TALBOT 2022 NOTES:
SQUARE K-THEORY AND CUT-AND-PASTE MANIFOLDS

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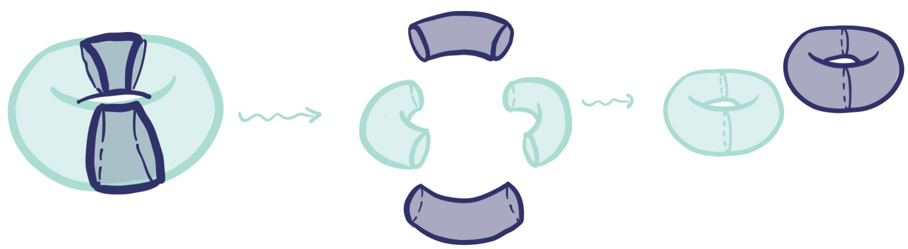
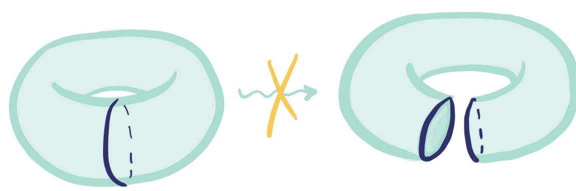
We've talked about scissors congruence for polytopes, and we can try to do the same thing for other kinds of spaces, like manifolds. We can (carefully) cut our manifold up into pieces, rearrange them, and paste them back together. This is called a SK-move (SK comes from *schneiden und kleben*, which means cut and paste in German), and two manifolds are scissors congruent or SK-equivalent if one can be obtained from the other by a finite sequence of these SK-moves.



Example: $T^2 \sim_{SK} T^2 \coprod T^2$

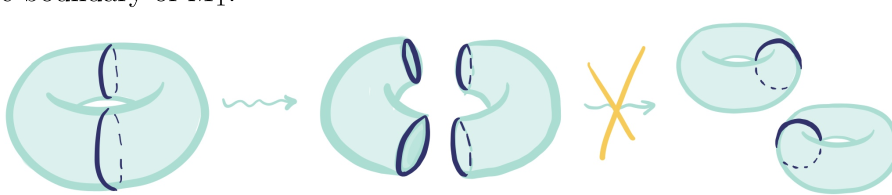
What does it mean to "carefully cut" a manifold M? Here are some non-examples/things to be aware of:

- Our cut must separate M into two (not necessarily connected) pieces M_0 and M_1.



Non-example: cut doesn't separate

- When pasting them back together, we need to glue the boundary of M_0 to the boundary of M_1.



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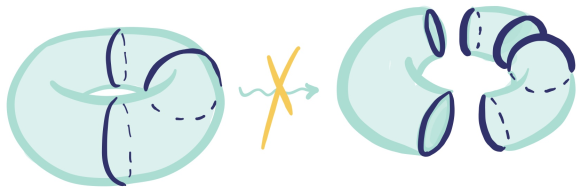
Non-example: can't glue ∂M_i to itself.

- Among other things, this means that we must have $\partial M_0 \cong \partial M_1$.



Non-example: can't write $\partial M_0 \cong \partial M_1 \cong (S^1)^{\amalg 3}$.

Example 1. In the first picture for $T^2 \sim_{SK} T^2 \coprod T^2$, we cut along four circles to separate T^2 into two (disconnected pieces): M_0 consists of one dark cylinder and one light cylinder, and M_1 consists of the two remaining cylinders. The boundaries of M_0 and M_1 are both diffeomorphic to $(S^1)^{\amalg 4}$, and we glue the dark piece of M_0 to the dark piece of M_1, and similarly for the light pieces. This satisfies all the rules for an SK-move for manifolds with boundary.

Here's the formal definition:

Definition 2. An SK-move on (smooth, closed, oriented) manifolds is defined as follows: cut an n-manifold M along a codimension-1 smooth submanifold N with trivial normal bundle which separates[1] M. Then paste the two pieces back together along an orientation-preserving diffeomorphism of N.

Definition 3. The SK-group for n-manifolds is

$$\mathbb{Z}[\text{diffeomorphism classes of } n\text{-manifolds}]/\sim_{SK}$$

Remark 4. The group SK_n can also be defined by a universal property. Let $\mathcal{M}_n$ denote the monoid of diffeomorphism classes of (smooth, closed, oriented) n-manifolds under disjoint union. Then SK_n is defined to satisfy the property that any Abelian group-valued map out of $\mathcal{M}_n$ which is a SK-invariant (i.e. respects SK-equivalence) must factor through it. The only SK-invariants for smooth oriented manifolds are the Euler characteristic, the signature, and their linear combinations.

