The Freudenthal Suspension Theorem

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1 Introduction

There are two roughly two sides to homotopy theory:
building machines, and using them to do computations.
—John Baez

The running theme of our workshop (and algebraic topology in general) has been the ways in which we can provide algebraic solutions into topological problems. The longstanding "heavy hitters" in this regard are homology, cohomology, and homotopy groups. By associating groups to a topological space, we can use the tools of algebra to gain insights that might otherwise be difficult to access. Moreover, there are a variety of options to choose from (such as Morse homology and de Rham cohomology we saw earlier in the summer). However, as we know, computing these algebraic invariants can be quite difficult. There is a trade-off, in many cases, between the complexity of the machine and the ease with which we can use it to do computations.

We can see how this trade-off plays out for spheres. Since spheres are some of the simplest spaces, we would hope that their (co)homology and homotopy groups would also be relatively simple. In the case of (co)homology groups, our wish is granted. Once we put in the work of understanding the construction of the (co)homology groups (which may require understanding singular chains or differential forms or gradient flow or...), we are rewarded with the knowledge that $H_i(S^n)$ and $H^i(S^n)$ are non-trivial just when $i = 0, n$.

In contrast, the homotopy groups of spheres are much harder to compute, although the construction of homotopy groups is much simpler than their (co)homological counterparts. In particular, while $\pi_i(S^n)$ is known to be trivial for $i < n$ and $\mathbb{Z}$ for $i = n$, the higher homotopy groups for $i > n$ are not known in general (although a lot of progress has been made, e.g. the fact that almost all these groups are finite, as mentioned on [Hat02, p.339, p.364, p.384]). The difficulty of computation is due, in part, to the lack of something like excision for homology (which we will discuss in Section 3) or the fundamental group's van Kampen theorem.

The Freudenthal Suspension Theorem gives us one way to better understand these elusive groups. This result, which finds its home as a cornerstone of stable homotopy theory, relates the homotopy groups of a sphere to that of its suspension, utilizing the construction of S^n as the suspension of the sphere one dimension lower. The remarkable implication of the theorem is that

$$\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$$

for $i < 2n-1$. The modern theorem statement deals with general spaces, although Freudenthal's original theorem only considered spheres. The goal of this write-up is to provide a relatively self-contained proof of the general Freudenthal Suspension Theorem.

In Section 2, we briefly review the basics of higher homotopy groups and relative homotopy groups. In addition to the long exact sequence for relative pairs, the most important ingredient for proving the suspension theorem is the homotopy excision (or Blakers-Massey) theorem, which we discuss in Section 3. The final section, Section 4, covers the main theorem, although we spend some time discussing suspensions (Section 4.1) before delving into the proof (Section 4.2). We briefly discuss some applications of the suspension theorem in Section 4.3, in particular to define stable homotopy groups.

2 Higher Homotopy Groups

There is much pleasure to be gained from useless knowledge.
—Bertrand Russell

Homotopy groups are one way that topologists can describe and classify topological spaces. Like homology groups, the homotopy groups provide a way to detect interesting higher dimensional characteristics of a space. The n^{th} homotopy group of a space records information about homotopically distinct maps from the n-sphere to the space.

Definition 2.1. Let X be a topological space with basepoint $x_0 \in X$. For a given basepoint $s \in S^n$, the n^{th} *homotopy group* of X is the set of homotopy classes of maps $f\colon S^n \to X$ that map s to x_0. In mathematical symbols,

$$\pi_n(X, x_0) = \{[f] \mid f\colon S^n \to X,\ f(s) = x_0\}.$$

Equivalently, we could ask about homotopy classes of maps $f\colon I^n \to X$ that take the boundary of the n-cube $I^n = [0,1]^n$ to x_0. If X is path-connected, then the choice of basepoint does not matter, and we may simply write $\pi_n(X)$.

A space is called *n-connected* if $\pi_i(X) = 0$ for all $i \leq n$. Thus 0-connected means path-connected, and 1-connected means simply connected. Since n-connectedness

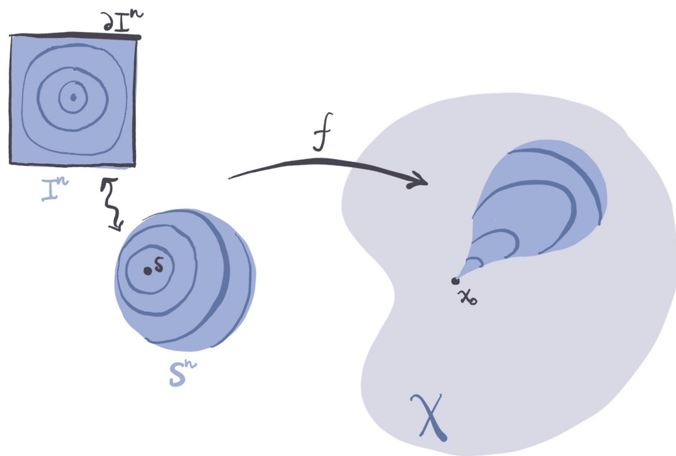

Figure 1: A representative of an element in $\pi_n(X)$.

implies 0-connectedness, the choice of basepoint x_0 is unimportant in any n-connected space.

The group operation is a generalization of the familiar "gluing" method in the fundamental group π_1. Given $f, g \in \pi_n(X)$ for $n \geq 2$, we have

$$(f+g)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, \ldots, t_n) & t_1 \in [0, \frac{1}{2}]; \\ g(2t_1 - 1, \ldots, t_n) & t_1 \in [\frac{1}{2}, 1]. \end{cases}$$

(Of course, we really mean to write $[f]$, $[g]$ and $[f+g]$, but we typically omit the brackets for the sake of simplicity, with the understanding that we are talking about homotopy classes of maps.) The additive notation above is justified by the fact that $\pi_n(X)$ is Abelian for $n \geq 2$. There is an illustrative proof of this fact in [Hat02, p.340], where the essential idea is that we have enough room within I^n (even just within a square I^2) to shrink, swap, and then expand the domains of f and g.

