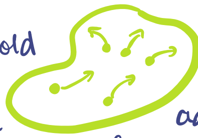
A Taste of Symplectic Geometry 04/18

A good question to start with is What is a symplectic manifold?
Wikipedia says: A smooth manifold with a closed, non-degenerate 2-form
But what does this mean? The motivation comes from classical mechanics: (which I won't pretend to know anything about, but here's the idea)

Say we have some manifold  $M = \mathrm{Conf}_1(M)$ and a little particle running around M. We can think of M as the conf. sp. of different possible states our baby system could be in. And we could add in more particles and complicate the system, also complicating our space of possible states, i.e. our manifold.

But our configuration space can't keep track of everything. It sees things "discretely" in the sense that it can't keep track of how the system evolves over time. In order to do this, what we really want to do is introduce dynamics on our configuration space, i.e. a vector field.

In classical mechanics, this might look like keeping track of a particle's momentum. The dynamics usually come from an energy function $H: M \to \mathbb{R}$, and we use H to generate a vector field X_H which

- depends only (linearly) on H — describes how energy changes locally
- H is constant along flow lines of X_H — "conservation of energy"

It turns out the "correct" abstract setting for this is symplectic geometry.
But why?

1. Want to associate H to v.f.:

$$\begin{tikzcd} T^*M \arrow[r, "\cong"] \arrow[d, shift left] & TM \arrow[dl, shift left] \\ M \arrow[u, bend left, "d_H"] \arrow[ur, bend right, "X_H"'] & \end{tikzcd}$$

equivalently, $\mathrm{Hom}_M(TM, T^*M) \cong$ sections of $\begin{matrix} T^*M \otimes T^*M \\ \downarrow \\ M \end{matrix}$

s.t. $\omega(X_H, -) = dH$.

2. ω is non-degen so we can always solve for X_H

3. Conservation of energy means $0 = (dH)(X_H) = \omega(X_H, X_H)$
   $\rightsquigarrow$ ω is alternating (2-form)

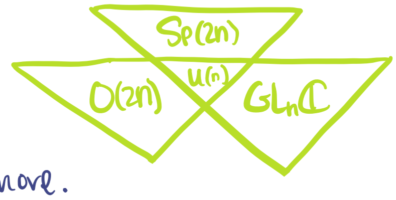
4. "laws of physics shouldn't depend on time" $\rightsquigarrow$ $d\omega = 0$.

This justifies the definition of symplectic from Wikipedia.
The word "symplectic" was coined by Weyl in the 1930's, and actually comes from
complectere
together weave or braid

Symplectic stuff has an interesting spot in the world of topology and geometry

rigid ⟵ Riemannian geometry — Symplectic — Complex geometry — topology ⟶ flexible

These geometries interact in an interesting way, which even shows up at the level of linear algebra:



$\subseteq GL_{2n}(\mathbb{R})$

Let me explain Sp(2n) a bit more.

Symplectic Linear Algebra

Defn - A symplectic structure on a v.s. V is a bilinear map $\Omega: V \times V \to V$ s.t.

(i) skew-symmetric: $\Omega(v,w) = -\Omega(w,v)$

(ii) non-degen: $\Omega(v,w) = 0 \ \forall w \in V \iff v = 0$.

We call the pair (V,Ω) a symplectic v.s.

Observations
- (ii) is equivalent to $V \xrightarrow{\cong} V^*$ for $v \mapsto \Omega(v,-)$ so V fin. dim'l
- $\dim V$ is even b/c $[\Omega]$ is invertible & skew symm so $\det[\Omega] = \det[-\Omega^t] = (-1)^{\dim V}\det[\Omega]$.
- We can choose a basis for V s.t.

$$[\Omega] = \left[\begin{array}{c|c} 0 & -I_n \\ \hline I_n & 0 \end{array}\right]$$

this is called a Symplectic basis. Conversely, every basis determines a symplectic form.

