obama. But is difficult to see how a modal articulation of mathematical structuralism would be helpful. We might try out the statement "The mathematical entities cannot vary independently of the structure they appear in." But recalling that we take mathematical entities to be just positions in a structure, this amounts to saying something like "The positions in the natural number structure cannot vary independently of the natural number structure," which seems to just be trivially true. Facts about mathematical entities are usually taken to be necessary, so the fact that the number 2 occupies its place in the natural number structure is true, necessarily. But this correspondence does not seem to follow from any priority of the natural number structure, but because we take mathematical facts to be necessarily true.

Sider takes this failing of modality to be indication that we should seek out better metaphysical tools to articulate our thesis. He claims that "A more satisfying statement of a structuralist position will no doubt *imply* a modal thesis, but that modal thesis would be due to some deeper non-modal thesis: nodes and patterns can't vary independently because nodes and patterns are tied together in some nonmodal way" [6, p.7]. A statement of structuralism in terms of modality can claim variations are impossible, but cannot explain *why* they are impossible, and so cannot conclude the impossibility arises from the structuralist nature of reality. Alongside other arguments regarding the inadequacy of modal tools, it seems we have more than enough reason to turn elsewhere.

Finding Sider's arguments convincing – as it seems we should – we now turn to postmodal concepts. Armed with the tools Sider gives us, like essence, ground, and fundamentality, we ask (nervously, perhaps) what kind of challenges we might face.[3]

Sider outlines three main obstacles for a postmodal structuralist. First off, there may be no coherent postmodal formulation of the structuralist thesis. If we are seeking an explanatory account of *why* we have "patterns without nodes," we cannot flat-footedly take the pattern to be facts involving nodes. This does not seem to be a big threat to the *ante rem* structuralist, as they are not claiming that there are *no* mathematical objects. Rather they could assert that 'what is ultimately going on' is that fundamentally, there are these objects, positions in structures, and there is a pattern that they are tied to (somehow!). Although this assertion likely would face

[3]For a more extensive discussion of the postmodal tools and the subsequent challenges for the structuralist, see Sider's brilliant first chapter [6]. Or, the whole book, really.

its own challenges, it would let the *ante rem* structuralist escape from the purely 'flat-footed' account.

Secondly, there may be a conflict with so-called 'postmodal logic.' If we appeal to ground, we might say "facts about the objects are grounded in facts about the pattern." This statement translates easily into the mathematical structuralist's lingo: "facts about the natural numbers (as positions in structures) are grounded in facts about the natural number structure." But, Sider argues, we can take facts about the pattern as existentially quantified facts about the nodes! And the logic of ground requires that existentials be grounded in their instances, not the other way around. So really the facts about the structure should be grounded in the facts about the objects. For example, something like the fact that the successor relation (or something similar) characterizes the natural number structure can naturally be rephrased as saying every object is the unique successor of some other object (besides one), no object is the successor of itself, and so on, until we have stated all the necessary conditions for the successor relation. How can we talk about grounding the relations between mathematical objects in terms of the structure they appear in when postmodal logic demands we do the opposite?

This does seem to be a serious question for Shapiro and friends. What grounds facts about the number 2 (e.g. that it is the only even prime)? As nothing more than a position in a structure, it seems these facts must be grounded in facts about the pattern. Perhaps we could say that the facts about the objects are equivalent to facts about the pattern, so maybe it makes no sense to talk about one grounding the other. Of course, this response hinges on our concept of "equivalence" – a tricky subject as we will soon see! So perhaps there is nothing better to do here than reject this additional imposed constraint from the logic of ground.

Sider's third complaint is that a reformulated structuralist thesis might be theoretically unattractive in a postmodal setting. For example, we might assert certain concepts as fundamental, but these concepts may be unable to meet epistemic conditions of fundamentality. It is difficult to see precisely how this third obstacle might show up for the mathematical structuralist, until we determine exactly what thesis we are working with.

In any case, Sider has presented quite the task for the postmodal structualist: somehow hold on to a distinctly structuralist metaphysics, wherein "structure is all there is," in this postmodal setting.

4. Theoretical Equivalence

Supposing we have not yet thrown up our hands and walked away from structuralism, we might return to some of the puzzles discussed earlier. Particularly, we might be interested to know when we have equivalent theories or structures, and what exactly it means when we do.

It seems that we have some intuitive ideas about what it means to have equivalent theories. We might say things like "A theory of mass in terms of grams can give a perfectly good description of the world, but we could have an *equivalent* theory in terms of kilograms!" or "We can describe the natural number structure using the successor relation, but we could describe it just as well using the less-than relation." What is a postmodal structuralist to make of such statements? What is it to say two theories are equivalent, in terms of a structuralist metaphysics?

Per usual any attempt to make our intuition more rigorous can lead us down rabbit holes of confusion. Following Sider, we might take a first stab at it and say that two theories (statements, models, representations) are *equivalent* when they represent the very same state of the world, such that any differences between them are notational or conventional.

Perhaps we could approach equivalence using symmetry. A symmetry is usually taken to be an operation on an object that leaves it unchanged in certain respects. So maybe we could say that two theories or representations of the world are equivalent when there is a symmetry map (of the laws) between them that maps solutions to the laws of nature to solutions, and non-solutions to non-solutions. But this condition may not be enough, since our symmetry could map between totally dissimilar worlds, so long as the solutions were properly preserved. Most philosophers who propose to work with symmetries posit some additional capital-X condition.

