Putting the "k" in Curvature: k-Plane Constant Curvature Conditions

Maxine Calle

Reed College

with Dr. Corey Dunn
at California State University, San Bernardino
REU 2018

callem@reed.edu

November 10, 2018

Overview

Who Cares?

Motivation:

We study curvature to generate representative numbers that can characterize model spaces.

Who Cares?

Motivation:

We study curvature to generate representative numbers that can characterize model spaces.

Goal:

We generalize the conditions known as constant sectional curvature and constant vector curvature.

Okay, what do I need to know?

- An **Algebraic Curvature Tensor (ACT)** is a multilinear function $R : V \times V \times V \times V \to \mathbb{R}$ with the following properties:
  - *Skew-symmetry in the first two slots, interchange symmetry, and the Bianchi Identity.*

  We say R is a **canonical ACT** if
  $R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$
  for some symmetric, bilinear form ϕ.

Okay, what do I need to know?

- An **Algebraic Curvature Tensor (ACT)** is a multilinear function $R : V \times V \times V \times V \to \mathbb{R}$ with the following properties:
  - *Skew-symmetry in the first two slots, interchange symmetry, and the Bianchi Identity.*

  We say R is a **canonical ACT** if
  $R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$
  for some symmetric, bilinear form ϕ.
- A **model space** $\mathcal{M} = (V, \langle\ ,\ \rangle, R)$ is defined as a vector space $V \subseteq \mathbb{R}^n$, a non-degenerate inner product on V, and an algebraic curvature tensor.

Okay, what do I need to know?

- An **Algebraic Curvature Tensor (ACT)** is a multilinear function $R : V \times V \times V \times V \to \mathbb{R}$ with the following properties:
  - *Skew-symmetry in the first two slots, interchange symmetry, and the Bianchi Identity.*

  We say R is a **canonical ACT** if $R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$ for some symmetric, bilinear form ϕ.
- A **model space** $\mathcal{M} = (V, \langle\ ,\ \rangle, R)$ is defined as a vector space $V \subseteq \mathbb{R}^n$, a non-degenerate inner product on V, and an algebraic curvature tensor.
- Let $\mathcal{M}$ be a model space and let $x, y \in V$ be tangent vectors. Let $\pi = span\{x, y\}$ be a non-degenerate 2-plane. The **sectional curvature** is a function $\kappa : V \times V \to \mathbb{R}$, where

$$\kappa(\pi) = \frac{R(x, y, y, x)}{\langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2}.$$

You've heard of 2-plane curvature...

Constant Sectional Curvature

A model space $\mathcal{M}$ has **constant sectional curvature** ε, denoted csc(ε), if $\kappa(\pi) = \varepsilon$ for all non-degenerate 2-planes π.

You've heard of 2-plane curvature...

Constant Sectional Curvature

A model space $\mathcal{M}$ has **constant sectional curvature** ε, denoted $\mathrm{csc}(\varepsilon)$, if $\kappa(\pi) = \varepsilon$ for all non-degenerate 2-planes π.

Constant Vector Curvature

A model space $\mathcal{M}$ has **constant vector curvature** ε, denoted $\mathrm{cvc}(\varepsilon)$, if for every $v \in V$, there is some 2-plane π where $v \in \pi$ and $\kappa(\pi) = \varepsilon$ for all non-degenerate 2-planes π.

...but get ready for k-plane curvature!!

- Let $M = (V, \langle\ ,\ \rangle, R)$ with $V \subseteq \mathbb{R}^n$ and non-degenerate inner product. Let L be a k-plane spanned by some orthonormal basis $\{f_1, \ldots, f_k\}$. Define the ***k*-plane scalar curvature** of L, $\mathcal{K} : L \to \mathbb{R}$, by

$$\mathcal{K}(L) = \sum_{j>i=1}^{k} \kappa(f_i, f_j).$$

...but get ready for k-plane curvature!!

