A Bit About Infinite Loop Spaces

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Abstract

We provide an expository overview of infinite loop space theory, discussing both the operadic and Γ-space perspective. In particular, we cover A_∞- and E_∞-operads and the Recognition Theorem for infinite loop spaces, and we also show how the machinery of Γ-spaces can be used to associate an infinite loop space to a symmetric monoidal category. We conclude with a discussion of infinite loop space machines and the uniqueness theorem of May and Thomason.

Contents

1 Introduction

Loop spaces are important objects in algebraic topology for a variety of reasons, including their connections to higher homotopy groups, spectra, and generalized cohomology theories. Even more fundamentally, loop spaces provide an example of topological spaces which admit a binary operation, thus allowing us to do algebra in the context of topology. Given a space with an operation, we can ask basic questions like *is this operation associative? commutative? does it have a unit?* Certainly the nicest sort of operation we could have is a unital, associative, and commutative one. The spaces with this sort of extremely well-behaved operation are the Abelian topological monoids. As every Abelian topological monoid has the homotopy type of a product of Eilenberg MacLane spaces (cf. [Hat02, Corollary 4K.7]), their theory is already quite well understood, so in order to generate more interesting examples we need to ask for an operation which is a bit more flexible. From a homotopy theory point of view, this can be achieved by asking for unity, associativity, or commutativity *but only up to homotopy.* A space with such a multiplication is called an H-space, and loop concatenation makes loop spaces a prototypical example.

In fact, not only is concatenation of loops associative up to homotopy, but it is associative *up to coherent higher homotopy*, meaning there are homotopies between the homotopies, and homotopies between those, and so on. A space with such a multiplication is called an A_∞-space, and the recognition principal of J. Stasheff [Sta63] tells us that this A_∞ property characterizes loop spaces in a certain sense. The theory of A_∞-spaces can be captured and generalized by operads, as developed and popularized by Peter May [May72]. In particular, there are certain operads which encode how well associativity and commutativity of an operation behave "up to coherent higher homotopy." These operads, known as E_n operads ($1 \leq n \leq \infty$), provide a way to characterize iterated and infinite loop spaces (Theorem 3.18).

Due to the interesting and fruitful nature of loop spaces, algebraic topologists in the 1960's and 70's were very interested in being able recognize iterated and infinite loop spaces when they appeared and "deloop" them. The machinery of operads provides one solution to this problem, and the second main approach is that of Segal's Γ-spaces, as developed in [Seg74]. Roughly, a Γ-space is a functor which takes values in spaces, and if it has some special extra structure, we can associate to it an Ω-spectrum (and hence an infinite loop space). This process can be generalized, essentially giving us a way to turn a given symmetric monoidal category into an Ω-spectrum (Theorem 4.10). Segal's machinery, like May's, comes with an explicit delooping of the associated infinite loop space, and we can ask to what extent the two theories will agree. It turns out that the two machines are equivalent, in the sense that they yield equivalent spectra when given the same data. This is known as the uniqueness theorem for infinite loop space machines (Theorem 5.8), as proven by May and Thomason in [MT78].

Outline and Assumptions

Section 2 begins with the basic definitions and properties of loop spaces, iterated loop spaces, and infinite loop spaces. Section 3 builds up to the recognition theorem for loop spaces using operads. After illustrating the general idea for loop spaces in Subsection 3.1, we give the formal definition of an operad before stating the full recognition theorem for iterated loop spaces in Subsection 3.3. Section 4 gives a brief overview of Segal's Γ-spaces and their function as a "delooping" machine, along with some interesting examples in Subsection 4.2. The final section, Section 5, covers infinite loop spaces machines more generally in the context of May and Thomason's uniqueness theorem.

We work in the convenient category of compactly generated weak Hausdorff spaces, which we denote **Top**, and we assume our spaces are connected unless explicitly stated otherwise. This assumption allows us to write things like ΩX and $\pi_1(X)$ for a based space (X, x_0), instead of the more accurate $\Omega(X, x_0)$ and $\pi_1(X, x_0)$.