Dasgupta takes up this banner in his structuralist approach to the physics of spacetime [2]. Using 'dynamical symmetries,' he argues, we can show that variant quantities under these symmetries are undetectable. We are then justified in removing the variant quantities from our theory. That is, a new theory that does not include variant quantities will give an equivalent description as one that does. Although the new theory will leave out facts about the variant quantities, Dasgupta's argument claims that we should favor the theory with 'less structure' on epistemic principles. His argument can be paraphrased roughly as follows:

> *A quantity can be shown to be undetectable if it is variant under and operation that maps worlds to worlds such that the worlds agree on the laws, are observationally equivalent, and are abductively equivalent. Theories that differ only with respect to their variant (and consequently undetectable) quantities are equivalent.*

There are a couple things to note here. Most importantly, perhaps, this account does not tell us what equivalence *is*. It is difficult to see how we could say what it means to be equivalent in terms of such symmetries. A symmetry account could help us with epistemic questions of equivalence. We seem to have good reason to think two theories are equivalent if they are related by some symmetry, and symmetry is likely a necessary consequence of equivalence. But it certainly does not seem to be sufficient. As mentioned earlier, without Dasgupta's 'observationally equivalent' capital-X condition, why think we could not get a symmetry that gives us a very dissimilar world? Even if we posit these extra conditions, we still run into problems in trying to characterize equivalence in terms of symmetry. First of all, Dasgupta's conditions involve the notion of equivalence already. Dasgupta himself notes "I leave open whether the definition contains this condition of abductive equivalence explicitly, or whether it contains some other condition from which abductive equivalence follows" [2, p.7, n.14]. To avoid sneaking in any hidden notion of equivalence, perhaps we toss the abductive condition and say "Well, what we really meant by observationally equivalent was perceptibly indistinguishable." But we run into trouble here as well, as Sider points out. By positing this additional capital-X condition, we are placing it outside of convention [6, see p.144].

All this to say, we should set symmetry aside in our quest for an answer to our metaphysical questions about equivalence. Plus, we want to talk about mathematical structures, not structures about the laws of nature generally, so maybe we should turn to stronger mathematical mappings. When mathematicians talk about different entities being equivalent, like in graph theory or combinatorial geometry, they use words like 'congruence' and 'isomorphism.' So perhaps we could look here for answers.

Many mathematical structuralists discuss structure equivalence or pattern equivalence in such terms. Two patterns are congruent if there is an isomorphism between them, that is, a relation-preserving bijection from the objects and relations of one to the objects and relations of the other. However both Resnik and Shapiro conclude that isomorphism / pattern congruence is too strong a condition. On this condition, we would have to conclude that structures like $\mathbb{N}$ with addition and multiplication and $\mathbb{N}$ with addition, multiplication, and less-than are not equivalent, as they are not isomorphic (with different sets of relations), although it seems that they do in fact exemplify the same structure.

Resnik turns to the idea of *structural equivalence*, which formulizes the intuitive idea that "Two patterns are like 'essentially the same' if they both encapsulate some 'bigger' pattern from which they can be obtained by deleting some of its relationships" [4, p.536]. Paraphrasing from Shapiro's charactization of this equivalence [5, p.91], we define P to be a *full subsystem* of a system R if they have the same objects[4] and if every relation of R can be defined in terms of the relations of P.

[4] Here I am not sure if Shapiro means 'same' in the way we usually take 'same' to mean as identity between individuals, or something more like 'same number of.'

So $\mathbb{N}$ with addition and multiplication is a full subsystem of $\mathbb{N}$ with addition, multiplication, and less-than, since the only difference is the omission of the definable relation 'less-than' in the former. We say that S_1 and S_2 are *structure equivalent* if there is a system S such that S_1 and S_2 are each isomorphic to full subsystems of S.

But here we run into the same problems as we did with symmetry. It seems that structural equivalence is a good candidate to tell us when systems have 'sameness of structure,' but it is hard to see how this characterization actually tells us anything about what equivalence *is*, and what it is for mathematical theories or structures to be saying the same thing about the world.

To continue our investigation, we delve back into general metaphysics with Sider. He presents two extreme approaches to equivalence: fundamentality and quotienting. Most of the following discussion will paraphrase Sider's work [6, Sections 5.3-5.11], although we wish to conclude (against Sider) that quotienting should be the favored option. In contrasting these opposing views, we can better weigh our options in defending a postmodal structuralist view of mathematics.

The Return of Ernie and Johnny

Infamous among the philosophical community (at least as far as philosophy of mathematics goes), Ernie and Johnny reunite after many years spent apart. They have long since renounced their old training, wherein they – oh so foolishly! – believed numbers to be sets. For old times' sake, they recount their now-classic disagreement [1].

Raised by militant logicians, Ernie and Johnny had both been taught the concepts of number-theory in entirely set-theoretic terms. Their education was so complete that they could sufficiently 'speak with the vulgar' who had learned about numbers the old-fashioned way. However, they soon discovered a dilemma. Ernie had learned the von Neumann definition of numbers, and Johnny the Zermelo defintion.

$$\emptyset, \quad \{\emptyset\}, \quad \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots \qquad \text{(von Neumann)}$$
$$\emptyset, \quad \{\emptyset\}, \quad \{\{\emptyset\}\}, \quad \{\{\{\emptyset\}\}\}, \ldots \qquad \text{(Zermelo)}$$

Consequently, Ernie and Johnny disagreed on many seemingly-innocuous questions, such as whether the statement '$3 \in 17$' is true (Ernie says yes, Johnny says no), or whether a set has n members if and only if there is a bijection with the number n itself (again, Ernie says yes and Johnny says no). But, given their educations, both of their definitions satisfy necessary and sufficient conditions for a correct account of number theory.

Benacerraf takes this conclusion to devastate the thesis that numbers are sets. "Although there are differences between the two accounts, it appears that both are correct in virtue of satisfying common conditions. If so, the differences are incidental and do not affect correctness... The two accounts agree in over-all structure" [1, p.56]. If we agree with Benacerraf on this point, we are faced with a crucial problem: can we decide which of the accounts is *really* correct, and if so, how? "Neither, of course!" says Benacerraf. Rather, both accounts adequately exemplify the structure of the abstract numbers, and it is in virtue of this fact that both are 'correct.'

Since this momentous occasion in their lives, Ernie and Johnny have committed themselves to a structuralist view of mathematics. For our purposes, we will suppose that they hold the *ante rem* view in particular, and so the desire for a post-modal structural metaphysics is well-motivated. Although they hold this general common ground, their approaches to equivalence differ wildly.

Ernie has been swayed by a fundamentalist-based approach. He holds that equivalent theories are those that say the same thing about the world at the fundamental level, in terms of fundamental concepts. In other words, if two theories build the same structures using the same 'building blocks' of the world, then they are equivalent. Not only that, but they are equivalent *because* of this agreement. What is *is* for two theories to 'say the same thing' about the world is for them to agree fundamentally.