- Let $M = (V, \langle\ ,\ \rangle, R)$ with $V \subseteq \mathbb{R}^n$ and non-degenerate inner product. Let L be a k-plane spanned by some orthonormal basis $\{f_1, \ldots, f_k\}$. Define the ***k*-plane scalar curvature** of L, $\mathcal{K} : L \to \mathbb{R}$, by

$$\mathcal{K}(L) = \sum_{j>i=1}^{k} \kappa(f_i, f_j).$$

k-Plane Constant Sectional Curvature

A model space $\mathcal{M}$ has ***k*-plane constant sectional curvature** ε, denoted k-csc(ε), if $\mathcal{K}(L) = \varepsilon$ for all non-degenerate k-planes L.

k-Plane Constant Vector Curvature

A model space $\mathcal{M}$ has ***k*-plane constant vector curvature** ε, denoted k-csc(ε), if for all $v \in V$ there is some non-degenerate k-plane L containing v such that $\mathcal{K}(L) = \varepsilon$.

General Results

1. If a model space $\mathcal{M}$ has k-csc(ε) then it has k-cvc(ε).
2. Let $M_1 = (V, \langle\ ,\ \rangle, R_1)$ have k-csc(ε) and $M_2 = (V, \langle\ ,\ \rangle, R_2)$ have k-cvc(δ). Then $M = (V, \langle\ ,\ \rangle, R = R_1 + R_2)$ has k-cvc$(\varepsilon + \delta)$.
3. Suppose $M = (V, \langle\ ,\ \rangle, R)$ has k-cvc(ε). Let $c \in \mathbb{R}$. Then $M = (V, \langle\ ,\ \rangle, cR)$ has k-cvc$(c\varepsilon)$.
4. Let $M = (V, \langle\ ,\ \rangle, R)$ where $\dim(\ker(R)) \geq k - 1$. Then $\mathcal{M}$ has k-cvc(0).

k-Plane Constant Sectional Curvature

Recall the Definition:

A model space $\mathcal{M}$ has ***k*-plane constant sectional curvature** ε, denoted k-csc(ε), if $\mathcal{K}(L) = \varepsilon$ for all non-degenerate k-planes L.

k-csc: Main Result and Some Nice Corollaries

Theorem

Set $2 \leq k \leq n-2$. Let $M = (V, \langle\ ,\ \rangle, R)$ be a model space. Suppose $\mathcal{K}(L) = 0$ for all k-planes L. Then $R = 0$.

k-csc: Main Result and Some Nice Corollaries

Theorem

Set $2 \leq k \leq n-2$. Let $M = (V, \langle\ ,\ \rangle, R)$ be a model space. Suppose $\mathcal{K}(L) = 0$ for all k-planes L. Then $R = 0$.

Corollaries

1. Suppose $\mathcal{K}_{R_1}(L) = \mathcal{K}_{R_2}(L)$ for all k-planes L. Then $R_1 = R_2$.
2. There is a unique R giving k-csc(ε) where $R = \frac{2\varepsilon}{k(k-1)} R_*$.
3. $\mathcal{M}$ has k-csc(ε) if and only if it has j-csc(δ), where $\delta = \varepsilon \frac{j(j-1)}{k(k-1)}$.

k-csc: Main Result and Some Nice Corollaries

Theorem

Set $2 \leq k \leq n-2$. Let $M = (V, \langle\ ,\ \rangle, R)$ be a model space. Suppose $\mathcal{K}(L) = 0$ for all k-planes L. Then $R = 0$.

Corollaries

1. Suppose $\mathcal{K}_{R_1}(L) = \mathcal{K}_{R_2}(L)$ for all k-planes L. Then $R_1 = R_2$.
2. There is a unique R giving k-csc(ε) where $R = \frac{2\varepsilon}{k(k-1)} R_*$.
3. $\mathcal{M}$ has k-csc(ε) if and only if it has j-csc(δ), where $\delta = \varepsilon \frac{j(j-1)}{k(k-1)}$.

