The Tambara Structure of the Trace Ideal

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February 1, 2020

The Big Idea

Ring Theory $\leftrightsquigarrow$ Tambara Functor Theory

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Trace homomorphism:	Dress map:
$A \to GW$	$\underline{A} \to \underline{GW}$
Kernel	Trace ideal $\mathcal{TI}$

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Trace homomorphism: $A \to GW$	Dress map: $\underline{A} \to \underline{GW}$
Kernel	Trace ideal $\mathcal{TI}$

Goal: Determine $\mathcal{TI}$

Then

$$\underline{GW} \cong \underline{A} \,/\, \mathcal{TI}$$

when the Dress map is surjective.

The Basic Ingredients

Our Main Example

Cyclic group with N elements: C_N.
Finite field with q elements: $\mathbb{F}_q$ (for q a power of an odd prime).

$\mathbb{F}_q \subseteq \mathbb{F}_p \iff q \mid p$, i.e. $p = q^N$. Then $\mathrm{Gal}(\mathbb{F}_p/\mathbb{F}_q) = C_N$.

The Less Basic Ingredients

Tambara functors (D. Tambara [4], 1993)

Specified by data:

- A commutative ring for each subgroup of G
- Tambara structure maps **restriction, transfer, norm,** and **conjugation** satisfying various compatibility conditions and commutative diagrams

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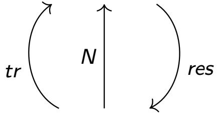
Examples we care about:

- **Burnside** Tambara functor $\underline{A}$
- **Grothendieck-Witt** (Galois) Tambara functor $\underline{GW}$

The Burnside functor $\underline{A}$ on C_p

$$\color{red}{n[C_p/e] + m[C_p/C_p] := nt_p + m \text{ and } t_p^2 = pt_p}$$

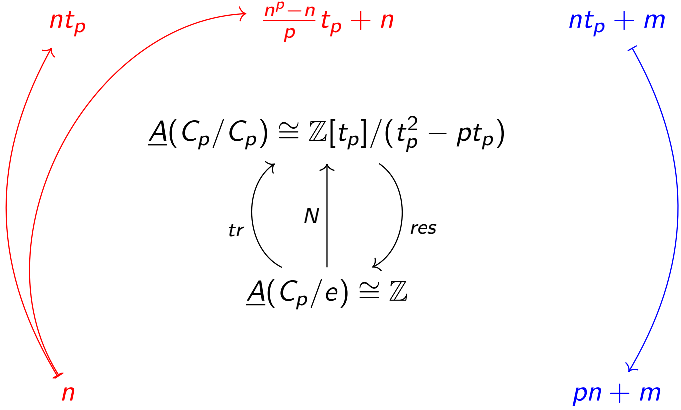
$$\underline{A}(C_p/C_p) \cong \mathbb{Z}[t_p]/(t_p^2 - pt_p)$$



$$\underline{A}(C_p/e) \cong \mathbb{Z}$$

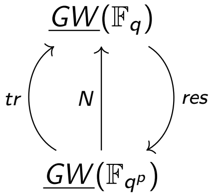
$$\color{blue}{n[e/e] := n}$$

The Burnside functor $\underline{A}$ on C_p



The Grothendieck-Witt functor $\underline{GW}$ on $\mathbb{F}_q \subseteq \mathbb{F}_{q^p}$

$$n\langle 1\rangle \oplus \langle \alpha\rangle$$



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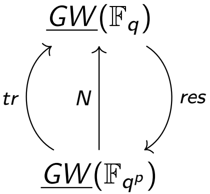

$$
\begin{aligned}
\text{restriction: } \langle 1\rangle &\mapsto \langle 1\rangle \\
\langle \alpha\rangle &\mapsto \langle \beta\rangle \\
\text{transfer: } \langle 1\rangle &\mapsto p\langle 1\rangle \\
\langle \beta\rangle &\mapsto (p-1)\langle 1\rangle \oplus \langle \alpha\rangle \\
\text{norm: } n\langle 1\rangle &\mapsto n^p\langle 1\rangle \\
(n-1)\langle 1\rangle \oplus \langle \beta\rangle &\mapsto (n^p-1)\langle 1\rangle \oplus \langle \alpha^n\rangle
\end{aligned}
$$

The Dress Map

Definition

- For rings, trace homomorphism (A. Dress [2], 1971)
- For Tambara functors, **Dress map** $\mathcal{D}$ is given by trace homomorphism at each level

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$$t_p \longmapsto p\langle 1\rangle$$

$$1 \longmapsto \langle 1\rangle$$

$$\begin{array}{ccc}
\underline{A}(C_p/C_p) & \xrightarrow{\mathcal{D}_{C_p}} & \underline{GW}(\mathbb{F}_q) \\
tr \uparrow \; N \uparrow \; \downarrow res & & tr \uparrow \; N \uparrow \; \downarrow res \\
\underline{A}(C_p/e) & \xrightarrow[\mathcal{D}_e]{} & \underline{GW}(\mathbb{F}_{q^p})
\end{array}$$

$$1 \longmapsto \langle 1\rangle$$

Example: The Whole Picture

$$
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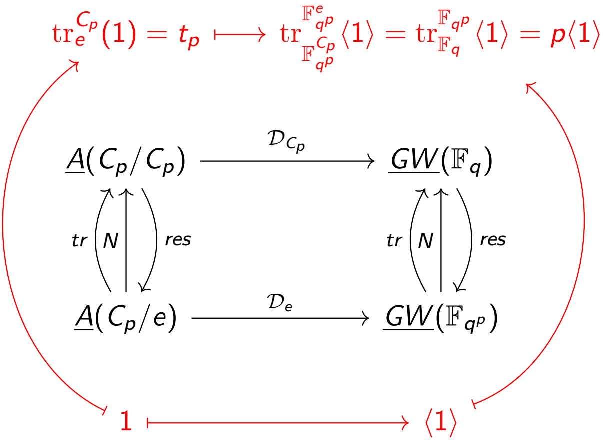
Example: The Whole Picture



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Definition

The **trace ideal** is the kernel of the Dress map,

$$\mathcal{TI} = \{\ker(\mathcal{D}_H)\}_{H \leq G}$$

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Goal

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Theorem

For cyclic groups, there is *one* generator!

Other Results and Future Work

1. Arbitrary cyclic extensions (non-finite fields)
   - For $\mathbb{C}/\mathbb{R}$, e.g., the trace ideal is 0, implying $\underline{GW} \cong \underline{A}$
2. Profinite extensions of finite fields
   - Quadratic closure $\mathbb{F}_{q^{2^\infty}}$ and the algebraic closure $\overline{\mathbb{F}_q}$
3. Prime ideals of $\underline{A}$
   - In progress...

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Thank You!

(Questions?)