We have seen that the fundamental group π_1 is a functor **Top** $\to$ **Gp**. It is perhaps unsurprising then that, for $n \geq 2$, the higher homotopy group π_n also defines a functor **Top** $\to$ **Gp** (or, more specifically, into **Ab**). Given a basepoint-preserving map of topological spaces $\phi\colon (X, x_0) \to (Y, y_0)$, we get an *induced map* on their higher homotopy groups

$$\phi_* := \pi_n(\phi)\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

which sends a homotopy class $f \mapsto \phi \circ f$. After a quick check, we can see that this functor sends homotopy equivalences to isomorphisms. In this spirit, a map $\phi\colon X \to Y$ is a *weak homotopy equivalence* if the induced map ϕ_* is an isomorphism (in **Set** for $n = 0$ and in **Gp** for $n \geq 1$). Weak homotopy equivalences play a key role in other interesting parts of algebraic topology (the most well-known of which is probably Whitehead's theorem [Hat02, p.346]). There are many more interesting facts and features related to higher homotopy groups that we will not go into here, but luckily the interested reader has many classic references to turn to, such as [Hat02, May99].

2.1 Relative Homotopy Groups

An important generalization of higher homotopy groups is the idea of the *relative homotopy groups*. These groups will give us a way to relate the homotopy groups of two spaces when one is a subspace of the other. The idea is that, for a given subspace $A \subseteq X$ with basepoint $x_0 \in A$, the relative homotopy group $\pi_n(X, A, x_0)$ will record information about homotopically distinct maps from the n-sphere to X without caring about the boundary of the sphere, which is mapped into A.

Definition 2.2. Let $A \subseteq X$ and $x_0 \in A$. The n^{th} *relative homotopy group* $\pi_n(X, A, x_0)$ is the collection of homotopy classes of based maps $D^n \to X$ which take the boundary $\partial D^n = S^{n-1}$ to A. That is, for a given basepoint $s \in S^{n-1}$,

$$\pi_n(X, A, x_0) = \{[f] \mid f\colon D^n \to X, f(S^{n-1}) \subseteq A, f(s) = x_0\}.$$

As before, we could instead define the relative homotopy groups in terms of maps $f\colon I^n \to X$ which take ∂I^n to A. There is one additional requirement: we also ask that f maps $J^{n-1} \subseteq \partial I^n$ to x_0, where J^{n-1} is defined to be the closure of $\partial I^n \setminus (I^{n-1} \times \{1\})$. In other words, f must map the top face $I^{i-1} \times \{1\}$ into A and the rest of the boundary to x_0. If A is path connected, we may safely write $\pi_n(X, A)$ without specifying a basepoint. For the sake of notational simplicity, we will write merely $\pi_n(X, A)$ unless there is a risk of ambiguity.

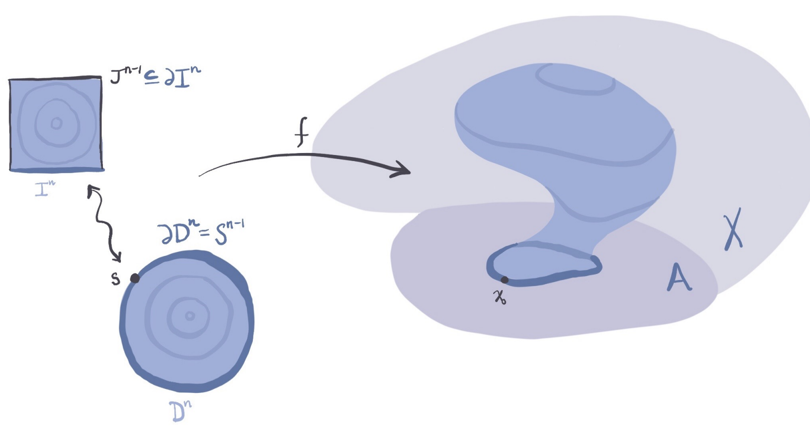

Figure 2: A representative of an element in $\pi_n(X, A)$.

Note that if $A = x_0$, then $\pi_n(X, A, x_0) = \pi(X, x_0)$, so indeed homotopy groups are a special case of relative homotopy groups. We say that the pair (X, A) is *n-connected* if $\pi_i(X, A) = 0$ for all $i \leq n$.

The sum operation for relative homotopy groups is defined by the same formula given for $\pi_n(X)$. Essentially the same proofs show that $\pi_n(X, A)$ is a group for $n \geq 2$ and an Abelian group for $n \geq 3$. For $n = 1$, $\pi_1(X, A)$ is is the set of homotopy classes of paths in X starting at the fixed basepoint $x_0 \in A$ and ending at some other point in A. In general, $\pi_1(X, A)$ is not a group in the usual way.

Two maps are called *homotopic relative to A* if they are homotopic via a base-preserving homotopy $H\colon D^n \times I \to X$ such that for any $x \in S^{n-1}$ and $t \in I$, we have $H(x, t) \in A$. A map $f\colon (D^n, S^{n-1}) \to (X, A)$ is considered trivial (i.e. represents zero) in $\pi_n(X, A)$ if and only if it is homotopic relative to A to a map whose image is contained in A. This is known as the *compression criterion* (cf. [Hat02, p.343]). We can think of continuously "compressing" the image of f into A via the given homotopy.

An important advantage of the relative homotopy groups is that they fit into a *long exact sequence*:

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \to \cdots \to \pi_0(X)$$

where i_* and j_* are the maps induced by the inclusions $A \hookrightarrow X$ and $(X, x_0) \hookrightarrow (X, A)$, respectively. The map ∂, called the *boundary map*, comes from restriction maps $(D^n, S^{n-1}) \to (X, A)$ to S^{n-1} (or, equivalently, restricting maps from $(I^n, \partial I^n, J^{n-1})$ to $I^{n-1} \times \{1\}$). The boundary map is a homomorphism when $n \geq 2$. For further details and proof of exactness, we point the reader to [Hat02, p. 344] or [Zha09, Theorem 1.3].