Although we hope to give a relatively self-contained account of these concepts, we assume the reader is familiar with the basics of algebraic topology and homotopy theory. Knowledge of simplicial sets, spectra, and generalized cohomology theories will be helpful but hopefully not necessary to understand the main ideas; comprehensive explanations of these topics can be read in [May99], [Hat02, §4], or [BM21]. Wonderful overviews of the content we cover can also be found in other write-ups such as [May77, Chapter I] and [Oso11] (although it should be noted the latter one is in Spanish), as well as the classic book [Ada78]. This write-up is meant to be a survey of the concepts and ideas surrounding loop spaces, and is by no means a complete account of the mechanisms involved. To borrow an expression from J.F. Adams [Ada78, p.37], this note will be written as an essay in machine appreciation and is not intended to qualify the reader for a mechanic's certificate.

Acknowledgements

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2 Loop Spaces

The study of loop spaces is motivated in part by a desire to study spaces with a binary operation which is not necessarily strictly "nice" (unital, associative, commutative) but is still well-behaved from the point of view of homotopy theory. The particular operation on loop spaces we utilize — familiar to the student of algebraic topology — is concatenation of loops. Given a topological space X, a path in X is just a continuous map from the interval $I = [0, 1]$ into X, and a loop is just a path whose start and end points are the same. In other words, a loop is a based map $S^1 \to X$, and we can assemble all of the loops on X into its *loop space.*

Definition 2.1. The (based) *loop space functor* $\Omega\colon Top_* \to Top_*$ sends a based space (X, x_0) to its loop space

$$\Omega X = \mathrm{Map}_*(S^1, X)$$

which is topologized under the compact-open topology. We can also write ΩX as the pullback[1]

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & * \arrow[d, "c_{x_0}"] \\ Map_*(I, X) \arrow[r, "\varepsilon_1"'] & X \end{tikzcd}$$

where we assume I is based at 0 and ε_1 means evaluation at 1. The basepoint of ΩX is c_{x_0}, the constant loop at x_0. A based map $f\colon (X, x_0) \to (Y, y_0)$ is sent to $f_*\colon p \mapsto f \circ p$.

There are a couple of eminent connections between the loop space ΩX and the fundamental group $\pi_1(X)$ to remark upon. The first crucial observation to make is that ΩX admits *the same* operation of loop concatenation as $\pi_1(X)$ (except, strictly speaking, the operation in π_1 is on homotopy classes of maps). Given two loops p_1 and p_2 based at $x_0 \in X$, recall that their *concatenation* is the loop

$$(p_2 \circ p_1)(t) = \begin{cases} p_1(2t) & t \in [0, 1/2]; \\ p_2(2t-1) & t \in [1/2, 1]. \end{cases}$$

This defines a binary operation $\Omega X \times \Omega X \to \Omega X$ which makes ΩX into something like a topological monoid, except concatenation is not *strictly* unital and associative, because of a difference in parametrization. The differently parametrized paths $c_{x_0} \circ p$ and p and $p \circ c_{x_0}$ become *the same path* after identifying things up to homotopy. Similarly, up to homotopy, the paths $p_3 \circ (p_2 \circ p_1)$ and $(p_3 \circ p_2) \circ p_1$ are also the same. In many algebraic topology textbooks, this idea is usually illustrated by something like Fig. 1.
This notion of a space being "like a monoid (up to homotopy)" is captured by the definition of H-spaces (Definition 3.1). Loop spaces are an important class of H-spaces, being particularly well-behaved from a homotopy-theory point of view, as we shall see in Subsection 3.1 as well.