Ex. $\mathbb{R}^{2n}$ w/ standard basis $e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}$ determines

$$\Omega_0((v_1,v_2),(w_1,w_2)) = \langle v_1, w_2\rangle - \langle w_1, v_2\rangle$$

Rmk. For any v.s. W, $V = W \oplus W^*$ has sympl. structure like this. Conversely(ish), if $W \subseteq V$ is Lagrangian, then (V,ω) is "the same" as $(W \oplus W^*, \Omega')$ where

$$\Omega'(w \oplus f, w' \oplus f') = f'(w) - f(w').$$

What does "the same" mean here?

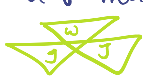
↳ "the same" = symplectomorphic : $(V,\Omega) \xrightarrow{\phi} (V',\Omega')$ iso s.t. $\phi^*\Omega' = \Omega$.

Defn $Sp(V,\Omega) = \{$symplectomorphisms $(V,\Omega) \to (V,\Omega)\}$

$Sp(2n) = Sp(\mathbb{R}^{2n}, \Omega_0)$.

So the diagram says that preserving any two of (cpx, i.p., symplectic) also preserves the third. This same idea works on the level of manifolds, where the structures hold at the level of tangent spaces.

e.g. Since every manifold admits a R. metric, every symplectic mfld admits an "almost complex structure."



$g(-,-) = \omega(-,J-)$.

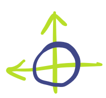
Said before that symplectic mflds have no local invariants. This is the content of Darboux's Thm:

Thm (Darboux). Let (M,ω) symplectic and $p \in M$. Then $\exists$ chart $(U; x_1,\dots,x_n,y_1,\dots,y_n)$ s.t. $\omega = \sum_i dx_i \wedge dy_i$ on U.

Rmk $(\mathbb{R}^{2n}, \omega_0)$ symplectic for $\omega_0 = \sum_i dx_i \wedge dy_i$ for x_i, y_i lin. coords of $\mathbb{R}^{2n}$

So just as every mfld is locally "the same" as $\mathbb{R}^{2n}$, so is every symplectic mfld locally "the same" as this example. Thus there's nothing like curvature which lets us distinguish btwn points. However, a symplectic structure does have some global implications.

Observations

- M must be orientable b/c $\omega^n = \omega \wedge \cdots \wedge \omega$ is a volume form
- A surface is symplectic $\iff$ orientable
- If M is compact, then $[\omega^k] \in H^{2k}_{dR}(M)$ is non-zero
  So if $H^{2k}(M) = 0$, then M not symplectic.
  So e.g. S^{2n} not symplectic for $n \geq 2$.
- Can distinguish submflds N based on how ω behaves.
  - ↳ <u>Symplectic</u> if $\omega|_N$ symplectic
  - ↳ <u>isotropic</u> if $i^*\omega \equiv 0$ ($\Rightarrow \dim N \leq \frac{1}{2}\dim M$)
  -  ↳ <u>Lagrangian</u> if isotropic and $\dim N = \frac{1}{2}\dim M$

Lagrangian submanifolds have interesting geometry and arise naturally everywhere... In fact, every manifold is a Lagrangian submanifold (just like all concepts are Kan extensions)

Ex. The cotangent bundle.

$T^*M = \{(p,\xi) \mid p \in M, \xi \in T_p^*M\}$ captures the idea of position + momentum

Has a canonical symplectic structure:

$$\omega = -d\alpha \text{ for } \alpha = \textstyle\sum_i \xi_i dx_i \text{ in coords } (U; x_1,\dots,x_n,\xi_1,\dots,\xi_n)$$
$$= \textstyle\sum_i dx_i \wedge d\xi_i$$

zero section $z(M) \subseteq T^*M$ is Lagrangian:

(i) $\dim z(M) = \dim(M) = \frac{1}{2}\dim(T^*M)$

(ii) $z^*\omega \equiv 0$ b/c $\xi_1 = \cdots = \xi_n = 0$ on $U \cap z(M)$.