An immediate corollary of the long exact sequence is that $j_*\colon \pi_n(X) \to \pi_n(X, A)$ is an isomorphism whenever A is contractible. Finally, if (X, A) is connected, then we can relate the homotopy groups of A and X.

Definition 2.3. Suppose (X, A) is n-connected, so $\pi_i(X, A) = 0$ for $i \leq n$. By the long exact sequence, this implies that the induced map $i_*\colon \pi_i(A) \to \pi_i(X)$ is an isomorphism for $i < n$ and a surjection for $i = n$. The inclusion $i\colon A \hookrightarrow X$ is called an *n-equivalence.*

More generally, a map $f\colon X \to Y$ is called an *n-equivalence* if the induced map $f_*\colon \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i < n$ and a surjection for $i = n$. This definition comes from thinking of X as a subspace of Y using f, and applying the definition above to the pair $(Y, f(X))$. In this vocabulary, a weak homotopy equivalence is an ∞-equivalence.

Example 2.4. Consider the inclusion $S^n \hookrightarrow D^{n+1}$. The ball D^{n+1} is contractible, and so $\pi_i(D^{n+1}) = 0$ for all i. By the cellular approximation theorem, we also have $\pi_i(S^n) = 0$ for $i < n$ ([Hat02, Corollary 4.9]). However, as we shall soon see,

$\pi_n(S^n) \cong \mathbb{Z}$. This means that the inclusion of the n-sphere into the solid $(n+1)$-ball is an n-equivalence.

3 Homotopy Excision

Keep computations to the lowest level of the multiplication table.
—David Hilbert

Homotopy groups are notoriously hard to compute. This difficulty is due, in part, to the fact that homotopy groups fail to satisfy the excision axiom.

Definition 3.1. An *excisive triad* $(X; A, B)$ consists of a topological space X along with two subspaces $A, B \subseteq X$ such that the interiors of A and B cover X, i.e. $X = A^\circ \cup B^\circ$.

The excision axiom in homology states that the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism on the homology groups. Excision is one of the main reasons that homology can often be effectively calculated, as it produces, among other things, the long exact Mayer-Vietoris sequence of an excisive triad, and is key to identifying the cellular and singular homologies of a CW-complex.

Unfortunately, the same does not hold of homotopy groups: the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ does not induce an isomorphism on homotopy groups, in general. We can see this unfortunate fact in a relatively simple example, using wedges of spheres.

Example 3.2. Take $X = S_2 \vee S_2$. Decompose X into two halves, with $A = C_+$ being the northern hemispheres of the two spheres (including the equator) and $B = C_-$ the southern hemispheres, as shown in Fig. 3. Note that $(X; A, B)$ is indeed an excisive triad. Then $A \cap B \cong S^1 \vee S^1$, so we have the inclusion

$$(C_+, S^1 \vee S^1) \hookrightarrow (S^2 \vee S^2, C_-).$$

We claim that we do *not*, in general, have $\pi_i(C_+, S^1 \vee S^1) \cong \pi_i(S^2 \vee S^2, C_-)$, in particular for $i = 2$.

Note that both C_+ and C_- are contractible, since both look like $D^2 \vee D^2$. From our earlier observations about the long exact sequence for pairs, this implies that $\pi_i(S^2 \vee S^2, C_-) \cong \pi_i(S^2 \vee S^2)$. On the other hand, the long exact sequence for $(C_+, S^1 \vee S^1)$ implies that $\pi_i(C_+, S^1 \vee S^1) \cong \pi_{i-1}(S^1 \vee S^1)$. Taking $i = 2$, we know that $\pi_1(S^1 \vee S^1)$ is the free group on two generators (by the usual application of van Kampen's theorem). This means that $\pi_2(C_+, S^1 \vee S^1) \cong \pi_1(S^1 \vee S^1)$ is *not Abelian*. Therefore it is impossible for $\pi_2(C_+, S^1 \vee S^1)$ to be isomorphic to $\pi_2(S^2 \vee S^2, C_-) \cong \pi_2(S^2 \vee S^2)$, since the higher homotopy group $\pi_2(S^2 \vee S^2)$ must be Abelian.

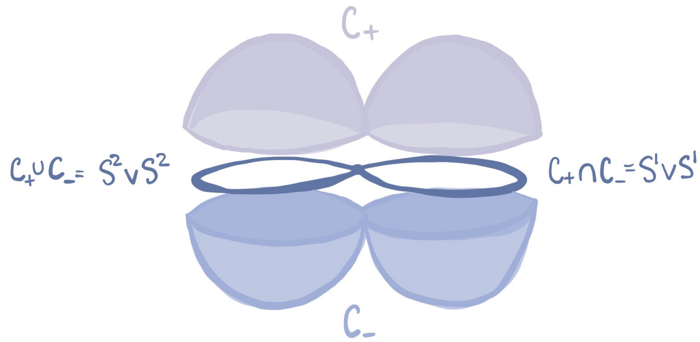

Figure 3: The decomposition of $X = S^2 \vee S^2$ into the upper/lower hemispheres $C_\pm$, which intersect along the equator $S^1 \vee S^1$.

So when *does* an excisive triad yield such an isomorphism? The Blakers-Massey theorem provides an answer to this question, stating that the homotopy groups will satisfy excision in a *range* of dimensions, and range which is roughly the sum of the connectivities of (X, B) and $(A, A \cap B)$.

Theorem 3.3 (Blakers-Massey). *Let $(X; A, B)$ be an excisive triad such that $C = A \cap B$ is non-empty. Suppose $(A, C, *)$ is n-connected and $(B, C, *)$ is m-connected for every choice of basepoint $* \in C$. Then, for every basepoint $* \in C$, the map*

$$\pi_i(A, C) \to \pi_i(X, B)$$

induced by the inclusions is an isomorphism for $i < n + m$ and a surjection for $i = n + m$. In other words, the inclusion is an $(n + m)$-equivalence.