Another link between ΩX and $\pi_1(X)$ is given by the similarity of their elements: loops and homotopy classes of loops, respectively. Specifically, since a path in ΩX is precisely a homotopy of loops in X, two loops in X will belong to the same equivalence class in $\pi_1(X)$ if and only if they are in the same connected component of ΩX. That is, there is a bijective correspondence between $\pi_0(\Omega X)$ and $\pi_1(X)$, as illustrated in Fig. 2. In fact this is an isomorphism of groups, since $\pi_0(\Omega X)$ inherits a group structure from ΩX (using the same operation modulo the equivalence relation of being path connected).

[1] For the reader familiar with homotopy limits, we note that we could also realize ΩX as the homotopy pullback of $* \to X \leftarrow *$.

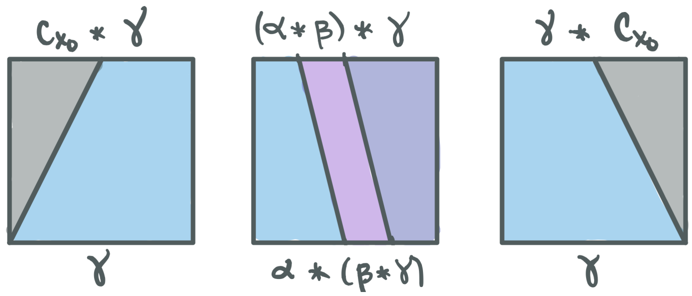


Figure 1: Concatenation of loops is unital and associative up to homotopy. A linear homotopy smoothly transitions between differences in parametrization by changing how fast the paths are traversed.

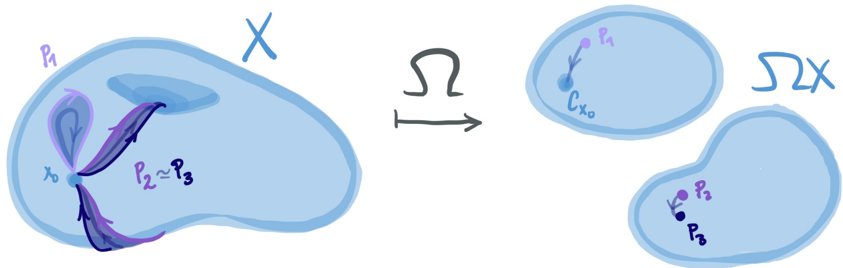


Figure 2: The correspondence between $\pi_0(\Omega X)$ and $\pi_1(X)$. Since p_1 is nullhomotopic in X, it is in the same connected component as the basepoint c_{x_0} in ΩX. The fact that the two loops p_2 and p_3 are homotopic in X is reflected by the existence of a path in ΩX between those elements.

Remark 2.2. More generally, there is a group isomorphism $\pi_n(\Omega X) \cong \pi_{n+1}(X)$. One way to see this is to use the adjunction with the suspension functor $\Sigma \dashv \Omega$. In particular, we have a group isomorphism

$$\mathbf{Top}_*(\Sigma S^n, X) \cong \mathbf{Top}_*(S^n, \Omega X).$$

Since ΣS^n is homeomorphic to S^{n+1}, this implies the desired result upon taking homotopy classes. Another method of proof is via the path-loop fibration, as in [BM21, §3.8]. The key is to use the pullback definition of ΩX to see that ΩX sits inside a fiber sequence

$$\Omega X \to Map_*(I, X) \to X.$$

and then use the long exact sequence for homotopy groups and the fact that $Map_*(I, X)$ is contractible.

2.1 Iterated and Infinite Loop Spaces

To form the k-fold loop space of a based space $X = (X, x_0)$, we can just continue to apply the loop space functor Ω to X, arriving at the k^{th} loop space $\Omega^k X$. The elements of $\Omega^k X$ (which are loops of loops of ... of loops in X) can also be understood as based maps $S^k \to X$, as illustrated for $k = 2$ in Fig. 3.