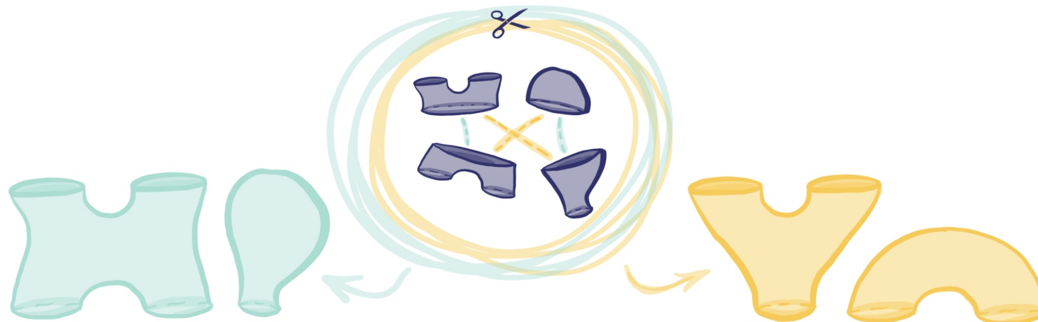
Proposition 5. *It's a group (under disjoint union).*

Remark 6. Because we are working with diffeomorphism classes, we don't have access to things like length, angles, scaling (which we needed for the Dehn invariant). This shows us already that scissors congruence of manifolds has a very different flavor than scissors congruence of polytopes!

Here's another difference from the world of polytopes: when we cut up a polytope, the pieces are all still polytopes, but when we cut up a manifold, we leave the category of manifolds and enter into the category of manifolds *with boundary.* This motivates us to work entirely in the setting of manifolds with boundary. The SK-relation is defined for manifolds with boundary just as it was for manifolds without

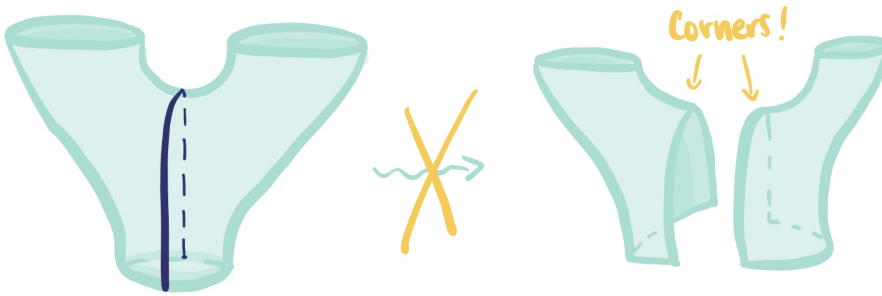
[1]This means $M \setminus N$ is a disjoint union of two components, each with boundary diffeomorphic to N.

boundary, with the additional condition that the codimension-1 separating manifold N must not intersect the boundary of M.

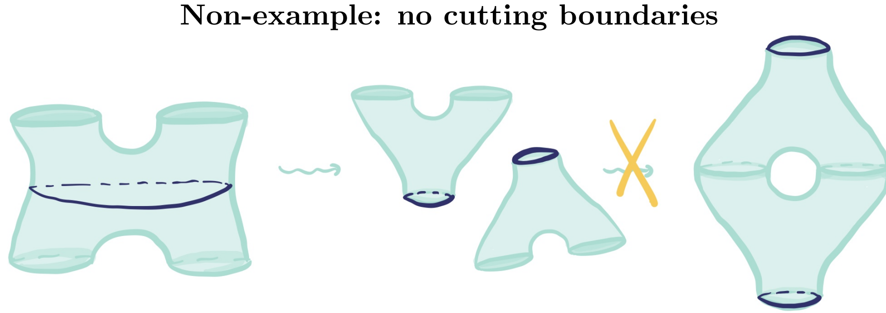


Example: two SK-equivalent manifolds with boundary

Remark 7. It is crucial that boundaries are *not allowed* to be cut in these SK-moves, and all boundaries which come from a cut must be pasted back together.