The Blakers-Massey theorem is the closest we can get to something like homotopy excision, so is often called the homotopy excision theorem. Since our focus is on the Freudenthal Suspension Theorem (a direct corollary of the Blakers-Massey Theorem), we provide only a sketch of the proof. We follow the typical strategy used to prove this result, which is to reduce from an arbitrary excisive triad $(X; A, B)$ to a simpler case. Our outline follows [Gro12], which is an expansion of the proof in [May99, §11.3]; in addition, [Hat02, §4.2], [Pér13, §1.1], and [Zha09, §5] follow a similar strategy. Different methods of proof are available in [tD08, §6.9] (using elementary homotopy techniques) and [Sch] (using the Hurewicz theorem and homotopy fibers).

Reduction 1. It suffices to prove the Excision Theorem when A is built from C by attaching cells of dimension $> n$ and B is built from C by attaching cells of dimension $> m$.

Sketch of Proof. We claim that an n-connected pair (A, C) can be replaced by another n-connected pair (A', C) such that the following diagram commutes

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, hook] & A' \arrow[dl, "\simeq"] \\ A & \end{tikzcd}$$

and A' is built from C by attaching cells of dimension $> n$ only. To show this, we build up a CW complex from C by adding cells which represent elements of $\pi_i(A)$ or gets rid of elements which should not be there. Since $\pi_i(C) \cong \pi_i(A)$ for all $i < n$, we only need to add cells of dimension $> n$ to make this work. This procedure can be carried out for (B, C) as well. □

Reduction 2. It suffices to prove the Excision Theorem when each of A and B is built from C by attaching one cell apiece.

Sketch of Proof. The proof is inductive. First, we claim that it is sufficient to prove the result when (A, C) has exactly one cell. Write $A = A' \cup e$ for $C \subset A' \subset A$, so that (A, A') has one cell, and (A', C) has one less cell than (A, C). Consider $X' = A' \cup_C B$, so X' is just X without the cell e. The heart of the proof is showing that if homotopy excision holds of both $(X'; A', B)$ and $(X; A, X')$ by induction, then it also holds of the triad $(X; A, B)$. This follows from an application of the five lemma (see [Gro12, Lemma 7]) to the exact sequence of triples ([Pér13, Proposition 1.1.5]) for both of the triples in the inclusion $(X'; A', B) \hookrightarrow (X; A, X')$. We can then conclude it is sufficient to consider (A, C) with one cell.

Next, we have to show the same claim holds for (B, C). As before, we write $B = B' \cup e'$ and $X'' = A \cup_C B'$ for $C \subset B' \subset B$. If homotopy excision holds for $(X'; A, B')$ and $(X; X', B)$, then it must hold for $(X; A, B)$ as well, since the inclusion $(A, C) \hookrightarrow (X, B)$ factors as $(A, C) \to (X'', B') \to (X, B)$. This proves the reduction. □

Thus far, we have shown that it suffices to consider $(X; A, B)$ with $A = C \cup e$ and $B = C \cup e'$ for cells e, e' of dimension $> n, > m$, respectively. The technical heart of the proof is showing that homotopy excision actually holds of such triads. The proof of this simplified case typically relies on either simplicial approximation or smooth approximation. We will provide but a brief overview, and the details can be found in the references provided at the beginning of the subsection.

Lemma 3.4. *Suppose that $X = A \cup_C B$, where*

$$A = C \cup e \text{ and } B = C \cup e'$$

are built from C by attaching cells of dimension $> n$ and $> m$, respectively. Then $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < n+m$ and a surjection for $i = n+m$.

Sketch of proof. For any interior points $x \in e^\circ$ and $y \in e'^\circ$, there is a diagram

$$\begin{tikzcd}
\pi_i(A,C) \arrow[r] \arrow[d, "\cong"'] & \pi_i(X,B) \arrow[d, "\cong"] \\
\pi_i(X\setminus\{y\}, X\setminus\{x,y\}) \arrow[r] & \pi_i(X, X\setminus\{x\})
\end{tikzcd}$$

whose vertical maps are isomorphisms. These isomorphisms follow from observing that $X \setminus \{x\}$ is homotopy equivalent to B by retracting $e \setminus \{x\}$ to its boundary, and similar retractions give $X \setminus \{y\} \simeq A$ and $X \setminus \{x, y\} \simeq C$.

We first discuss surjectivity. Consider a representative of $\pi_i(X, B)$, that is, a map $f\colon (I^i, \partial I^i) \to (X, B)$ which takes J^{n-1} to the basepoint $* \in C$. In other words, f maps the top face of I^i into B and the rest of the boundary to $*$. By the diagram above, it suffices to prove that f is homotopic to a map f' via a homotopy h, such that

(i) the image of f' is in $X \setminus \{y\}$,

(ii) for every $t \in I$, the restriction of h_t to the top face of I^i avoids x,

(iii) for every $t \in I$, h_t maps J^{i-1} to $*$.

If we can find such a map f' and homotopy h, then we have shown that every representative in $\pi_i(X, B) \cong \pi_i(X, X \setminus \{x\})$ is homotopic to some representative in $\pi_i(X \setminus \{y\}, X \setminus \{x, y\})$, which is to say $\pi_i(A, C) \to \pi_i(X, B)$ is surjective for $i \leq n + m$. For the proof that we can actually *find* such f' and h, we point the reader to [Gro12, Proposition 8].

Showing that $\pi_i(A, C) \to \pi_i(X, B)$ is injective for $i < n + m$ follows essentially the same argument. Suppose that we have two representatives g, g' of $\pi_i(A, C)$ such that $[g] = [g'] \in \pi_i(X, B)$ via a homotopy $H\colon I^i \times I \to X$. Now replace f in the argument above with the homotopy H. Now we claim that we can find a *new* map H', homotopic to H via a homotopy G, such that H' avoids y, the restriction of G_t to the top face of I^{i+1} avoids x, and G_t maps J^i to $*$. This means that there is a homotopy from f to g in $X \setminus \{y\}$, relative to $X \setminus \{x, y\}$, i.e. $[g] = [g'] \in \pi_i(X \setminus \{y\}, X \setminus \{x, y\})$. This works for $i + 1 \leq n + m$ (the domain of H is the $(i + 1)$-cube while the domain of f is the i-cube), which is to say we have injectivity for $i < n + m$. $\square$

4 The Freudenthal Suspension Theorem

There are no solved problems; there are only problems that are more or less solved.
—Henri Poincaré

Homotopy excision does not help us directly compute homotopy groups, but does promise us a way to relate the homotopy groups of different spaces within a certain *stable* range. The Freudenthal Suspension Theorem uses this idea to explain the consequence of simultaneously suspending the space and increasing the index of its homotopy group, declaring that

$$\pi_i(X) \cong \pi_{i+1}(\Sigma X)$$

for certain i, where ΣX is the reduced suspension of the space X. This result is an important foundational theorem for stable homotopy theory.