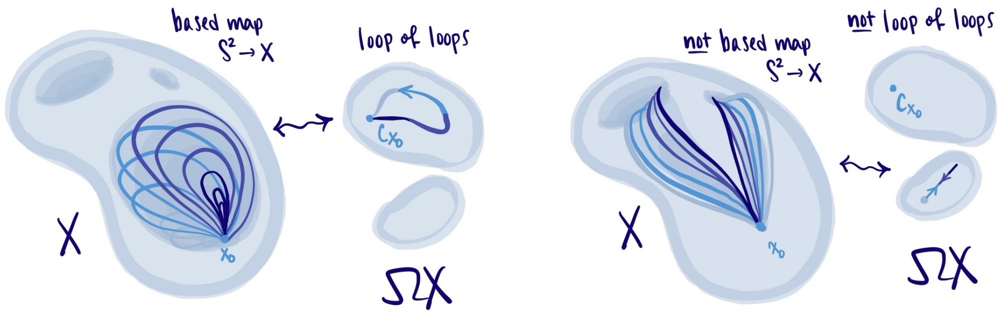


Figure 3: Loops of loops. The left part of the figure illustrates how a loop in ΩX (that is, loop of loops in X) can be seen as a based map $S^2 \to X$. The right part of the figure illustrates how a path in ΩX which is *not* a loop will not result in a continuous based map from $S^2 \to X$. In particular, the given path in ΩX is not a loop (although it begins and ends at the same point) since it is not based at the constant path c_{x_0}.

Definition 2.3. The k^{th} *loop space* of $X = (X, x_0)$ is $\Omega^k(X) = \Omega(\Omega^{k-1}(X)) = \mathrm{Map}_*(S^k, X)$.

By Remark 2.2, we know $\pi_0(\Omega^k X) \cong \pi_k(X)$, which demonstrates how the iterated loop spaces encode a lot of information about the homotopy of X. Moreover, this means that $\pi_0(\Omega^k X)$ has the structure of an *Abelian* group for $k \geq 2$. Thus $\Omega^k X$ should be something like an Abelian topological monoid for $k \geq 2$, where commutativity, associativity, and unity are only required to hold up to homotopy. Iterated loop spaces will therefore have even more structure than regular loop spaces, an idea which will be made more precise in our discussion of operads in Section 3.

Given a k-fold loop space $\Omega^k X$, we know how to "deloop" it in the sense that we know $\Omega^k X$ is just loops on $\Omega^{k-1}X$, which in turn is just loops on $\Omega^{k-2}X$, and so on. That is, $\Omega^k X$ comes with a sequence of spaces $\Omega^k X, \Omega^{k-1}X, \ldots, X$ such that $\Omega^k X = \Omega^j(\Omega^{k-j}X)$. It may not seem like there is much to this observation, but it helps to contextualize the definition of infinite loop spaces as the natural generalization of iterated loop spaces.

Definition 2.4. A space X is an *infinite loop space* if it admits an infinite chain of deloopings. That is, if there is a sequence of spaces $X = X_0, X_1, X_2, \ldots$ with weak equivalences

$$X_n \xrightarrow{\simeq} \Omega X_{n+1}$$

for all $n \geq 0$. In particular, X_0 is weakly equivalent to $\Omega^k X_k$ for all $k \geq 0$.

Given an infinite loop space, we can apply the adjunction $\Sigma \dashv \Omega$ mentioned in Remark 2.2 to get a map

$$\Sigma X_n \to X_{n+1}$$

for each n. In other words, an infinite loop space gives us a spectrum, in fact, an Ω-spectrum.[2] Thus another way to define an infinite loop space is as the 0^{th} term of an Ω-spectrum. Moreover, given a non-Ω-spectrum E, we can turn it into an equivalent Ω-spectrum by replacing E_n with $\operatorname{colim}_k \Omega^k E_{n+k}$. In this sense, the study of infinite loop spaces is the same as the study of spectra, which is the same as the study of generalized cohomology theories by Brown Representability (see [Hat02, §4E] or [PÍ3]).