Regarding their original childhood disagreement, Ernie would say that their reductions of numbers to sets were equivalent (with respect to a satisfactory account of number theory) because there was this underlying, more fundamental natural number structure. Both their definitions merely exemplify the structure, because they are models of that structure.

For another example, suppose we hold that there is a *distinguished* successor relation that characterizes the natural number structure.[5] Perhaps there are other relations that could do the job, like the less-than relation, and we still accept facts about such relations. But we hold that the successor relation is special somehow, and when people give less-than facts they are *really* talking about successor facts in disguise. "Aha!" Ernie says, "You think the successor relation is a fundamental concept!" The distinguished concepts of a theory are exactly the fundamental ones. For the successor-loving folk, the fact that '3 is the successor of 2' would be fundamental, whereas '2 is less than 3' is *not fundamental* as it holds in virtue of the successor-fact. Consequently, the theory that says the successor relation is

[5]We probably should not hold such a position, since it results in a number of unfortunate consequences. For example, switching 0 and 1 in the succession counts as a genuinely different scenario, although the structure would stay pretty much the same. Sider raises this point in his discussion of 'bare particulars' [6, p.106].

distinguished will *not* be equivalent to the theory that takes the less-than relation to be fundamental, since their description that '2 is less than 3' is missing some structural facts, namely that 3 is *really* the successor of 2.

"Hold on Ernie," says Johnny, "What does it really mean to say that theories 'say the same thing' at the fundamental level?" Sider, in Ernie's defense, suggests we speak of the dependence of the non-fundamental on the fundamental in terms of ground. "We might say that facts F_1 and F_2 are equivalent when, necessarily, for any facts that involve only fundamental concepts, those facts ground F_1 if and only if they ground F_2" [6, p.147-148]. Leaving aside mystifying appearance of a modal operator, this claim needs some clarification.

Sider himself raises two concerns about this characterization, both due in part to the postmodal logic of ground [6, p.148, n.6 and n.7]. One concern points to how the non-transparency of ground will lead to a certain non-transparency of equivalence. Someone could accept one fact but deny an equivalent one (for facts with the same grounds) without being irrational or conceptually confused. The other concern is that facts like 'snow is white' will be counted as equivalent to the fact that 'either snow is white or $0 \neq 0$,' since grounding is transitive and disjunctions are grounded by their (true) disjuncts. Sider seeks to address this issue with his notion of biconditional ground, but this move is not necessary as it is not clear that a fundamentalist would be dissatisfied with such a consequence. It makes some sense that, metaphysically, whatever it is about reality that makes 'snow is white' true would also make the other statment true. Perhaps the trouble here is that we could then create very very long disjunctions that would still be equivalent to the statement 'snow is white.' But this is not the biggest issue the fundamentalist will have to face.

Ultimately, what Ernie wants to say is that the distinguished concepts are the fundamental ones, but it can be tricky to determine how and why the former concepts are uniquely singled out by the latter. Returning to the case of $\mathbb{N}$ with the successor relation versus $\mathbb{N}$ with the less-than relation, we surely want to draw some equivalence between these structures. In order to do so, Ernie is committed to deciding which relation is more fundamental, and justifying why, else he cannot claim equivalence.

Alternatively, suppose Ernie wants to say that neither the successor nor the less-than relation is more fundamental, but maintains that the two corresponding theories are equivalent. "Okay then, Ernie," says Johnny, "So what *are* the fundamental properties or relations of the natural number structure?" In this case, Ernie must seek out a third, more fundamental theory that says what the natural number structure is and explains why the other relations are so often used. Although such a third theory can be found in some cases (as Sider develops for a theory of mass), Ernie will often be stuck with hard choices.[6] However the fundamentalist account of equivalence does seem to meet our need for an account of what equivalence *is*: it is just to give the same fundamental description of the world.

Johnny, on the other hand, has been hanging out with a different sort of crowd, and advocates for a quotienting-based approach to equivalence. "We can say *that* theories are equivalent," Johnny explains, "Without having to say *why* they are equivalent in terms of any sort of underlying, 'more fundamental' third theory."

Johnny thinks of the old disagreement quite similarly to Ernie, without positing that the natural number structure is fundamental. Johnny thinks their reductions were equivalent because they belong to the same equivalence class (the class formed by systems that exemplify the natural number structure) once the conventional and notational content is removed.

The goal here is to just separate our the representational content from the conventional content (sometimes called the 'artifacts of the model'), but it is okay to have a perfectly good model with features that are not part of the representational content. While Ernie determined to seek out artifact-free models in order to justify equivalence, Johnny is content to say there may not be any such model. In the statement '$2 < 3$,' the less-than relation is just an artifact of the model, the result of choosing one relation and not another. Given a set of theories with conventional differences, we can 'quotient out' conventional content and get an equivalence class of theories. This equivalence relation does not have to come from any fundamental theory, but can simply be stipulated.

An illustration of quotienting can be taken from Shapiro. As we saw before, isomorphism is too strict a relation to properly characterize equivalent mathematical structures. Shapiro does, however, take isomorphism as the identity relation between structures [5, p.93].

> *The identity relation we need is more a matter of decision or invention, based on convenience, rather than a matter of discovery... We take identity among structures to be primitive, and isomorphism is a congruence among structures. That is, we stipulate that two structures are identical if they are isomorphic. There is little need to keep multiple isomorphic copies of the same structure in our structure ontology, even if we have lots of systems that exemplify each one.*

Recall that he made a similar move with cross-structural identification. He gives no argument that isomorphism is somehow the 'fundamental' relation of identity

[6]A particularly troubling case (not strictly within the context of mathematical structuralism) is the quantifiers. As Sider points out, Ernie will face the question of which of $\exists$ or $\forall$ is fundamental. This genuine question, given different answers, will result in genuinely non-equivalent theories. Sider notes that perhaps some idea of 'weakly' versus 'strongly' equivalence could help in this case [6, p.151, n.11].

between structures, although perhaps he could do so. He does not present any third, underlying theory to justify such claims, but says it can merely be stipulated.

Having each presented their approaches, Ernie and Johnny can now articulate their dissatisfaction with the other's view. Johnny maintains that Ernie is requiring too much, while Ernie thinks Johnny is requiring not enough.