We will derive the suspension theorem from the homotopy excision theorem. Freudenthal's original proof in his 1938 paper "Über die Klassen der Sphärenabbildungen I. Große Dimensionen" actually pre-dates the Blakers-Massy theorem (which was published in 1952) and only considers the special case of spheres. There are other proofs which use different techniques (see, for example, Milnor's proof in [Mil63, §2.2] which utilizes Morse theory), but our proof will follow the standard method, as in [Sch, Hat02, Pér13, Zha09].

4.1 Suspensions

Suspension provides us with a way to construct a new space out of an old one (a favorite game of topologists). For a given space X, the suspension of X looks like two cones over X glued together at the base. By also defining a way to suspend maps between topological spaces, we can extend suspension to a functor.

Definition 4.1. *Suspension* is a functor $S\colon \mathbf{Top} \to \mathbf{Top}$ which sends a space X to its suspension SX, which is the space $X \times I$ with $X \times \{0\}$ and $X \times \{1\}$ each collapsed to a point. A map $f\colon X \to Y$ in **Top** is sent to $Sf\colon SX \to SY$ which maps $[x,t] \mapsto [f(x),t]$.

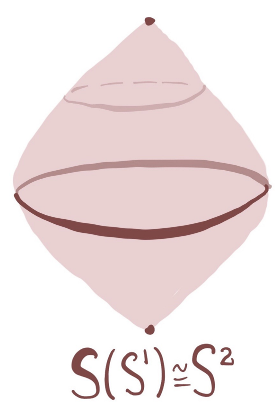

Figure 4: Suspension of S^1.

Suspending a space can be thought of as "increasing the dimension by 1." The well-known, motivating example of suspension is the n-sphere, for which $S(S^n) \cong$

S^{n+1} (cf. [Hat02, p.8]) as illustrated for $n = 1$ in Fig. 4. More generally, we have $S^{n+k} \cong S^n(S^k)$ where the S^{n+k} on the left is the $(n+k)$-sphere and the S^n on the right is the n-iterated suspension functor.

The Freudenthal Suspension Theorem deals with a variation on the suspension functor, called reduced suspension. The *reduced suspension* of a based space (X, x_0) is the quotient space ΣX which is SX with the line segment $\{x_0\} \times I$ collapsed to a single point. If X is a CW complex, then the reduced suspension is (weakly) homotopy equivalent to SX, so one might wonder why we bother with reduced suspensions at all. One possible response is that Σ allows us to make the suspension of a based space (X, x_0) again a based space; the reduced suspension defines a functor from $\mathbf{Top}_*$, the category of based spaces and based maps, to itself. Moreover, the reduced suspension actually results in a simpler space than the unreduced suspension. As illustrated in Fig. 5, the ordinary suspension of $S^1 \vee S^1$ looks like two spheres pinched together along the line $\{x_0\} \times I$, whereas the reduced suspension collapses that line to a point, yielding $S^2 \vee S^2$, a much simpler space.

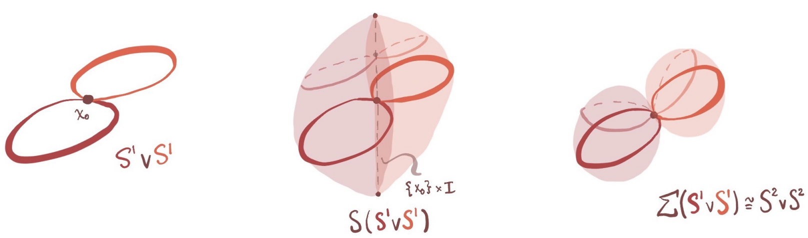

Figure 5: Comparing the suspension and reduced suspension of $S^1 \vee S^1$.

One can also think of ΣX as the smash product $S^1 \wedge X$. This identification follows from the observation that both spaces are the quotient of $X \times I$ with $X \times \{0, 1\} \cup \{x_0\} \times I$ collapsed to a single point. Fig. 6 uses this homeomorphism to show that $\Sigma S^n \cong S^{n+1}$ for $n = 1$. Recall that $S^1 \wedge S^1 = S^1 \times S^1 / S^1 \vee S^1$; in Fig. 6, $S^1 \times S^1$ is illustrated as the torus and $S^1 \vee S^1$ is the red equator and orange meridian which intersect at a point x_0. Collapsing $S^1 \vee S^1$ to the point x_0, done in two steps in Fig. 6 by first contracting the equator and then the meridian, we get a space homeomorphic to S^2.

Another interesting feature of the reduced suspension is that Σ is left-adjoint to the loop space functor Ω. Recall that the loop space of Y is the (based) space ΩY of continuous pointed maps $(S^1, s_0) \to (Y, y_0)$ under the compact open topology. The adjunction between the two functors says that

$$\mathbf{Top}_*(\Sigma X, Y) \cong \mathbf{Top}_*(X, \Omega Y),$$

which means every map $\Sigma X \to Y$ can be bijectively associated with a map $X \to \Omega Y$. Every non-basepoint $x \in X$ lives on a "loop" $[x, I]$ in ΣX which is attached to the basepoint x_0, as illustrated in the left of Fig. 7. Given a map $f \colon \Sigma X \to Y$, we look