In any case, given a spectrum E, we write $\Omega^\infty E$ for its infinite loop space (the 0^{th} term of the equivalent Ω-spectrum). Thus Ω^∞ gives us a way to turn spectra into spaces. By construction, the homotopy groups of $\Omega^\infty E$ (as a space) are equal to the homotopy groups of E (as a spectrum). In symbols:

$$\pi_n(E) = \operatorname{colim}_k \pi_{n+k}(E_n) \cong \operatorname{colim}_k \pi_k(\Omega^n E_n) \cong \operatorname{colim}_k \pi_n(E_0) = \pi_n(\Omega^\infty E),$$

since E_0 is weakly equivalent to $\Omega^n E_n$ for all n.

Remark 2.5. Recall that the suspension spectrum $\Sigma^\infty X$ of a based space X is the spectrum with $(\Sigma^\infty X)_n = \Sigma^n X$; although this is not an Ω-spectrum in general, we can replace it with an equivalent Ω-spectrum to get the *free infinite loop space* on X, $\Omega^\infty \Sigma^\infty X$. In fact, $\Omega^\infty \Sigma^\infty$ (sometimes denoted Q) is an endofunctor on $\mathbf{Top}_*$ landing in infinite loop spaces. The display above implies that the (unstable) homotopy groups of this space $\Omega^\infty \Sigma^\infty X$ are precisely the homotopy groups of the suspension spectrum $\Sigma^\infty X$, which are in turn the stable homotopy groups of X. This consequence, although straightforward, is a bit mind-boggling. For instance, it means there is a *single space* $\Omega^\infty \Sigma^\infty S^0 = \Omega^\infty \mathbb{S}$ which contains all the information about the stable homotopy groups of spheres. This space is undoubtedly incredibly complex and difficult to understand, so in all likelihood studying this space is not really a tractable approach to stable homotopy theory.

Example 2.6. One of the most fundamental examples of these ideas is the Eilenberg-Mac Lane spectrum. Recall that an ordinary cohomology theory (i.e. a cohomology theory isomorphic to singular cohomology) with coefficients in an Abelian group G is represented by the spectrum formed by the Eilenberg-Mac Lane spaces $\{K(G,n)\}_{n\geq 0}$, where $K(G,n)$ is uniquely specified (up to weak equivalence) by

$$\pi_k(K(G,n)) = \begin{cases} G & k = n, \\ 0 & k \neq n, \end{cases}$$

[2] Recall that a spectrum is called an Ω-*spectrum* if the adjoints of the structure maps are all weak equivalences.

We can explicitly construct $K(G,n)$ as the n-fold bar construction on G (see [BM21, §8]). By Remark 2.2, we have an isomorphism $\pi_{k+1}(\Omega K(G,n)) \cong \pi_k(K(G,n))$ for all $k \geq 0$, which implies that $\Omega K(G,n)$ is weakly equivalent to $K(G,n-1)$. Thus for any $n \geq 0$, we have weak equivalences

$$K(G,n) \xrightarrow{\simeq} \Omega K(G,n+1),$$

so $\{K(G,n)\}$ forms an Ω-spectrum. Therefore any Eilenberg-Mac Lane space $K(G,n)$ is an infinite loop space, with explicit deloopings given by $K(G,n) \simeq \Omega^k K(G,n+k)$.

There are many other examples of infinite loop spaces arising from Ω-spectra which are important in algebraic topology, in particular related to K-theory and cobordism. For a survey of these examples, we point the reader to [Ada78, §1.8].

3 Recognition Theorem via Operads

Given the nice properties of these loop spaces, we would love to have some sort of recognition theorem so we can know one when we see it. Arguably one the most popular approaches to the recognition theorem is May's operads (which are descended from Boardman and Vogt's PROPs). Before diving into operads in Subsection 3.2, we will discuss the recognition theorem for $k=1$ and Stasheff's approach [Sta63] to A_∞-spaces. The more general E_n-spaces and the full recognition theorem for $\Omega^k X$ are covered in Subsection 3.3, although we do not provide the full and detailed proof, pointing the reader to [May72] instead. Parts of our exposition are inspired by [Bae02, Bel17] and [Oso11] as well.