Anticipating objections from Johnny, Ernie begins his defense. "You might not like my approach because we are faced with difficult choices," he says, "But you can never know that we *won't* make progress on answering them." Certainly this is true, but neither do we know that Ernie *will* make progress on the questions. "I don't see why this point means we should favor your theory above mine." replies Johnny, "In the event that you *do* make some progress, great! Then that is further evidence that my posited equivalence is a good one. But in the event that you can't, or that the 'progress' you make doesn't actually give you an artifact-free model, then you're stuck stubbornly denying equivalence when common sense might tell us we should allow it."

"But you're not really even answering the question." protests Ernie, "You haven't told me what equivalence is, and what it is about reality that enables it to be represented by multiple different theories! You have to tell me why *this* equivalence relation should be stipulated." This demand that Johnny provide Ernie with some explanation may just be reiterating the demand for some deeper, more fundamental account of reality, which is exactly what the quotienter claims they do not need to seek. In many cases, Johnny might be able to give a partial answer of why particular descriptions are equivalent. But Johnny would certainly reject Ernie's demand for specification as to *which* relations are fundamental. "That's just an illegitimate demand," Johnny says.

Perhaps Johnny (as a milder quotienter) admits that sometimes it is a metaphysical improvement to explain equivalence in terms of an underlying theory. Maybe sometimes we can distinguish between fundamental and non-fundamental facts, but in the cases where we cannot, we should utilize quotienting instead of continuing a potentially endless search for such a distinction. "Aha!" says Ernie, "Then you're admitting that we *should* seek a quotienting-free theory!"

"Further," Ernie continues, "You've admitted that quotienting should be a last resort. We prefer to seek coordinate-free formulations of our geometric theories, not coordinate theories with content just 'quotiented out.'[7] Further, if you admit

[7]Sider makes another argument on this point that ontology can become really tricky once we start quotienting. Since Ernie and Johnny are both *ante rem* structuralists, it seems like they are committed to the same ontology: the existence of abstract structures and positions in them. Perhaps we could take Sider's point with respect to an ontology about relations, i.e. what kind of relations our theory says exists, but I am not exactly sure how to apply such an argument here.

some talk of fundamentality, then its hard to avoid difficult choices like those I'm faced with."

The example Sider gives here recalls the gruesome predicates.[8] We could rewrite our theories in such predicates, and although quotienter Johnny might be fine with saying the cooked-up theories are equivalent to the original ones, Ernie would say "No! The objects are attracted to each other because they had opposite charges, not because they had opposite *smarges* and were first observed before 3000 A.D.!" If we are at all sympathetic with Ernie on this point and admit that a distinction like this is legitimate, then it is hard to say where to draw the line, and so we are probably stuck with all of Ernie's difficult questions.

It is difficult to see how exactly we could apply this gruesome argument to mathematical structuralism, because it is difficult to think of such predicates the *ante rem* structuralist would accept. Since structures (and their positions) are taken to exist in an abstract sense, and are not 'observed' in the same way colors or objects with charge are, it seems we could not make the usual '3000 A.D.' choice.

In any case, it seems the thing for Johnny to do here is reiterate the point that yes, maybe we should *seek* models with fewer artifacts, but that does not mean we should refrain from asserting equivalence until we are certain we have reached rock-bottom, especially given that it may be that no such bottom exists.

Sider gives Ernie two more points in defense. First off, we do not need to commit to only one fundamental theory, but could instead "sacrifice parsimony to avoid arbitrariness" [6, p.163]. Such a commitment would help explain the lack of metaphysical asymmetry between concepts that seem to give equivalent theories.

Finally, there may just be no way to know whether the correct answer to the question 'Which concept is more fundamental?' is choosing one concept, the other, both, or neither. "Unknowability," Ernie argues (channeling Sider), "may be a sign that the concepts were using arent in good-standing, but that doesnt really seem to be the case with fundamentality. Rejecting fundamentality is itself a metaphysical stance that would need to be justified."

Having thus paraphrased Sider's arguments for and against the two views, it seems what it really comes down to is this: what should a structuralist demand from their metaphysics? Given that Sider made many convincing arguments as to why fundamentality-based accounts of structuralist metaphysics are inadequate, it seems strange that he should settle on Ernie's view over Johnny's.

Consider again our example of the natural number structures, one with the successor relation $(\mathbb{N}, S)$ and the other with the less than relation $(\mathbb{N}, <)$. It seems we would want to say that these are equivalent structures. In order to make such a conclusion, Ernie must choose which of S or $<$ is more fundamental, or find a third

[8] N. Goodman, *Fact Fiction and Forecast*, Harvard University Press (1983).

theory. It is hard to think of a third theory Ernie could provide of "the natural number structure" that is not characterized by some relation or another, and as it stands, there does not seem to be an argument for favoring any one relation over another. So to avoid the arbitrariness accused of Johnny, perhaps Ernie says that both are fundamental. Then Johnny might push back, "But what about the predecessor relation P, or the greater-than relation $>$?" and likely Ernie would be forced to say those are fundamental as well. Suddenly we have a much larger set of fundamental 'building blocks' for our natural number structure. We have four different fundamental structures when it seems we really only wanted one. But how can Ernie escape such a fate?

At this point, Johnny seems to be sitting-pretty. Johnny does not need to seek out a certain fundamental account of reality that says whether the successor relation is fundamental, and why, but can merely say "Descriptions of the natural number structure that differ solely by a variation in choice of relation, yet leave the pattern intact, are equivalent," and not need to say anything more.[9] The relations are equivalent simply because they characterize the same structure. In fact, this seems to be precisely what Resnik and Shapiro had in mind for the structural equivalence relation.

What about the issue of cross-structure identification? Can we draw equivalence between the statements '$2_{\mathbb{N}} + 2_{\mathbb{N}} = 4_{\mathbb{N}}$' and '$2_{\mathbb{Q}} + 2_{\mathbb{Q}} = 4_{\mathbb{Q}}$' (and similarly for the integers, reals, and complex numbers)? To make a claim about equivalence, it seems that Ernie might want to say "Ah, well, we can say that the natural numbers are fundamental, so the statement about the rationals holds in virtue of the statement about the rationals." But then what about trying to say '$\frac{1}{4}_{\mathbb{Q}} + \frac{1}{2}_{\mathbb{Q}} = \frac{3}{4}_{\mathbb{Q}}$' is equivalent to '$\frac{1}{4}_{\mathbb{R}} + \frac{1}{2}_{\mathbb{R}} = \frac{3}{4}_{\mathbb{R}}$'? Following the same strategy as before, Ernie would want to find underlying fundamental facts. We previously took statements about the rationals to hold in virtue of statements about the reals, yet it is difficult to see which fundamental facts about naturals might let us draw this equivalence.[10] So maybe Ernie has to posit the rationals to be fundamental as well. But then how can we make the first assertion about equivalence between the naturals and rationals? The whole business is tricky, and Ernie would need to be very careful.