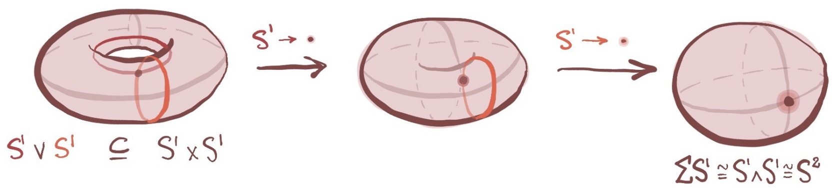

Figure 6: Showing that $\Sigma(S^1) \cong S^1 \wedge S^1$ is homeomorphic to S^2.

where the circle associated to x is sent under f: this will be the loop in Y given by $t \mapsto f[x,t]$. Similarly, if we are given $g\colon X \to \Omega Y$, we can associate the circle $[x,I]$ with the loop $g(x)$; the new map $\Sigma X \to Y$ sends $[x,t] \mapsto g(x)(t)$. This adjunction is actually a specific instance of the adjunction for smash products, which says that $\mathbf{Top}_*(X \wedge A, Y) \cong \mathbf{Top}_*(X, \mathrm{Map}_*(A,Y))$. Here, $\mathrm{Map}_*(A,Y)$ is the collection of basepoint-preserving maps $A \to Y$ under the compact open topology.

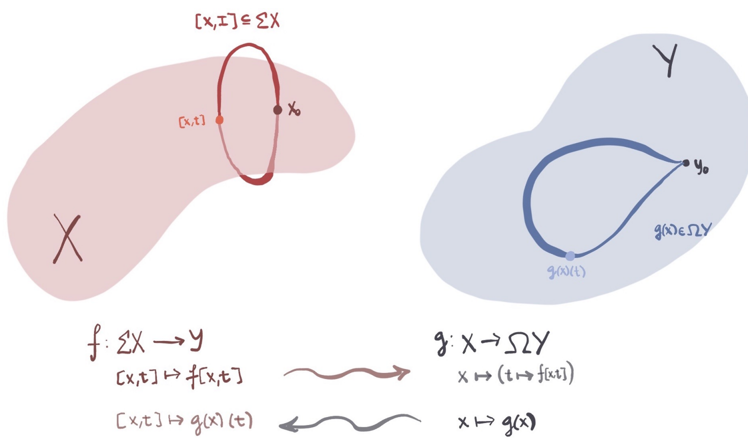

Figure 7: The adjunction between Σ and Ω.

Note that this adjunction implies that $\pi_{i+1}(X) \cong \pi_i(\Omega X)$, since every map $S^i \to \Omega X$ is in bijection with a map $S^{i+1} \cong \Sigma S^i \to X$.

4.2 The Suspension Theorem

The adjunction in the previous subsection gives us one way to define the suspension homomorphism. In categorical language, the suspension homomorphism is the map on the homotopy groups induced by the adjunction unit of $\Sigma \dashv \Omega$. Before we unpack what that means, it will do us good to introduce a concrete definition of this homomorphism.

Definition 4.2. Let (X, x_0) be a based space. The *suspension homomorphism* is the map $\Sigma_*\colon \pi_i(X) \to \pi_{i+1}(\Sigma X)$ which sends $[f] \mapsto [\Sigma f]$, where

$$\begin{aligned} \Sigma f := f \wedge \mathrm{id}_{S^1} \colon S^{k+1} &\to \Sigma X \\ [s,t] &\mapsto [f(s),t]. \end{aligned}$$

The *unit* of the adjunction $\Sigma \dashv \Omega$ is the natural transformation $\eta\colon \mathrm{id}_{Top_*} \to \Omega\circ\Sigma$. This natural transformation sends X to $\Omega\Sigma X$ by mapping a point $x \in X$ to the loop $t \mapsto [x,t]$. The induced map on the homotopy groups $(\eta_X)_*\colon \pi_i(X) \to \pi_i(\Omega\Sigma X)$ sends $f\colon S^i \to X$ to a map

$$\begin{aligned} (\eta_X)_* f\colon S^i &\to \Omega\Sigma X \\ s &\mapsto (t \mapsto [f(s),t]) \,. \end{aligned}$$

Passing through the isomorphism $\pi_i(\Omega X) \cong \pi_{i+1}(X)$ mentioned at the end of the previous subsection, we see that $(\eta_X)_* f$ is in fact Σf. That is, the map induced by the unit is the suspension homomorphism. The Freudenthal Suspension Theorem states that this homomorphism is an isomorphism within a certain range of dimensions.

Theorem 4.3 (Freudenthal)**.** *Suppose X is an $(n-1)$-connected based space. The suspension homomorphism $\Sigma_*\colon \pi_{i-1}(X) \to \pi_i(\Sigma X)$ is an isomorphism for $i < 2n$ and a surjection for $i = 2n$.*

In other words, the suspension homomorphism is a $2n$-equivalence. We will prove this result by providing a suitably nice excisive cover of ΣX, applying homotopy excision, and examining some long exact sequences for relative pairs. This strategy is pretty standard, albeit with small variations between authors. We primarily follow [Sch].

Proof. The suspension ΣX has an excisive cover given by two reduced cones on X, one over and one under, identified along their bases. We will denote these cones by Y_+ (over X) and Y_- (under X), with $Y_0 = Y_+ \cap Y_- = X \times \{1/2\}$. This space is clearly homotopy equivalent to X (a claim which is verified by the homotopy equivalence $x \mapsto [x, 1/2]$). For the sake of simplicity, we will denote a chosen basepoint $[x_0, 1/2] \in Y_0$ by x_0.

