The Tambara Structure of the Trace Ideal for Cyclic Groups

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Overview

Preliminaries
- The Basic Ingredients

The Trace Ideal
- The Dress Map
- The Trace Ideal

The Main Results
- Example Result
  - Generators for cyclic extensions
- Other Results

The End

G-Sets

G-Action

Given a group G an (left) action of G on a set X is function $\cdot : G \times X \to X$ such that for all $g, h \in G$ and $x \in X$

1. $e \cdot x = x$
2. $(gh) \cdot x = g \cdot (h \cdot x)$

G-Set

A G-set is a set X along with a group action of G on X.

Example

Given a group G and a subgroup $H \leq G$ we can make G/H a G-Set by giving the action $g \cdot (g'H) = (gg')H$.

G-Set Category

Equivariant Map

Given two G-sets X and Y an G-equivariant map $f : X \to Y$ is a function on the underlying sets satisfying $f(g \cdot x) = g \cdot f(x)$

G-Set Category

The G-Set Category for a *finite* group G is the category whose objects are finite G-Sets and whose morphisms are G-equivariant Maps.

Important Remark

Any G-set is isomorphic to a G-set formed by taking a finite direct sum of G-sets of the form G/H with action defined as in the previous slide.

Mackey Functors (Abstract Version)

A Mackey Functor M on a finite group G is a pair of a covariant functor M_* and a contravariant functor M^* from G-Set to the category of abelian groups such that.

1. M_* and M^* agree on objects
2. $M(X \sqcup Y) \cong M(X) \oplus M(Y)$
3. Given a pullback diagram in G-Set.

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\delta} \\ W & \xrightarrow{\gamma} & Z \end{array}$$

We have that $M_*(\beta)M^*(\alpha) = M^*(\gamma)M_*(\beta)$

Mackey Functors (Concrete Version)

A Mackey Functor M on a finite group G is an abelian group $M(G/H)$ for each $H \leq G$ and for each $H \leq K \leq G$ and $g \in G$ group homomorphisms $\mathrm{res}^K_H : M(G/K) \to M(G/H)$, $\mathrm{tr}^K_H : M(G/H) \to M(G/K)$ and $c_g : M(G/H) \to M(G/{}^gH)$

1. $\mathrm{res}^H_H = \mathrm{tr}^H_H = c_e = \mathrm{id}$
2. $\mathrm{res}^K_H \circ \mathrm{res}^L_K = \mathrm{res}^L_H$ and $\mathrm{tr}^L_K \circ \mathrm{tr}^K_H = \mathrm{tr}^L_H$
3. $c_g \circ c_h = c_{gh}$, $c_g \circ \mathrm{res}^K_H = \mathrm{res}^{^gK}_{^gH} \circ c_g$ and $c_g \circ \mathrm{tr}^K_H = \mathrm{tr}^{^gK}_{^gH} \circ c_g$
4. $\mathrm{res}^H_J \mathrm{tr}^H_K = \sum_{x \in J\backslash H/K} \mathrm{tr}^J_{J \cap {}^xK} \circ c_x \circ \mathrm{res}^K_{J^x \cap K}$

Translation

Given $H \leq K$ we have a quotient map $q : G/H \to G/K$ and $\mathrm{con}_g : G/H \to G/{}^gH$. We define $\mathrm{tr}^K_H = M_*(q)$, $\mathrm{res}^K_H = M_*(q)$ and $c_g = M_*(\mathrm{con}_g)$

Tambara Functor (Abstract Version)

A Tambara funtor T on a finite group G is a collection of a contravariant functor T^*, and two covariant functors T_+, $T_\cdot$ from the burnside category on G to the category of sets which satisfies

1. (T^*, T_+) is a Mackey Functor
2. $(T^*, T_\cdot)$ is a Semi-Mackey Functor (functor into abelian monoids instead of abelian groups)
3. Given an exponential diagram in G-Set

$$\begin{array}{ccccc} X & \xleftarrow[p]{} & A & \xleftarrow[\lambda]{} & Z \\ \downarrow f & & & & \downarrow \rho \\ Y & & \xleftarrow[q]{} & & B \end{array}$$

We have $T_\cdot(f)T_+(p) = T_+(q)T_\cdot(\rho)T^*(\lambda)$

Tambara Functors (Concrete)