Non-example: no cutting boundaries



Non-example: cuts must be reglued

There are other variants of the definition of SK-moves in the literature which allow for these sorts of things (see [HMM+21, Remark 2.6]).

Definition 8. The group SK_n^∂ is the Grothendieck group (group completion) on the monoid of diffeomorphism classes of (smooth, compact, oriented) n-manifolds with boundary under disjoint union, modulo the SK-relation. Explicitly,

$$SK_n^\partial = \mathbb{Z}[\text{diffeomorphism classes } [M] \text{ of } n\text{-manifolds with boundary}]/\sim$$

where the relations are generated by

(i) $[M \coprod N] = [M] + [N]$,

(ii) Given manifolds M, M' with closed submanifolds $\Sigma \subseteq M$, $\Sigma' \subseteq M'$ and orientation-preserving diffeomorphisms $\phi, \psi \colon \Sigma \to \Sigma'$,

$$[M \cup_\phi \overline{M'}] = [M \cup_\psi \overline{M'}],$$

where $\overline{M'}$ is M' with the opposite orientation.

This group looks like it should be K_0 of something...and this is exactly what R. Hoekzema, M. Merling, L. Murray, C. Rovi, and J. Semikina show in [HMM+21] using the of machinery *square K-theory.*

Theorem 9. $SK_n^\partial \cong K_0^\square(\mathrm{Mfld}_n^\partial)$.

Our goal for this talk is to understand this theorem. The benefit of using K-theory is we have access to more structure; the higher K-groups of Mfld_n^∂ can be interpreted as "higher scissors congruence groups" for manifolds with boundary. As of now, there is no known way to realize SK_n as K_0 of some category.

Remark 10. In [HMM+21, Theorem 2.1], the authors show there is a short exact sequence

$$0 \to SK_n \xrightarrow{i} SK_n^\partial \xrightarrow{\partial} C_{n-1} \to 0.$$

Here, C_{n-1} is the group of diffeomorphism classes of nullbordant[2] $(n-1)$-manifolds, i is the inclusion $M \mapsto (M, \emptyset)$, and ∂ sends $(N, \partial N)$ to its boundary ∂N. In fact, this short exact sequence splits and so $SK_n^\partial \cong SK_n \oplus C_{n-1}$. In the "$SK$-book" [KKNO73], they compute

$$SK_n \cong \begin{cases} 0 & n \text{ odd}, \\ \mathbb{Z}[S^n] & n \equiv 2 \pmod 4, \\ \mathbb{Z}[S^n] \oplus \mathbb{Z}[\mathbb{CP}^{n/2}] & n \equiv 0 \pmod 4. \end{cases}$$

Square K-theory. We have discussed how higher algebraic K-theory can be constructed in settings where we have some way to "chop things up":

exact categories	**Waldhausen categories**	**CGW categories**
short exact sequences	cofiber sequences	spans⇝squares
$0 \to A \to B \to C \to 0$	$A \rightarrowtail B \twoheadrightarrow B/A$	$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \square & \downarrow \\ \bullet & \rightarrowtail & C \end{array}$
Q	$S_\bullet$	Q
$[B] = [A] + [C]$	$[B] = [A] + [B/A]$	$[B] = [A] + [C] - [\bullet]$

We talked about how we can interpret the K-theory of CGW categories as the "combinatorial" analogue of the algebraic K-theory of exact categories. In this talk, we'll develop the combinatorial analogue of the algebraic K-theory of Waldhausen categories: square K-theory.

Categories with squares
squares
$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \square & \downarrow \\ C & \rightarrowtail & D \end{array}$
$C^{(\bullet)}$
$[D] = [B] + [C] - [A]$

As the name suggests, our setting will be *categories with squares* and we will decompose objects according to these squares. The square K-theory space is built using something similar to the $S_\bullet$-construction, denoted $C^{(\bullet)}$. This is forthcoming work of Campbell-Zakharevich.

We've already seen the idea that "$[D] = [B] + [C] - [A]$" from the square

$$\begin{array}{ccc} A \cap B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \cup B \end{array} .$$

An exercise in a first set theory class may be to prove

$$|A \cup B| = |A| + |B| - |A \cap B| .$$

In general, the squares under consideration may not be pushouts or pullbacks (or both, like the example above), but it is a helpful intuition to keep in mind. The benefit of a category with squares is that we get to specify exactly what kinds of squares we want to work with, subject to a few conditions.