"What's the *deal* with $(n-1)$-csc(0)?"

- Weird things happen!
- Conjecture: $R \neq 0$.

k-Plane Constant Vector Curvature

Recall the Definition:

A model space $\mathcal{M}$ has ***k*-plane constant vector curvature** ε, denoted k-csc(ε), if for all $v \in V$ there is some non-degenerate k-plane L containing v such that $\mathcal{K}(L) = \varepsilon$.

k-cvc: Calculating k-cvc Values

Methods:

- Work in model spaces with canonical tensors,
- Use the eigenspaces!

For a model space with k-cvc(ε), we can...

- Calculate multiple values for ε for given k,
- Set loose bounds for ε,
- Rotate $(k-1)$-planes in $v^{\perp}$ to get a connected set of curvature values.

k-cvc: Example

Let $M = (V, \langle\ ,\ \rangle, R)$ be a model space such that $V \subseteq \mathbb{R}^n$, the inner product is positive definite, and $R = R_\phi$ where ϕ is represented by

$$\begin{bmatrix} I_2 & 0_2 & \dots & 0_2 \\ 0_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0_2 & 0 & \dots & 0 \end{bmatrix}$$

where I_2 is the 2×2 identity matrix and 0_2 is the 2×2 matrix whose entries are all 0. Note that $\lambda_1 = 1$ where $\dim(E_1) = 2$ and $\lambda_2 = 0$ where $\dim(E_2) = n - 2$.

- For $k \geq 3$, $\mathcal{M}$ has k-cvc(0) and k-cvc(1).
- If $\mathcal{M}$ has 3-cvc(ε), then $\varepsilon \in [0, 1]$.
- For $k \geq 4$, if $\mathcal{M}$ has k-cvc(ε), then $\varepsilon \in [0, k - 1)$.

k-cvc: Connected Sets of Curvature Values

For any $v \in V$, rotate $(k-1)$-planes in $v^\perp$ to get a connected set of curvature values. Let the linear transformation $A_\theta : [0, \frac{\pi}{2}] \to SO(n)$ be represented by

$$A_\theta = \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & I_{n-k-1} \end{bmatrix}$$

where

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

and I_m is the $m \times m$ identity matrix.

- Let $v \in V$. For all $\varepsilon \in [0,1]$, there is a k-plane L containing v and some θ such that $\mathcal{K}(A_\theta L) = \varepsilon$.
- Audience Participation: give me an ε, any ε![1]

[1] *Any $\varepsilon \in [0, 1]$.

In Conclusion,

Goals:

- To study k-csc and k-cvc in Riemannian model spaces.
- To generalize some previously known results from 2-plane constant curvature conditions.

Results:

- For k-csc:
  - k-csc(ε) uniquely determines R for $2 \leq k \leq n-2$.
  - k-csc(ε) $\Leftrightarrow$ j-csc(δ), where $\delta = \varepsilon \frac{j(j-1)}{k(k-1)}$.
  - $(n-1)$-csc(0) is strange and interesting.
- For k-cvc:
  - ε can be found in terms of products of eigenvalues.
  - ε can be bounded based on sectional curvatures.
  - For any $[a, b] \subset \mathbb{R}$, there exists $\mathcal{M}$ with k-cvc of at least that interval.

References

B.Y. Chen (1999)
Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimension
Glasgow Math. J. 41, 33 – 41.

P. Gilkey (2001)
Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor
World Scientific Pub.

R. Klinger (1991)
A Basis that Reduces to Zero as many Curvature Components as Possible
Abh. Math. Sem. Univ. Hamburg 61, 243 – 248.

M. Beveridge (2017)
Constant Vector Curvature for Skew-Adjoint and Self-Adjoint Canonical Algebraic Curvature Tensors
CSUSB REU.

Thank You!

(The End.)