A Tambara Functor on a group G is given by,

1. A ring $T(G/H)$ for each $H \leq G$
2. For each $H \leq K \leq G$ and $g \in G$ the maps
   2.1 $\mathrm{res}_H^K : T(G/K) \to T(G/H)$
   2.2 $\mathrm{tr}_H^K : T(G/H) \to T(G/K)$
   2.3 $\mathrm{N}_H^K : T(G/H) \to T(G/K)$
   2.4 $c_g : T(G/H) \to T(G/{}^gH)$

   Where res is a ring homomorphism, tr is a homomorphism on the additive abelian group and N is a homomorphism on the multiplicative abelian monoid.
3. Satisfying a bunch of compatibility conditions on the three maps from which we can recover the original definition.

Examples

Constant Tambara Functor

Given a ring R and a group G we define the constant Tambara functor $\underline{R}$ at R on G by

1. $\underline{R}(G/H) = R$
2. res^K_H and c_g are the identity maps
3. $\mathrm{tr}^K_H(a) = |K : H|a$
4. $\mathrm{N}^K_H(a) = a^{|K:H|}$

Galois Correspondence

Given a field extension K/F with Galois group G. The Galois Correspondence which maps G/H to K^H along with inclusion maps, field trace and field norm maps gives a Tambara Functor. This is a subcase of a more general "fixed point Tambara Functor".

The Burnside Tambara Functor

The Burnside Ring

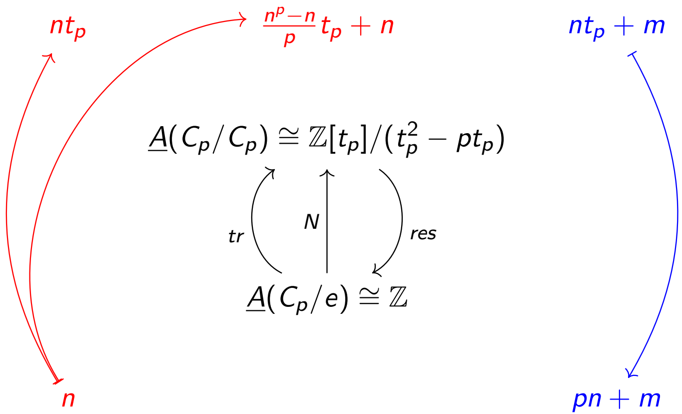
Given a finite group G the Burnside ring $A(G)$ on G is the ring of finite G-Sets with addition given by direct sum and multiplication given by cartesian product.

The Burnside Tambara Functor

Given a finite group G the burnside Tambara functor $\underline{A}(G)$ is given by

1. $\underline{\mathrm{A}}(\mathrm{G}/\mathrm{H}) = \mathrm{A}(\mathrm{H})$
2. restriction is given by restricting the action
3. $\mathrm{tr}_H^K(H/J) = K/J$, $c_g(H/K) = c_g({}^gH/{}^gK)$
4. $\mathrm{N}_H^K(X) = \mathrm{Map}_H(K, X)$

Example: C_p


nt_p
$\frac{n^p-n}{p}t_p + n$
$nt_p + m$
$\underline{A}(C_p/C_p) \cong \mathbb{Z}[t_p]/(t_p^2 - pt_p)$
tr
N
res
$\underline{A}(C_p/e) \cong \mathbb{Z}$
n
$pn + m$

Grothendieck-Witt Ring

Quadratic forms

A quadratic form is a function of the form $q(X_1, X_2, \dots X_n) = aX_1X_1 + bX_1X_2 + \cdots + dX_nX_n$ where the input/output is taken to be in some field F. Two quadratic forms are considered equal if there is a linear coordinate transformation taking one to the other. Any quadratic form is equivalent to one of the form $q = a_1X_1^2 + \cdots + a_nX_n^2$ which we write as $\langle a_1, \dots, a_n\rangle$

Grothendieck-Witt Ring

Given a field F the Grothendieck Witt ring $\mathrm{GW}(F)$ is the ring of all quadratic forms over F where addition is given by $\langle a_1, \dots, a_n\rangle + \langle b_1, \dots, b_m\rangle = \langle a_1, \dots a_n, b_1, \dots b_n\rangle$ and multiplication by $\langle a_1, a_1 \dots, a_n\rangle\langle b_1, \dots, b_m\rangle = \langle a_1b_1, a_1b_2 \dots a_nb_n\rangle$