Johnny, once again, can say such statements are equivalent by merely defining the appropriate equivalence relation, and that is all there is to say about that.

[9] "That's cheating!" says Ernie. "No, it isn't." says Johnny.

[10] Maybe Ernie can think of the rationals as 'built up from' the naturals somehow, and give a proper account of division in terms of fundamental concepts from the natural number structure.

5. Conclusions

Support for mathematical structuralism stems from an observation like 'all mathematics cares about is structure.' If we take this statement to heart, particularly regarding claims of equivalence, it seems we are led straight into the arms of the quotienter. Indeed, many *ante rem* structuralists like Shapiro seem to have used quotienting to make claims of identity or equivalence, by merely stipulating the relation as 'a matter of decision.'

Consider one last claim of Shapiro's [5, Section 5]. After distinguishing between the three ontological schools of thought for mathematical structuralists, he argues that they are in fact *equivalent*, utilizing different primitives to deliver the same 'structure of structures' at the end of the day. Of course, Johnny agrees. Per Sider, the worrisome consequence here is that ontological primitives are taken to be conventional content, mere 'artifacts of the model,' which will undoubtedly raise some eyebrows amongst metaphysicians.

But what can Ernie make of such a claim? What fundamental concepts could be appealed to? Perhaps Ernie maintains that *his* ontological school of thought is fundamental, and all the others depend on it, somehow. "Fundamentally," Ernie argues, "There *are* these structures, and the objects are the positions in them." But then which ones? Else it seems Ernie must commit to three fundamental structuralist accounts (a much bigger ontological commitment than just taking one!) or somehow find a grand, underlying vision of this 'structure of structures.' Unfortunately, Ernie is stuck between fulfilling this tall order or simply denying the equivalence. If a postmodal mathematical structuralist wants to make their bed with Sider and Ernie, perhaps they should instead go back and consider one of the eliminitivist views, one that is not required to have a distinctively structuralist metaphysics. Otherwise, quotienting seems to be their best shot, albeit a slightly antimetaphysical one.

References

1. P. Benacerraf, "What Numbers Could Not Be," *The Philosophical Review* **74** (1965). (p.47-73)

Benacerraf raises a problem for realism in ontology of mathematical object through the story of Ernie and Johnny, who each have been taught about mathematical concepts in terms of set theory. Numbers, arithmetic operations, and applications of mathematics are explained to them using strictly logical concepts like predicates and relations. Benacerraf goes into some detail as to how such number-theoretic notions can be defined using set-theoretic terms.

The dilemma that Benacerraf highlights is that there are several possible reductions of number theory to set theory. If Ernie is taught the von Neumann account of numbers $(\emptyset, \{\emptyset\}, \{\emptyset\{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \dots)$ and Johnny the Zermelo account

$(\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \dots)$ then Ernie and Johnny will disagree on many mathematical statements, such as whether it is true that $3 \in 17$. Given that the accounts differ only in "places where there is no connection whatever between features of the accounts and our uses of the words in question," there seems to be no justification of favoring one reduction over the other. These troubling disagreements seemingly cannot be resolved, and Benacerraf argues that numbers cannot be sets at all.

To resolve some of the problems, Benacerraf proposes that expressions of identity such as '$x = y$' should only make (semantical) sense when x and y are of the same category. So statements like 'Julius Caesar= 3' are simply senseless in that the two terms do not refer to things of the same category. He further argues that reducibility of arithmetic to set theory is not a reason to assert that numbers are sets. Mathematics stops at the level of structure, and does not care whether individuals of one theory really are the individuals of a second theory that the first can be reduced to. It is the mistaken impulse of the philosopher, Benacerraf says, to push these questions that miss the point of what mathematics is all about.

Since any set of objects (sets or otherwise) that forms a recursive relation could adequately model the natural numbers, what is important is the structure that is exhibited rather than the individuality of each element. To be a number is merely to play a certain role within a structure, and set-theoretic Platonism cannot succeed as a philosophy of mathematics. Extending the argument that numbers cannot be sets, Benacerraf concludes that numbers cannot be objects, against ontological realism.

2. S. Dasgupta, "Symmetry and Superfluous Structure: A Metaphysical Overview," *Companion to the Philosophy of Physics* (draft of February 2018).

Dasgutpa seeks to develop a method of symmetry that can be used in metaphysical applications. He considers a main role of symmetry to be as a guide to superfluous structure, that is, quantities of a theory that change under a certain symmetry. He claims that the variance of these quantities should indicate to the metaphysician that the quantity is unreal, and so conclusions about the nature of reality can be informed by considerations of symmetry.

First, he takes up dynamic symmetries, using the paradigm case of absolute velocity and the uniform boost operation. Uniform boosts are then dynamical symmetries in the context of Newtonian physics, since for any world that obeys Newton's laws, the boosted world will as well. Dasgupta claims that variance of absolute velocity under dynamical symmetry implies unreality, and so we have reason to think absolute velocity is unreal in the context of Newtonian physics. He claims that metaphysicians could extend this method of symmetry to argue for there being 'no such X' in reality, for some X that is variant under a dynamical symmetry.

To justify dispensing of variant qualities, Dasgupta turns to three possibilities: that variants are not objective, are physically redundant, and undetectable. He identifies this third possibility as the best justification, given the epistemic virtue of theories that do not positing undetectable quantities.

The main challenge is to show that variant quantities are undetectable. Returning to the absolute velocity example, Dasgupta claims that the symmetry of the laws implies that it is impossible to build a device that could measure mere differences in absolute velocity, and further that this argument is stronger than the standard skeptical argument. This symmetry argument can be extended to show that any quantity altered by a certain type of transformation is undetectable. In particular, these transformations must map between worlds that follow the same laws, and are observationally and abductively equivalent.