Definition 11. A *category with squares* consists of a category $\mathscr{C}$ with coproducts, a chosen distinguished object 0, and two subcategories $c\mathscr{C}$ and $f\mathscr{C}$ called cofibrations ($\rightarrowtail$) and cofiber ($\twoheadrightarrow$) maps along with a collection of distinguished squares

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \square & \downarrow \\ C & \rightarrowtail & D \end{array}$$

which satisfy the following:

(i) distinguished squares are closed under coproducts:

$$\text{if } \begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \square & \downarrow \\ C & \rightarrowtail & D \end{array} \text{ and } \begin{array}{ccc} A' & \rightarrowtail & B' \\ \downarrow & \square & \downarrow \\ C' & \rightarrowtail & D' \end{array}, \text{ then } \begin{array}{ccc} A \coprod A' & \rightarrowtail & B \coprod B' \\ \downarrow & \square & \downarrow \\ C \coprod C' & \rightarrowtail & D \coprod D' \end{array},$$

[2] By nullbordant M, we mean that $M = \partial W$ for some W.

(ii) distinguished squares are commutative in $\mathscr{C}$ and can be composed vertically and horizontally,
(iii) the subcategory $iso\mathscr{C}$ of isomorphisms ($\xrightarrow{\sim}$) is contained in both $c\mathscr{C}$ and $f\mathscr{C}$,
(iv) all squares of the form

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \sim\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow\sim \\ C & \rightarrowtail & D \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \overset{\sim}{\rightarrowtail} & B \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ C & \underset{\sim}{\rightarrowtail} & D \end{array}$$

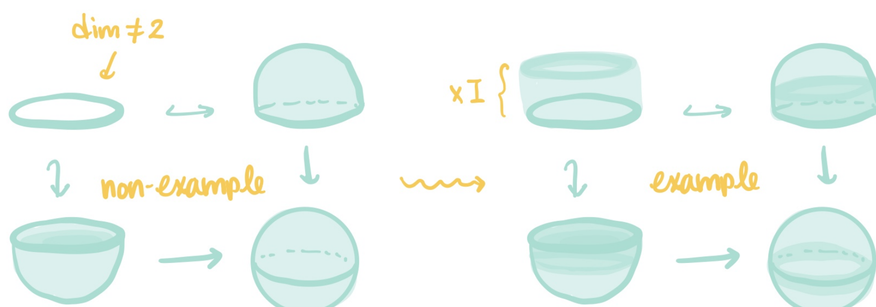
are distinguished.

Example 12. The objects of Mfld_n^∂ are (nice) n-manifolds with boundary and the morphisms are closed embeddings plus a condition[3] on the boundary. Both $c\text{Mfld}_n^\partial$ and $f\text{Mfld}_n^\partial$ are all morphisms. Distinguished squares are pushouts

$$\begin{array}{ccc} N & \rightarrowtail & M \\ \downarrow\!\!\downarrow & \square & \downarrow\!\!\downarrow \\ M' & \rightarrowtail & M' \cup_N M \end{array} \quad .$$

The distinguished object for Mfld_n^∂ is the empty manifold $\emptyset$.

Remark 13. Note that the requirement that $M' \cup_N M$ be a *smooth manifold* imposes restrictions on our squares, some of which may feel unfamiliar.



(Non-)example: "thicken" S^1 to get distinguished pushout square

Definition 14. Define a simplicial category $\mathscr{C}^\bullet$ where $\mathscr{C}^{(n)}$ is the subcategory of $\mathbf{Cat}([n], \mathscr{C})$ whose objects are length n cofibration sequences

$$c_0 \rightarrowtail c_1 \rightarrowtail \cdots \rightarrowtail c_n$$

[3]Specifically a map $M \to M'$ must map each component of ∂M either entirely into the interior of M' or diffeomorphically onto a component of $\partial M'$.

and whose morphisms are natural transformations in which every square is distinguished. Then $N_*\mathscr{C}^\bullet$ is a bisimplicial set whose (m,n)-simplicies look like diagrams

$$\begin{array}{ccccc} c_{00} & \rightarrowtail & \dots & \rightarrowtail & c_{0m} \\ \downarrow\!\!\!\!\twoheadrightarrow & & & & \downarrow\!\!\!\!\twoheadrightarrow \\ \vdots & & \ddots & & \vdots \\ \downarrow\!\!\!\!\twoheadrightarrow & & & & \downarrow\!\!\!\!\twoheadrightarrow \\ c_{n0} & \rightarrowtail & \dots & \rightarrowtail & c_{nm} \end{array}$$

of distinguished squares.