Grothendieck-Witt Tambara Functor

Given a field extension K/F with finite Galois group G we define the Grothendieck Witt Tambara Functor via

1. For each $H \leq G$, $\mathrm{GW}_F^K(G/H) = \mathrm{GW}(K^H)$
2. Restriction is given by inclusion maps
3. transfer and norm are ..... complicated (part of the project was figuring out how to actually calculate them)

Example

Finite Fields and Cyclic Groups

Let $\mathbb{F}_q$ denote the finite field of order q (power of odd prime).

$$\mathbb{F}_q \subseteq \mathbb{F}_r \iff q \mid r \iff r = q^N$$

Then $\mathrm{Gal}(\mathbb{F}_{q^N}/\mathbb{F}_q) = C_N$.

Example

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Prime Extensions

For $\mathbb{F}_q \subseteq \mathbb{F}_{q^p}$ with p prime, only two 'levels.'

Example: $\mathbb{F}_q \subseteq \mathbb{F}_{q^p}$ for an odd prime p

$$\begin{array}{c} \underline{GW}(\mathbb{F}_q) \cong \mathbb{Z} \oplus \langle \alpha \rangle \\ tr \uparrow \quad N \uparrow \quad \downarrow res \\ \underline{GW}(\mathbb{F}_{q^p}) \cong \mathbb{Z} \oplus \langle \beta \rangle \end{array}$$

$$\begin{aligned}
\text{restriction: } \langle 1 \rangle &\mapsto \langle 1 \rangle \\
\langle \alpha \rangle &\mapsto \langle \beta \rangle \\
\text{transfer: } \langle 1 \rangle &\mapsto p\langle 1 \rangle \\
\langle \beta \rangle &\mapsto (p-1)\langle 1 \rangle \oplus \langle \alpha \rangle \\
\text{norm: } n\langle 1 \rangle &\mapsto n^p \langle 1 \rangle \\
(n-1)\langle 1 \rangle \oplus \langle \beta \rangle &\mapsto (n^p - 1)\langle 1 \rangle \oplus \langle \alpha^n \rangle
\end{aligned}$$

Tambara Functor Morphisms

Tambara Functor Morphism (Abstract)

$\varphi : T \to S$ is a collection of ring homomorphisms $T(X) \to S(X)$ that forms a natural transformation with respect to each of the component Functors of the Tambara functor.

Tambara Functor Morphism (Concrete)

A Tambara functor morphism $\varphi : T \to S$ is a ring homomorphism $\varphi_H : T(G/H) \to S(G/H)$ for each $H \leq G$ such that

$$\begin{array}{ccc} T(G/K) & \xrightarrow{\varphi_K} & S(G/K) \\ \downarrow{\scriptstyle \mathrm{res}^K_H} & & \downarrow{\scriptstyle \mathrm{res}^K_H} \\ T(G/H) & \xrightarrow{\varphi_H} & S(G/H) \end{array}$$

commutes. Similar diagrams for tr, N, c_g

The Dress Map

Definition

- For rings, trace homomorphism (A. Dress [2], 1971)

$$\mathcal{D}\colon A(G) \to GW(F)$$

maps $G/H \mapsto \mathrm{tr}_F^{K^H}\langle 1\rangle$.

- For Tambara functors, **Dress map**

$$\mathcal{D}\colon \underline{A}_G \to \underline{GW}_{K/F}$$

is given by trace homomorphism at each level.

Example

$$t_p \longmapsto p\langle 1\rangle$$

$$1 \longmapsto \langle 1\rangle$$

$$\begin{array}{ccc}
\underline{A}(C_p/C_p) & \xrightarrow{\mathcal{D}_{C_p}} & \underline{GW}(\mathbb{F}_q) \\
tr \uparrow \ N \uparrow \ \downarrow res & & tr \uparrow \ N \uparrow \ \downarrow res \\
\underline{A}(C_p/e) & \xrightarrow[\mathcal{D}_e]{} & \underline{GW}(\mathbb{F}_{q^p})
\end{array}$$

$$1 \longmapsto \langle 1\rangle$$

Example

$$\begin{array}{ccc} \underline{A}(C_p/C_p) & \xrightarrow{\mathcal{D}_{C_p}} & \underline{GW}(\mathbb{F}_q) \\ tr \uparrow \; N \uparrow \; \downarrow res & & tr \uparrow \; N \uparrow \; \downarrow res \\ \underline{A}(C_p/e) & \xrightarrow{\mathcal{D}_e} & \underline{GW}(\mathbb{F}_{q^p}) \end{array}$$