Since dynamical symmetries are spoken of as operations on possible worlds, it is important to clarify the notion of possibility that is used. Dasgupta argues for epistemically possible worlds over metaphysically possible worlds, given the advantage that the results about what can or cannot be detected are epistemic in nature, and that there are possibilities we want to consider that may be epistemically possible but not metaphysically possible.

The argument can be extended from dynamical to empirical symmetries, where instead of requiring the two worlds to have the same laws, it is enough to have abductively equivalent laws. He claims the method of empirical symmetry ensures that the new theory (without the undetectable quantities) is empirically adequate if the old theory was. Metaphysicians could make use of either kind of symmetry, dynamical or empirical, depending on the purpose of use.

3. G. Hellman, "Structuralism Without Structures," *Philosophia Mathematica* **4** (1996). (p.100-123)

Hellman is an advocate of modal structuralism, and seeks to build upon his previous account by filling in some gaps, particularly relating to the treatment of certain mathematical structures. He outlines four main approaches to structuralism: model theory (structures are models), category theory (structures are 'objects' of a category), a *sui generis* approach (structures are patterns or universals in their own right), and his own modal structuralist approach. His goal is to find a modal structuralist approach that does not rely on set theory or category theory. Such a view, he argues, would avoid an ontological commitment to abstracts (although modality would be taken as primitive).

He first reviews his previous work in "Mathematics Without Numbers," wherein he developed a modal-structural framework for arithmetic and real analysis. On this conception, his use of second-order comprehension requires him to have more of a Platonist commitment than he would like. To resolve this issue, he posits only the logical possibility of an infinitude of atoms, and it is from here that he can build up the natural number structure and everything that follows from there. Within this 'nominalist modal structuralist' system, postulating a countable infinity of atoms, he can attain third-order number theory and (equivalently) second-order analysis. This account can be extended one level further, to fourth-order number theory and third-order analysis, given uncountable rather than countable infinity.

For the remainder of the paper, Hellman exemplifies how his framework sufficiently fits the structuralist enterprise by examining specific examples in mathematics. In particular, he examines metric spaces (including Banach and Hilbert spaces), measure theory and measure spaces, sigma-rings, topological spaces, and sheafs.

4. M. Resnik, "Mathematics as a Science of Patterns: Ontology and Reference," *Nos* **15** (1981). (p.529-550)

Resnik seeks an account of mathematics as the study of abstract entities, and cites two main problems that arise for such a Platonist-like philosophies of mathematics. The first is epistemic, questioning how we can acquire knowledge of abstract entities, and the second is metaphysical, arising from the difficulty of giving definitive statements about what mathematical objects are. Resnik posits that the objects of mathematics are merely positions in structures, and so have no identity or features outside of a structure. The patterns (structures) and positions in them are abstract entities. This philosophy of mathematics, he claims, will solve both the epistemological and metaphysical problems for the platonist.

His main focus of the paper is on the ontological question, but he gives a sketch of the epistemic discussion as well. Exposure to instances of a pattern can help us try to describe the pattern itself, by developing a theory of the pattern and of the data fitting it. From a partial description, we can infer other features. We get infinite patterns by considering finite patterns and extending them indefinitely. Resnik draws an analogy between mathematical knowledge and musical or linguistic knowledge, wherein we abstract from experience to arrive at the unexperienced.

Resnik argues that a pattern is a complex entity consisting of positions that stand in various relations to each other. A position has no distinguishing features other than those it has in virtue of being *that* position in the pattern. Patterns can be related to each other by structural isomorphisms or congruence relations. Instantiation of a pattern occurs when some arrangement of objects 'occupy the positions' of the pattern, so the pattern and arrangement are congruent, although the objects may have some features external to those given in virtue of the arrangement.

A pattern can also occur within another pattern, with a special case being the *subpattern relation*. For example, the natural number sequence could be represented by an endless line of dots, and we could get a subpattern of this sequence by considering the same line of dots, starting at the n^{th} to the left. Similarly, we say the even number sequence is a subpattern of the natural number sequence.

He argues that pattern occurrence is too weak of a condition to give pattern equivalence, and instead we need something like *definitionally* equivalent. He spends some time trying to develop this relation, attempting to articulate when we have identities between patterns or positions in patterns. Resnik next turns to reduction, particularly the reduction of number theory to set theory, before examining the question of reference. He takes up a variant of Quines views, a version of referential relativity, in order to explain the connection between mathematical statements and objective reality. This view has the corollary of ontological relativity, that it only makes sense to speak of the ontology (positions) of a theory of a pattern relative to some fixed occurrence of a pattern.

5. S. Shapiro, *Philosophy of Mathematics: Structure and Ontology*, Oxford University Press (1997). (Part 2, Chapter 3: Structure)
Shapiro opens Section 1 by noting that structuralism can have interesting consequences regarding ontology. A structuralist posits objects to be secondary to the structure they appear in, yet this thesis cannot be articulated without invoking the notion of an object. He asks for the readers 'dialectic indulgence' on this point. Shapiro characterizes himself as a realists in ontology and in truth-value, that mathematical objects exist independently of the mathematician and arithmetical assertions have objective truth-values in reference to these objects.

He first outlines a traditional Platonist conception of numbers. Similar to a structuralist, a platonist identifies numbers to have some sort of ontological independence. However structuralists would disagree with a platonist who characterizes this independence by saying "the *essence* of each number can be stated without reference to any other number." A structuralist similarly rejects the distinction between Plato's arithmetic and theoretical logisitic. The individual numbers are completely characterized by their relations to each other. A *system* is defined to be a collection of objects with certain relations, and a *structure* to be the abstract form of a system. On an epistemological note, Shapiro says that there are many ways structures could be apprehended, such as through a process of pattern method or abstraction or via direct description, possibly in terms of a previously understood structure.

Shapiro seeks to characterize mathematics as the science of structure. He discusses an error in Field's 'nominalist' theory (wherein there are no abstract objects, so everything is concrete). Field desires to show that science can proceed without committing to the existence of numbers and other abstract objects, by postulating a rich physical space. Many skeptical mathematicians claim that mathematics has not really been eliminated from Field's system, and Shapiro agrees.