Recall the *no. 1 fact about bisimplicial sets*:

$$|[m] \mapsto |N_m\mathscr{C}^\bullet|| \cong |diag(N_*\mathscr{C}^\bullet)| \cong \left|[n] \mapsto \left|N_*\mathscr{C}^{(n)}\right|\right|.$$

That is, it does not matter whether we realize horizonally then vertically, or vice versa, since both are homeomorphic to the realization of the diagonal simplicial set $[n] \mapsto N_n\mathscr{C}^{(n)}$.

Definition 15. The *K-theory space* is this realization (with a shift)

$$K^\square(\mathscr{C}) = \Omega\,|N_*\mathscr{C}^\bullet|\,,$$

and its K-groups are the homotopy groups of $K^\square(\mathscr{C})$

$$K_i^\square(\mathscr{C}) = \pi_i(K^\square(\mathscr{C})).$$

Example 16. Every Waldhausen category $\mathscr{C}$ is naturally a category with squares. In fact, there are sometimes multiple ways to do this:

(1) When $w\mathscr{C} = iso\mathscr{C}$, then we can take $c\mathscr{C} = co\mathscr{C}$, $f\mathscr{C}$ =cofiber maps, $\square$ =all commutative squares, $0 = 0$.
(2) No matter what $w\mathscr{C}$ is, we can take $c\mathscr{C} = co\mathscr{C}$, $f\mathscr{C}$ =all maps, $\square$ =pushouts up to weak equivalence,[4] $0 = 0$.

In both cases, a comparison at the level of simplicial objects shows $K^\square(\mathscr{C}) \simeq K^W(\mathscr{C})$.

Typically, K_0 of a category can be described very concretely the free Abelian group on objects modulo some relations (often as the Grothendieck group of some monoid). Campbell-Zakharevich prove that this works for square K-theory as long as we put some (reasonable) assumptions on $\mathscr{C}$.

Theorem 17. *Suppose $\mathscr{C}$ is a category with squares, with distinguished object O. If*

- *O is initial or terminal in $c\mathscr{C}$,*
- *O is initial or terminal in $f\mathscr{C}$,*

[4]This means that a square is distinguished when $B \cup_A C \xrightarrow{\sim} D$ is a weak equivalence.

- *for all objects $A, B \in \mathscr{C}$, there is an object $X \in \mathscr{C}$ so that the squares*

$$\begin{array}{ccc} O & \rightarrowtail & B \\ \downarrow\!\!\!\downarrow & \square & \downarrow\!\!\!\downarrow \\ A & \rightarrowtail & X \end{array} \quad \text{and} \quad \begin{array}{ccc} O & \rightarrowtail & A \\ \downarrow\!\!\!\downarrow & \square & \downarrow\!\!\!\downarrow \\ B & \rightarrowtail & X \end{array}$$

  are distinguished,

then

$$K_0^{\square}(\mathscr{C}) \cong \mathbb{Z}[\mathrm{Ob}\mathscr{C}]/\sim$$

where $\sim$ is generated by $[O] = 0$ and $[A] + [D] = [B] + [C]$ for every distinguished square

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow\!\!\!\downarrow & \square & \downarrow\!\!\!\downarrow \\ C & \rightarrowtail & D \end{array}.$$

Their proof (forthcoming) is very similar to the proof for the Q-construction, basically showing that $K_0^{\square}(\mathscr{C})$ has the right generators and relations. The assumptions in this theorem are pretty reasonable to ask for (note the similarities with CGW categories), and hence make $K_0^{\square}$ computable in many cases.

Exercise 18. For a Waldhausen category $\mathscr{C}$, recall that $K_0^W(\mathscr{C})$ is the free Abelian group on objects modulo the relation $[A] = [A']$ for every weak equivalence $A \xrightarrow{\sim} A'$ and $[B] = [A] + [B/A]$ for every cofiber sequence $A \rightarrowtail B \twoheadrightarrow B/A$. Using Example 16(2), show that $K_0^{\square}(\mathscr{C}) \cong K_0^W(\mathscr{C})$ directly.