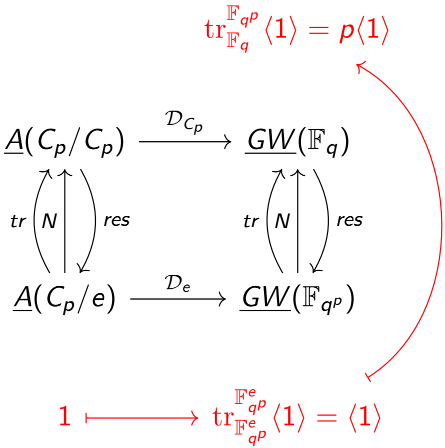
1

Example

$$\begin{array}{ccc} \underline{A}(C_p/C_p) & \xrightarrow{\mathcal{D}_{C_p}} & \underline{GW}(\mathbb{F}_q) \\ tr \uparrow \; N \uparrow \; \downarrow res & & tr \uparrow \; N \uparrow \; \downarrow res \\ \underline{A}(C_p/e) & \xrightarrow{\mathcal{D}_e} & \underline{GW}(\mathbb{F}_{q^p}) \end{array}$$

$$1 \longmapsto \mathrm{tr}_{\mathbb{F}^e_{q^p}}^{\mathbb{F}^e_{q^p}} \langle 1 \rangle = \langle 1 \rangle$$

Example



$\mathrm{tr}_{\mathbb{F}_q}^{\mathbb{F}_{q^p}}\langle 1\rangle = p\langle 1\rangle$
$\underline{A}(C_p/C_p)$
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$\underline{GW}(\mathbb{F}_q)$
tr
N
res
tr
N
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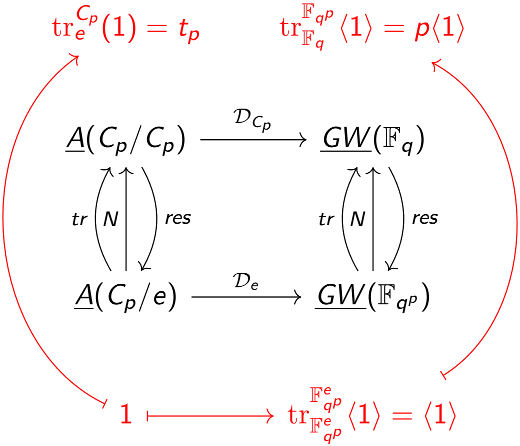
$\mathrm{tr}_e^{C_p}(1) = t_p$
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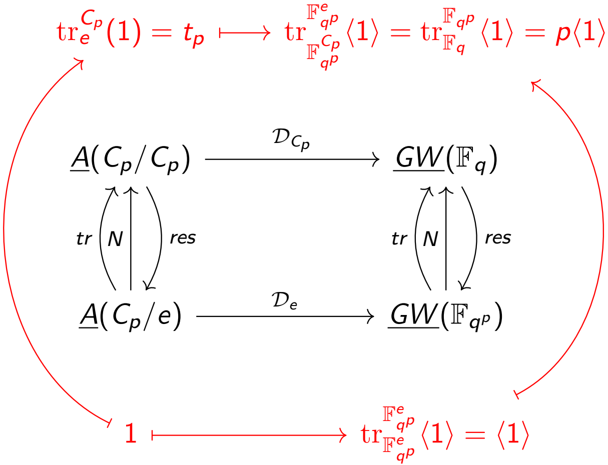
$\mathrm{tr}_e^{C_p}(1) = t_p \longmapsto \mathrm{tr}_{\mathbb{F}_{q^p}^{C_p}}^{\mathbb{F}_{q^p}^e}\langle 1\rangle = \mathrm{tr}_{\mathbb{F}_q}^{\mathbb{F}_{q^p}}\langle 1\rangle = p\langle 1\rangle$
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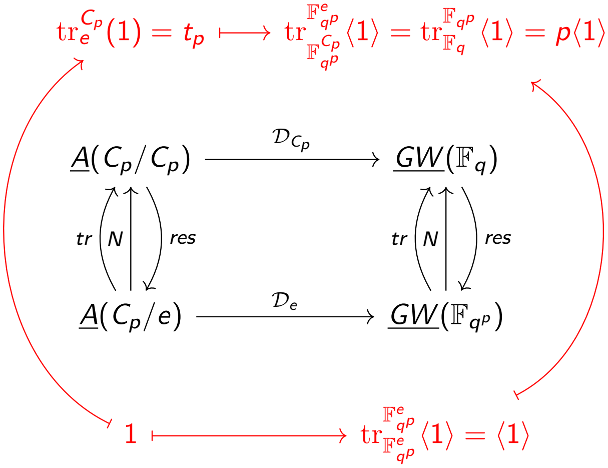

Surjectivity?