We should also note that both Y_+ and Y_- are contractible onto their "poles" (the classes of the collapsed subsets $X \times \{0\}$ and $X \times \{1\}$), via the homotopies

$$\begin{aligned} Y_- \times I &\to Y_- & \qquad Y_+ \times I &\to Y_+ \\ ([x,t],s) &\mapsto [x,ts] & ([x,t],s) &\mapsto [x, t + s(1-t)]. \end{aligned}$$

Our goal, at this point, is to relate the homotopy groups of X to the homotopy groups of ΣX. This is when our old friend, the long exact sequence for relative homotopy groups, comes in handy. We first look at the long exact sequence for the pair $(\Sigma X, Y_\pm)$:

$$\cdots \to \pi_i(Y_\pm, x_0) \to \pi_i(\Sigma X, x_0) \to \pi_i(\Sigma X, Y_\pm, x_0) \to \pi_{i-1}(Y_\pm, x_0) \to \dots$$

Plugging in what we know, namely that $\pi_i(Y_\pm, x_0) = 0$, we get a short exact sequence

$$0 \to \pi_i(\Sigma X, x_0) \to \pi_i(\Sigma X, Y_\pm, x_0) \to 0.$$

In other words, $\pi_i(\Sigma X, x_0) \cong \pi_i(\Sigma X, Y_\pm, x_0)$. Similarly, if we consider the long exact sequence for the pair $(Y_\pm, Y_0)$:

$$\cdots \to \pi_i(Y_\pm, x_0) \to \pi_i(Y_\pm, Y_0, x_0) \to \pi_{i-1}(Y_0, x_0) \to \pi_{i-1}(Y_\pm, x_0) \dots$$

we see that $\pi_i(X, x_0) \cong \pi_{i+1}(Y_\pm, Y_0, x_0)$ as well. Moreover, since X is $(n-1)$-connected, meaning $\pi_i(X, x_0) = 0$ for $i \leq n-1$, the pairs $(Y_\pm, Y_0)$ are n-connected. Applying Blakers-Massey, we have that the map

$$i_*\colon \pi_i(Y_-, Y_0) \to \pi_i(\Sigma X, Y_+)$$

induced by the inclusion is an isomorphism for $i < 2n$ and a surjection for $i = 2n$. Thus we have a commutative diagram

$$\begin{tikzcd}
\pi_i(Y_-, Y_0) \arrow[r, "i_*"] \arrow[d, "\partial"', "\cong"] & \pi_i(\Sigma X, Y_+) \\
\pi_{i-1}(X) \arrow[r] & \pi_i(\Sigma X) \arrow[u, "\cong", "j_*"']
\end{tikzcd}$$

where the left isomorphism is the boundary map $\partial\colon \pi_i(Y_-, Y_0, x_0) \to \pi_{i-1}(Y_0, x_0)$ and the right isomorphism is given by the inclusion $j_*\colon (\Sigma X, x_0, x_0) \to (\Sigma X, Y_+, x_0)$. Our work thus far tells us that the bottom horizontal map is an isomorphism for $i < 2n$ and a surjection for $i = 2n$. It remains to show that this map is in fact the suspension homomorphism Σ_*.

To do so, we need to understand where an element $[f] \in \pi_{i-1}(X)$ is sent to in $\pi_i(Y_-, Y_0)$ under the inverse ∂^{-1}. Given a representative $f\colon S^{i-1} \to X$, we wish to find a map $g\colon (D^k, S^{i-1}) \to (Y_-, Y_0)$ so that $[g|_{S^{i-1}}] = [f]$. A natural choice for this map is

$$g\colon D^k \to Y_-, \quad t \cdot x \mapsto [f(x), t/2]$$

where $t \in [0,1]$ and $x \in S^{i-1}$. This map is continuous at $0 \in D^k$ and its restriction to S^{i-1} (i.e. when $t = 1$) is the composite

$$S^{i-1} \xrightarrow{f} X \xrightarrow{[-,1/2]} Y_0,$$

so indeed $[g] \in \pi_i(Y_-, Y_0)$. Thus $\partial[g] = [f]$ and so $\partial^{-1}[f] = [g]$. This association illustrated in Fig. 8.

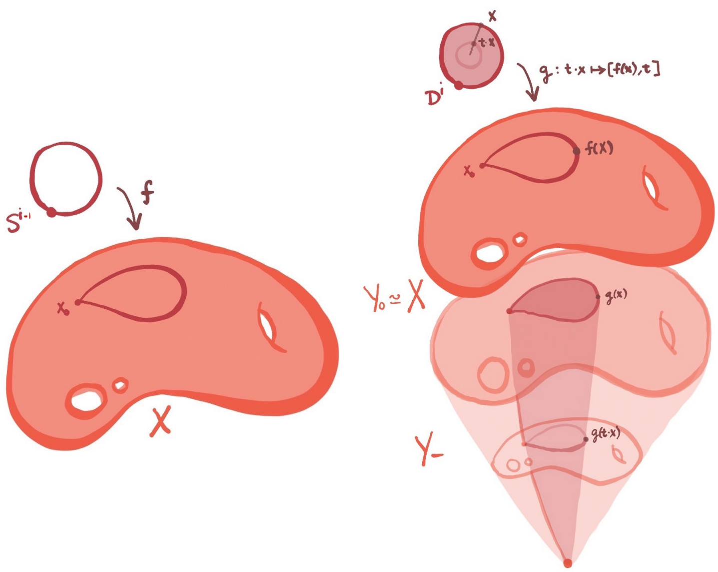

Figure 8: A map $f\colon S^{i-1} \to X$ on the left corresponds (via ∂) to a map $g\colon (D^i, S^{i-1}) \to (Y_-, Y_0)$. The image of g looks like the bottom half of the suspension of f. A radial line $I \cdot x$ in D^i (for fixed $x \in S^{i-1}$) gets sent to the line segment $[\{x\} \times [0, 1/2]]$ in ΣX. For the sake of illustration, we have drawn Y_- as an unreduced cone, and we leave it to the reader to imagine collapsing the line segment $\{x_0\} \times [0, 1/2]$ to a point.

Once we include into $\pi_i(\Sigma X, Y_+)$, we have enough wiggle room in Y_+ to homotope g to the map

$$\hat{g}\colon D^i \to \Sigma X, \quad t \cdot x \mapsto [f(x), t].$$

(One can check that the homotopy $(t \cdot x, s) \mapsto [f(x), t/(2-s)]$ does the trick.)