On the ontological front in Section 2, there are concerns regarding the ontological status of whole structures and the status of mathematical objects. Regarding the latter, Shapiro argues that a mathematical object is a place in a particular structure, and that the structure is prior to the object it contains. He claims that such structuralism can treat problems like Frege's classic Caesar problem and Benacerraf's variant problem in "What Numbers Could Not Be." Shapiro argues that structuralism renders questions like 'is Julius Caesar= 3?' as incoherent and indeterminant.

Mathematical objects are tied to the structures that constitute them, which points towards a relativity of ontology. Against Benacerraf, Shapiro argues that identifying positions can be convenient and compelling. Similarly, sometimes places of a structure are discussed in the context of one or more systems that exemplify the structure. He differentiates between the *places-are-offices* perspective, that presupposes a background ontology supplying objects to take places in the structures, and the *places-are-objects* perspective, where statements are about the respective structure can be independent of any exemplifications. Identity between objects can then be differentiated accordingly for the two perspectives, and Shapiro asserts that the structuralist means to use the latter.

In Section 3, Shapiro argues that a structure is one-over-many, as the same structure can be exemplified by more than one system. He distinguishes different kinds of structuralist positions, based on distinct views of universals. The 'hardheaded' or elimintivist structuralist thinks that there is no more to structures than the systems that exemplify them, an *in re* approach. A 'mystical' structuralist takes an *ante rem* stance, where structures have real but abstract, immaterial existence. The *in re* structuralist might run into trouble, since their view requires a very robust background ontology. Responses to this issue could utilize the 'ontological option,' modality (as in Hellman), or turn to *ante rem* realism toward structures (as Shapiro believes we should).

Section 4 begins by articulating equivalence relations between systems that amounts to 'having the same structure.' First, Shapiro claims that an isomorphism is too fine-grained a relation, as it requires that two systems have the same number of relations. Building off of Resnik, we can define 'structural equivalence' in terms of subsystems and isomorphisms. Shapiro gives a treatment of equivalence within each of the three structuralist positions, developing a structure theory for the *ante rem* view. He argues that *some* background theory will be required in a structuralist program, and between the options of set theory, modal model theory, and structure theory, he is inclined to favor the latter. He concludes that there are several ways to rigorously render structuralist theory, all of which are equivalent at the end of the day, delivering the same 'structure of structures' using different primitives.

The favored *ante rem* structuralist view is further developed in Section 5, where Shapiro first wants to differentiate between mathematical and ordinary, non-mathematical structures.

He tries to formulate a notion of 'formal' relations and articulate the 'freestanding' nature of mathematical structures. Next, he turns to a discussion of different levels of *abstracta*, in response to Parsons. He claims that the *ante rem* structuralist should hold both numbers and sets on par, as objects (in the places-as-objects sense). Shapiro claims that structuralists cannot eliminate the use of quasi-concrete objects, and should not seek to do so. The concrete or quasi-concrete, he argues, does not undermine the *ante rem* ontological thesis but serves to motivate and justify the existence of structures with certain properties.

In the final section, Shapiro discusses Block's three types of functionalism, spending some time on *metaphysical functionalism*, wherein mental states are characterized in terms of their causal roles. Shapiro argues that metaphysical functionalism is a structuralist thesis of sorts, although the structures utilized by metaphysical functionalism are not freestanding, and many of the places are not formal.

6. T. Sider, *Tools of Metaphysics and the Metaphysics of Science*, Oxford University Press (draft of January 25, 2018). (Chapters 1, 4, and 5)

Aptly named, *Tools of Metaphysics and the Metaphysics of Science* discusses how and why choices of metaphysical tools, those core concepts that are used to express and solve problems in metaphysics, matter to structuralist positions in the metaphysics of science and mathematics. The first chapter outlines the transition from modal metaphysics to post-modal metaphysics and the implication for structuralists.

Structuralism is roughly the idea that the patterns or structure is primary and the entities that appear in the pattern are secondary. Forms of structuralism appear throughout metaphysics of science and mathematics, such as nomic (dispositional, causal) essentialism, as pertaining to properties, quantities, or individuals. A modal structuralist claims that the patterns cannot vary independently of the nodes (or, alternatively, the other way around). Sider argues that modality is not a satisfying tool for metaphysical inquiry due to its crudeness and asymmetry, as well as arguably its non-fundamentality, superficiality, and minimality in describing reality. In light of these shortcomings, structuralists should instead turn to post-modal tools like fundamentality, essence, and grounding.

A post-modal structuralist seeks an explanation for *why* entities cannot vary independently of the pattern, an account of the fundamental facts underlying the pattern. Sider outlines three obstacles the post-modal structuralist might face. The first is that there may simply be no coherent thesis, the second obstacle arises in the post-modal logic of grounding, and the third is that the resulting theory might be unattractive with regards to the natural epistemology for fundamentality.

In the remaining sections of the chapter, Sider presents various post-modal concepts: essence, ground, and fundamentality. Essence, as discussed by Fine, cannot be defined in terms of necessity, but distinguishes between those features that are part of somethings nature and those that are not. Grounding, some philosophers claim, is an irreplaceable conceptual tool in philosophy that allows us to speak about facts (or propositions) holds 'in virtue of' or 'being explained by' other facts. Sider examines particular details of this concept, such as the 'levels' view of grounding, Wilson's objection that grounding is unsuitable for articulating forms of structuralism, and whether grounding facts have their own grounds. For fundamentality, Sider claims that the fundamental facts, which are formulated using fundamental concepts, are the lowest level upon which all other facts rest and the concept. He concludes the chapter by comparing these notions structural differences, such as factual/sub-factual and comparativeness, and a closing discussion of the epistemology of concept fundamentality claims that realism about fundamental concepts pairs nicely with seeking laws that are simple yet powerful.

In Chapter 4, Sider argues that structuralist positions desiring to reject individuals are the most difficult to articulate in the post-modal setting. Rejecting individuals has been argued to better resolve some metaphysical puzzles or to avoid certain ontological questions, but Sider claims these arguments are not entirely persuasive. Structural realists only commit to believing in the structural content of scientific theories, rather than holding that the nature of the unobservables posited by science is correctly described by the best theories (scientific realism). Advocates of this position have argued that structural realism solves problems related to pessimistic metainduction and metaphysical underdetermination, although Sider does not find either of these arguments convincing. The remained of the chapter presents particular anti-individualist positions.