The Trace Ideal

Definition

The **trace ideal** is the kernel of the Dress map,

$$\mathcal{TI}_{K/F} = \{\ker(\mathcal{D}_H)\}_{H \leq G}$$

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Who cares?

Via isomorphism theorems,

$$\underline{GW}_{K/F} \cong \underline{A}_G / \mathcal{TI}_{K/F}$$

when the Dress map is surjective.

Hang on, what are Tambara ideals?

Definition

An **ideal** I of a Tambara functor:

1. A ring-theoretic ideal $I(G/H) \subseteq T(G/H)$ for all $H \leq G$,
2. Closed under structure maps.

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Generators

- The ideal **generated** by a subset S is the intersection of all ideals of T containing S, denoted $(\!(S)\!)$.
- An ideal is **principal** if $I = (\!(a)\!)$ for some $a \in T$.
- An ideal is **strongly principal** if there exists $H \leq G$ and $a \in T(G/H)$ such that $I = (\!(a)\!)$.

The Agenda

Main Example

Finite extensions of finite fields $\mathbb{F}_q \subseteq \mathbb{F}_{q^p}$ with Galois group C_p (odd prime p).

Goal

Describe the trace ideal for cyclic extensions and find generators.

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Finite extensions of finite fields $\mathbb{F}_q \subseteq \mathbb{F}_{q^p}$ with Galois group C_p (odd prime p).

Goal

Describe the trace ideal for cyclic extensions and find generators.

Spoiler: It's strongly principal!

First Thoughts

The bottom level

Since $\mathcal{D}(n) = n\langle 1 \rangle$,

$$n \in \mathcal{TI}_{\mathbb{F}_{q^p}/\mathbb{F}_q} \iff n = 0.$$

So $\mathcal{TI}_{\mathbb{F}_{q^p}/\mathbb{F}_q}(C_p/e) = (0)$.

First Thoughts

The bottom level

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The top level

Have $\mathcal{D}(nt_p + m) = np\langle 1\rangle + m\langle 1\rangle = (np + m)\langle 1\rangle$, so

$$nt_p + m \iff np - m = 0 \iff \text{multiples of } t_p - p$$

In general...

Theorem

If n is odd, then

$$\mathcal{TI}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(C_n/C_m) = (t_{p^k} - p^k \ : \ p \text{ prime}, \ p^k \mid m)$$

is generated as a Tambara ideal by $t_p - p \in \underline{A}(C_n/C_p)$ for each prime divisor p of n.

In general...

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If n is odd, then

$$\mathcal{TI}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(C_n/C_m) = (t_{p^k} - p^k \ : \ p \text{ prime}, \ p^k \mid m)$$

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Main Theorem for Cyclic Groups

There is *one* generator!

Other Results and Future Work

1. Arbitrary cyclic extensions (non-finite fields)
   - For $\mathbb{C}/\mathbb{R}$, e.g., the trace ideal is 0, implying $\underline{GW} \cong \underline{A}$

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   - For $\mathbb{C}/\mathbb{R}$, e.g., the trace ideal is 0, implying $\underline{GW} \cong \underline{A}$
2. Profinite extensions of finite fields
   - Quadratic closure $\mathbb{F}_{q^{2^\infty}}$ and the algebraic closure $\overline{\mathbb{F}_q}$

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1. Arbitrary cyclic extensions (non-finite fields)
   - For $\mathbb{C}/\mathbb{R}$, e.g., the trace ideal is 0, implying $\underline{GW} \cong \underline{A}$
2. Profinite extensions of finite fields
   - Quadratic closure $\mathbb{F}_{q^{2^\infty}}$ and the algebraic closure $\overline{\mathbb{F}_q}$
3. Prime ideals of $\underline{A}$
   - In progress...

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Thank You!

(Questions?)