In fact, up to symplectomorphism, every Lagrangian submfld looks like this:

Thm - Let $i: N \to M$ be a compact Lagrangian Submanifold. Then $\exists$

$$\begin{tikzcd} (T^*N, \omega_0) \supseteq U_0 \arrow[rr, "\phi", "\cong"'] & & U \subseteq (M,\omega) \\ & N \arrow[ul, "z"] \arrow[ur, "i"'] & \end{tikzcd}$$

s.t. $\phi^*\omega = \omega_0$.

So our prototypical example of a Lagrangian Submfld comes from a section of $T^*M \to M$. Will this work in general? That is,

Q. If $\mu: M \to T^*M$ is a section, is $M_\mu = \{(p, \mu(p))\} \subseteq T^*M$ Lagrangian?

(i) holds b/c $\mu(M) \cong M$.
(ii) holds iff $d\mu = 0$.

Ex. $\mu = df$ for $f \in C^\infty(M) \Rightarrow M_\mu$ is Lagrangian

Note that if $H^1_{dR}(M) = 0$ (e.g. M simply cnt'd) then $d\mu = 0 \Rightarrow \mu = df$ so every Lagrangian graph is generated by some f. There are also plenty of examples of Lagrangian submflds which are not graphs.

But graphs generate lots of interesting/important examples, even outside of T^*M.

If $\phi: M_1 \to M_2$ is a diffeomorphism, its graph $\Gamma_\phi \subseteq M_1 \times M_2$,

$$\omega = (\pi_1)^*\omega_1 - (\pi_2)^*\omega_2.$$

Prop Γ_ϕ Lagrangian $\Leftrightarrow$ ϕ is symplectomorphism

Pf/
$$\begin{aligned} i^*\omega &= i^*(\pi_1)^*\omega_1 - i^*(\pi_2)^*\omega_2 \\ &= (\pi_1 \circ i)^*\omega_1 - (\pi_2 \circ i)^*\omega_2 \\ &= id^*\omega_1 - \phi^*\omega_2 \\ &= \omega_1 - \phi^*\omega_2. \end{aligned}$$

Ex. $id: M \to M \Rightarrow \Delta(M) \subseteq M \times M$ is Lagrangian

Study fixed pts of $\phi: M \to M$ by studying $\Gamma_\phi \cap \Delta(M)$ "Lagrangian intersection problem"
need stronger symmetry than just ϕ symplecto.

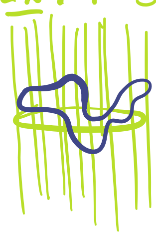
Arnold Conjecture If $\phi: M \to M$ is Hamiltonian diffeom, then

$$\#\{\text{non-degen fixed pts of } \phi\} \geq \#\text{crit pts of a non-degen smooth fn}$$
$$\geq \textstyle\sum_i \dim H^i(M) \quad \text{from Morse theory}$$

e.g. on S^2, any Hamiltonian diffeom has ≥ 2 fixed points

An interesting contrast to the Lefschetz fixed pt thm $\sum_i (-1)^i \dim H^i(M)$
So the geometry of certain Lagrangians somehow implies they intersect more than predicted by topology (rigidity)

Ex. $T^*S^1 \cong S^1 \times \mathbb{R}$ $\quad \omega = d\theta \wedge dt$



coords: (θ, t)

Take smooth fn $f: S^1 \to \mathbb{R}$
Then Γ_{df} is Lagrangian in T^*S^1
and $\Gamma_{df} \cap Z(S^1) = Crit(f)$.