Hint: consider the square

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow\!\!\!\downarrow & \square & \downarrow\!\!\!\downarrow \\ 0 & \rightarrowtail & B/A \end{array}$$

and the two cofiber sequences

$$A \rightarrowtail B \twoheadrightarrow B/A$$
$$C \rightarrowtail D \twoheadrightarrow D/C.$$

Theorem 19. $K_0^{\square}(\mathrm{Mfld}_n^{\partial}) \cong SK_n^{\partial}$.

Proof idea. Since the distinguished object $\emptyset$ is initial in both $c\mathrm{Mfld}_n^{\partial}$ and $f\mathrm{Mfld}_n^{\partial}$, we may apply Theorem 17 (noting that the third condition is clearly satisfied by disjoint union). This gives a description of $K_0^{\square}(\mathrm{Mfld}_n^{\partial})$ in terms of specific generators and relations and we can show that SK_n^{∂} is described by the same generators and relations. □

The authors of [HMM+21] also show that the Euler characteristic lifts (as an SK-invariant) to the level of K-theory.

Theorem 20. *There is a map of square K-theory*

$$K^{\square}(\mathrm{Mfld}_n^{\partial}) \to K(\mathbb{Z})$$

which on π_0 agrees with the Euler characteristic $\chi\colon \mathrm{SK}_{\partial}^n \to \mathbb{Z}$.

Proof idea. To prove the theorem, the authors use the intermediary category $\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}$ consisting of homologically bounded chain complexes (i.e. quasi-isomorphic to bounded finitely-generated $\mathbb{Z}$-complexes). Recall that $\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}$ has the structure of a Waldhausen category, where cofibrations are level-wise injective maps and weak equivalences are quasi-isomorphisms. By Example 4, we can also give $\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}$ the structure of a category with squares. The map $S\colon \mathrm{Mfld}_n^{\partial} \to \mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}$ is just the *singular chain functor* which sends a compact manifold with boundary to its singular chain complex. There are two things to show:

(1) S is a map of categories with squares,
(2) $K(\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}) \simeq K(\mathbb{Z})$ in such a way that S corresponds to χ on π_0.

For (1), the trickiest part is showing that a diffeomorphism $C \cup_A B \xrightarrow{\sim} D$ implies $S(C) \cup_{S(A)} S(B) \xrightarrow{\sim} S(D)$ is a quasi-isomorphism. The idea is to model the pushout using $S(B+C)$[5] and use stuff from Hatcher to show the inclusion $S(B+C) \to S(D)$ is a quasi-isomorphism. One thing to note here is that the choice of distinguished squares in $\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}$ is crucial for the proof, which would not have worked if we only allowed cofiber maps as the vertical maps.

For (2), the authors use various theorems of higher algebraic K-theory to show that all the maps

$$\mathrm{Mod}_{\mathrm{f.g.}}^{\mathrm{proj}}(\mathbb{Z}) \xrightarrow{i} \mathrm{Mod}_{\mathrm{f.g.}}(\mathbb{Z}) \xrightarrow{t} \mathrm{Ch}_{\mathbb{Z}}^{b} \xrightarrow{j} \mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}$$

realize to isomorphisms on K-theory. Here, $\mathrm{Mod}_{\mathrm{f.g.}}^{\mathrm{proj}}(\mathbb{Z}) \xrightarrow{i} \mathrm{Mod}_{\mathrm{f.g.}}$ is the inclusion of projective finitely generated $\mathbb{Z}$-modules into finitely generated ones, t maps a finitely generated $\mathbb{Z}$-module A to the bounded chain complex with A in degree 0 and 0's everywhere else, and j is the inclusion of bounded complexes into homologically bounded ones. The final step is to show that the inverse of this map coincides with the Euler characteristic on K_0. □

References

[HMM+21] Renee S. Hoekzema, Mona Merling, Laura Murray, Carmen Rovi, and Julia Semikina. Cut and paste invariants of manifolds via algebraic K-theory, 2021. arXiv.2001.00176v2 [math.AT].

[KKNO73] U. Karras, M. Kreck, W.D. Neumann, and E. Ossa. *Cutting and pasting of manifolds; SK-groups.* Number 1 in Mathematics Lecture Series. Publish or Perish, Inc., 1973.

[5]For each n, $S_n(B+C)$ is the subgroup of $S_n(D)$ consisting on n-chains in D which are sums of n-chains in B and n-chains in C.