Finally, we want to understand $[\hat{g}]$ as an element of $\pi_i(\Sigma X)$, rather than as an element of $\pi_i(\Sigma X, Y_+)$. By our earlier work, we know that j_* is an isomorphism, induced by the contraction of Y_+ onto the basepoint x_0. By the construction of $\hat{g}$, any boundary point $s \in S^{i-1}$ is sent to $[f(s), 1] = [x_0, 1] = x_0$. So $[\hat{g}]$ is already an element of $\pi_i(\Sigma X)$, except now we think of $\hat{g}$ as a map out of $\Sigma S^i \cong S^{i+1}$ by precomposing with the homeomorphism

$$\Sigma S^i \cong D^i / S^{i-1}, \quad [x, t] \mapsto t \cdot x.$$

The resulting map is precisely the suspension of f, as shown in Fig. 9. All in all, this implies that the following diagram commutes:

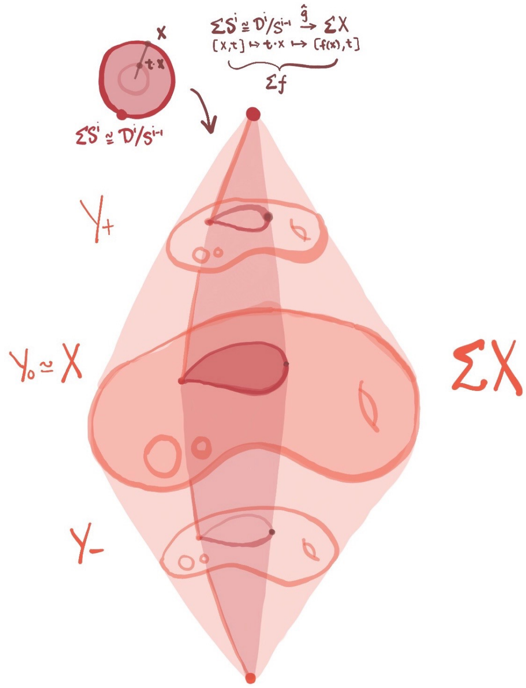

Figure 9: In $\pi_i(\Sigma X, Y_+)$, we can continuously deform g (from Fig. 8) into $\hat{g}$, by stretching the image of $\partial D^i \cong S^{i-1}$ up to the top of the reduced cone. A radial line $I \cdot x$ in D^i is now sent to $[\{x\} \times I]$. The result is (essentially) the suspension of f. As in Fig. 8, we have illustrated the unreduced cones for the sake of simplicity.

$$\begin{tikzcd}
{[g\colon t\cdot x \mapsto [f(x), t/2]]} \arrow[r, mapsto] & {[g] = [\hat{g}\colon t\cdot x \mapsto [f(x), t]]} \arrow[d, mapsto] \\
{[f\colon x \mapsto f(x)]} \arrow[u, mapsto] \arrow[r, mapsto, dashed] & {[\Sigma f\colon [x,t] \mapsto [f(x), t]],}
\end{tikzcd}$$

which means that the bottom horizontal arrow is indeed the suspension homomorphism just as we suspected all along. □

4.3 Applications

The original Freudenthal Suspension Theorem was stated for $X = S^n$, with the intent of calculating the higher homotopy groups of spheres. Specifically, we apply the theorem to see that $\Sigma\colon \pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$ for $i < 2n-1$. In particular, since we already know that $\pi_1(S^1) \cong \pi_2(S^2) \cong \mathbb{Z}$, the suspension theorem tells us that

$$\mathbb{Z} \cong \pi_1(S^1) \cong \pi_2(S^2) \cong \pi_3(S^3) \cong \ldots \cong \pi_n(S^n) \cong \ldots$$

This theorem has also motivated the study of stable homotopy groups of spaces. For an arbitrary CW complex X, the Freudenthal Suspension Theorem and the fact

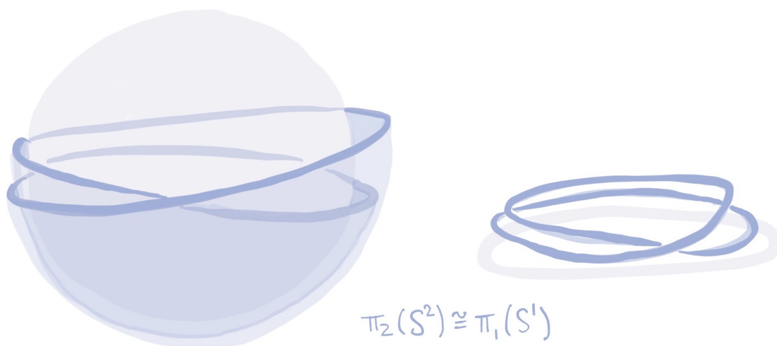

Figure 10: Illustrating wrapping (the bottom half of) a 2-sphere twice around S^2, corresponding to an element of $\pi_1(S^1) \cong \mathbb{Z}$.

that ΣX is connected implies that $\Sigma^n X$ is $(n-1)$ connected, where $\Sigma^n X$ is the n^{th}-iterated reduced suspension of X, i.e. $\Sigma^n X = \Sigma(\Sigma^{n-1}(X))$. Thus the map

$$\Sigma_* \colon \pi_i(\Sigma^n X) \to \pi_{i+1}(\Sigma^{n+1} X)$$

is an isomorphism for $i < 2n-1$. This then implies that for some fixed value of i, the maps in the sequence

$$\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \pi_{i+2}(\Sigma^2 X) \to \cdots \to \pi_{i+n}(\Sigma^n X) \to \pi_{i+n+1}(\Sigma^{n+1} X) \to \dots$$

become isomorphisms. The eventual value where this sequence stabilizes is called the i^{th} stable homotopy group of X.

Definition 4.4. The i^{th} *stable homotopy group* of X is the colimit

$$\pi_i^s(X) = \mathrm{colim}_n(\pi_{i+n}(\Sigma^n X)).$$

Our earlier observation that $\pi_n(S^n) \cong \mathbb{Z}$ tells us that $\pi_0^s(S^0) \cong \mathbb{Z}$; a table for other known values of $\pi_i^s := \pi_i^s(S^0)$ is given on [Hat02, p.384]. The (Abelian) stable homotopy groups of a space are often easier to calculate than their unstable counterparts, although undoubtedly the computations are still quite difficult. Stable homotopy groups are important objects in algebraic topology, and motivate the development of stable homotopy theory.

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