Can interpret Γ_{df} as deformation of $Z(S^1)$ via a time-1 Hamiltonian flow. The intersection pts are the fixed pts of the flow

Recall: Given $H: M \to \mathbb{R}$, get X_H s.t. $\omega(X_H, -) = dH$.
$\rightsquigarrow X_H$ w/ flow $\rho_t: M \to M$ symplectomorphism solves diff. eqn

Defn - H is a Hamiltonian fn, X_H is Hamiltonian v.f., ρ_1 is Hamiltonian diffeom.

Rmk X is Hamiltonian v.f. $\iff$ $\omega(X, -)$ is exact $\quad$ Sympl v.f. $\iff \iota_X \omega$ closed.
$\mathcal{L}_X \omega = 0$

Note. $\phi = \rho_1$ has fixed pt p $\iff$ $\{x(t) = \rho(t,p) \mid t \in [0,1]\}$ closed orbit.
(non-degen)
$\hookrightarrow \det(id - d\rho_1(p)) \neq 0$
$\Uparrow$
$p \in Crit(H)$, i.e. $\rho(t,p) \equiv p$.
(non-degen)

Proves Arnold Conjecture for time independent flows.

More generally: $H: M \times \mathbb{R} \to M$ generates $X_t := X_{H_t}$ and ρ_t $\quad$ diff. eqn not autonomous

$\phi = \rho_1$ has fixed pt p $\Rightarrow$ $\{\rho_t(t,p) \mid t \in I\}$ closed orbit
$\Leftarrow$ if $H_t = H_{t+1}$.
$\nearrow$ $p \in Crit(H)$ $\quad$ $\nwarrow$ need $p \in Crit(H_t)$ $\forall t$.

Upshot: For 1-periodic Hamiltonians,

closed period-1 orbits $\leftrightsquigarrow$ fixed pts $\leftrightsquigarrow$ $\underbrace{\Gamma_\phi \cap \Delta(M)}_{\text{Lagrangian Floer homology}}$ $\leftarrow$ $\exists$ sympl. (not H. diffeos) s.t. this is empty.

Idea of Lagrangian Floer Homology often called "infinite dim'l MT" + new complications (not always $\partial^2 = 0$)

Let $L_0, L_1 \subseteq M$ cpt Lagrangians s.t. $L_0 \pitchfork L_1$. Define

$$\mathcal{P}(L_0, L_1) = \{\gamma: I \to M \text{ smooth} \mid \gamma(0) \in L_0, \gamma(1) \in L_1\}$$

Note: $\{\text{constant paths}\} \leftrightarrow L_0 \cap L_1$

Idea: Define action fn'l on $\tilde{\mathcal{P}}(L_0, L_1; \tilde{\gamma}) = \{[\gamma, w] \mid w: I \times I \to M \text{ path } \tilde{\gamma} \to \gamma\}$ $\quad$ ($\tilde{\gamma}$: chosen basept)

To make this well-defn'd, Assume: L_0, L_1 cnt'd, M "aspherical": $\int_{S^2} f^*\omega = 0$ $\forall [f] \in \pi_2$

$$A([\gamma, w]) = \int_{I^2} w^*\omega \quad \text{symplectic area of } w(I^2) \in (M, \omega).$$

Do Morse Theory on $(\tilde{\mathcal{P}}, A)$:

To define gradient flow, choose R. metrics... this is equivalently a choice of almost-cpx structure

$$\text{grad}\,\mathcal{A}([\gamma,w]) = J_t \frac{\partial \gamma}{\partial t} \qquad J_t \text{ compat a-cpx}$$

ends up being indep of this choice

$\text{grad}\,\mathcal{A}([\gamma,w]) = 0 \iff \gamma$ constant so $\text{Crit}(\mathcal{A}) =$ constant paths

gradient flow lines are actually "J-holomorphic strips"

Mimic Morse homology ideas, do a bunch of analysis to show $\partial^2 = 0$.

<u>Show</u> $\#(L \cap \Phi(L)) \geq \sum_i \dim H_i(L, \mathbb{Z}/2)$.