Sider first examines Dasgupta's anti-individualist thesis, which posits that we should reject individuals given that they are physically redundant and empirically undetectable. Parallel to his argument for discarding absolute velocity under Newtonian Gravitational Theory, Dasgupta justifies adopting a metaphysics in which permutations of individuals makes no difference.

Mathematical structuralists, similar to structural realists, believe that structure is all that matters in mathematics. Such a position can be used to address many problems for philosophers of mathematics, including Benacerraf's famous problem from "What Numbers Could Not Be," Frege's Caesar problem, and questions like whether $2_{\mathbb{N}}$ is the same as $2_{\mathbb{Q}}$. There are various answers to these puzzles, each of which requires varying degrees of a structural metaphysics. Sider argues that in order to adopt 'non-eliminativist' structuralism (like Shapiro), to claim that mathematical objects are just positions in mathematical structures, one must also adopt a distinctively structuralist metaphysics.

Antihaecceitism, the thesis that the non-qualitative supervenes on the qualitative,has been used to solve problems of modal nature. However, given post-modal view of modality as a sort of epiphenomenon, Sider argues that an antihaecceitists solutions to problems are unsatisfying, and we should turn to post-modal articulations of anti-individual structuralism.

Anti-individualist positions can vary between radically rejecting a predicate-logic apparatus of reference and moderately admitting some 'entities' whilst rejecting some other 'individuals.' In eliminativist structural realism (ESR), objects do not exist: only structure. Sider, quoting Ladyman and French extensively, argues that ESR does not give a satisfying account of what reality is ultimately like. Next on the chopping block, bundle theory takes individuals to be just bundles of properties, and Sider faults them for their silence on relations. For a structuralist who seeks to deny intrinsic properties, Sider argues that they should posit entities that are 'bare particulars,' which merely instantiate fundamental relations. But it is not clear whether bare particulars could meet all the problems a structuralist wants to solve. Turning to post-modal metaphysical tools, Sider argues that an individuals-free description of fundamental reality is still difficult to articulate, so perhaps instead we should argue that facts about individuals are grounded in facts about structures. An extreme view of this sort is monism, per Schaffer, that posits everything to be grounded in the entire Cosmos. To this, Sider asks for an account of what enables the Cosmos to ground sub-Cosmos facts, and doubts that there is any attractive monistic account of the nature of fundamental facts.

One problem for mathematical structuralists in particular is cross-structural identification, as exemplified by the classic 'Is Julius Caesar= 3?' problem. Although a 'no fact of the matter' approach can be taken perhaps for non-fundamental languages, Sider is hesitant that such an approach would serve mathematical structuralists. Sider is similarly skeptical of weak discernibility, wherein two objects can be differentiated if a binary predicate applies to them in different patterns.

The last section of this chapter examines generalism, in particular Dasgupta's term functorese and quantifier generalism. Dasgupta develops a name-free language with the expressive power of standard predicate logic, and so this term functorese is intertranslatable with predicate logic. According to quantifier generalism, fundamental facts (like 'every electron has negative charge') are the predicate logic translations of Dasgupta's fundamental facts. Sider concludes the chapter with objecting to Dasgupta's argument that individuals are redundant (arguing instead that they are replaceable) before examining several different types of generalism.

In Chapter 5, the final chapter of Siders book, he addresses the question of theoretical equivalence: *What does it mean for two theories to be equivalent?* He claims that we say two theories are equivalent when they represent the same state of the world, so any differences between them are merely conventional or notational. He differentiates his metaphysical question (what *is* equivalence, really?) from the epistemic one, using symmetry as an example of criteria for the latter but not the former.

Sider argues that when we claim that two theories are equivalent when, for example, they differ only in what unit of measure they employ, we are agreeing that there is no distinguished unit of measure. The metaphysical question of equivalence leads us directly to ask what 'sameness' or 'difference' in the world consists in, and what we really mean by 'distinguished.' Sider outlines two very different approaches to this dilemma: fundamentality and quotienting.

Sider favors the fundamentality-based approach. On this view, the distinguished concepts are the fundamental ones, and equivalent theories will say the same thing about the world at the fundamental level in terms of fundamental concepts. A feature of this view is that we must identify which concepts are fundamental. When deciding whether two theories are equivalent, we must be able to say whether one or the other is fundamental, whether they are both fundamental, or whether there is a third, underlying fundamental theory. This is a serious question for the fundamentalist, and different answers result in non-equivalent theories. While this requirement might seem like an uncomfortable consequence, Sider argues that it is not a reason to abandon the theory.

The other view Sider presents, quotienting, takes an entirely different stance on theoretical equivalence. Given theories with only conventional differences, we can quotient out the conventional content and regard the best description as an equivalence class of theories, without needing to refer to fundamentality or underlying third theories. Reality is such as to be well-described by any of the equivalent theories. While the friend of fundamentalism might claim that we should be able to find some model without conventional content, with no artifacts, the quotienter claims that it is enough to say which of the features in a theory are representational and which are artifacts. Quotienting seems to address many of the structuralists problems, and Sider examines instances where a quotienting-like approach has proven helpful. However Sider argues against adopting a quotienting-based approach, claiming that it is often a last resort, intuitively unsatisfying as to what equivalence *is*, and that a quotienter who acknowledges that fundamentality-based explanations can be a metaphysical improvement is admitting that a quotienting-free theory would indeed be superior.

Sider concludes his chapter in defense of fundamentality. First off, he says that we can never know that we will *not* make progress on difficult questions like which quantifier is more fundamental. Secondly, he says that difficult questions like these are hard to avoid. It is difficult to admit sympathy to a fundamentalist approach, even to a small degree, without admitting that such questions are legitimate. Third, there is no reason we could not say that there is more than one fundamental theory, and sacrifice parsimony to avoid arbitrariness. Finally, Sider says that not always having answer to the question "Which concepts are fundamental?" is no reason to sacrifice the whole theory. Unknowability, he argues, may be a sign that the employed concepts are not in good-standing, however this does not seem to be the case with fundamentality. He points out that rejecting fundamentality is itself a metaphysical stance that requires justification, and there does not seem to be a good argument for doing so.