3.1 H-Spaces, A_∞-Spaces, and the Recognition Principle

We saw in Section 2 that ΩX is some sort of generalization of a topological monoid, thanks to our ability to continuously "multiply" loops together via concatenation. The exact nature of this generalization is captured by the concept of an H-space.[3]

Definition 3.1. A topological space X is an *H-space* if there are continuous maps $\mu\colon X \times X \to X$ and $e\colon * \to X$ along with homotopies

$$\mu(e(*), -) \simeq \mathrm{id}_X \simeq \mu(-, e(*)).$$

That is, $e(*)$ acts as a two-sided unit for the multiplication μ, up to homotopy. An H-space is *homotopy associative* if there are homotopies

$$\mu(\mu(-,-),-) \simeq \mu(-,\mu(-,-))$$

[3]The name "H-space" is due to Serre, who chose the letter H in homage to Hopf's work on the topology of Lie groups, cf. [Ada78, p.13].

and *homotopy commutative* if there are homotopies

$$\mu(-,-) \simeq \mu \circ \sigma(-,-),$$

where σ is the "swap" map which sends $(x, y) \mapsto (y, x)$.

Remark 3.2. If a homotopy associative H-space also admits a continuous map $i\colon X \to X$ which acts like a two-sided inverse up to homotopy, then X is called an *H-group*. These objects can be characterized their represented functor $[-, X]_*$. In general, $[-, X]_*$ defines a functor $\mathbf{Top}_* \to \mathbf{Set}_*$, but when X is an H-group, $[-, X]_*$ actually lands in the category $\mathbf{Grp}$ of groups. In fact, $[-, X]_*\colon \mathbf{Top}_* \to \mathbf{Grp}$ if and only if X is an H-group. There is a similar characterizing result for H-spaces, but it is a bit more nuanced, see [Sta70, Chapter 3].

In general, the H-space structure of a space X will only give $\pi_0(X)$ a monoid structure; those special H-spaces whose π_0 is a group are called *grouplike*. Note that π_0 of an H-group will be a group, so every H-group will be grouplike. In particular, loop spaces are grouplike. An important fact (which will become especially relevant in Section 4) is that any non-grouplike homotopy commutative H-space X can be *group completed* to a grouplike one. As a brief aside, we will recall the basics of group completion.

Definition 3.3. The *group completion* of a commutative monoid M is an Abelian group A along with a morphism of commutative monoids $i\colon M \to A$ which satisfies the following universal property: If A' is an Abelian group and $f\colon M \to A'$ is a morphism of commutative monoids, then there is a unique group homomorphism $\hat{f}\colon A \to A'$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "f"'] & A \arrow[d, dashed, "\hat{f}"] \\ & A' \end{tikzcd}$$

An explicit construction of the group completion is given by the *Grothendieck group* of M, defined as

$$G(M) := \text{free Abelian group}\{[m] : m \in M\}/\sim$$

where $[m + n] \sim [m] + [n]$ for all $m, n \in M$.

Given any space X with a homotopy commutative operation, its *group completion* is a homotopy commutative H-space Y such that $\pi_0(Y)$ is the group completion of $\pi_0(X)$ in the sense above, and $H_*(Y)$ is the localization of $H_*(X)$ at $\pi_0(X)$. If X is a topological monoid, then an explicit model is given by $Y = \Omega BX$, where BX is the classifying space of X. See the paper of Segal and McDuff [MS76] for more details.

A loop space is a homotopy associative (but not homotopy commutative) H-space, with $e(*) = c_{x_0}$ and $\mu =$ concatenation of loops.[4] Any candidate space which hopes to be a loop space must therefore at least be a homotopy associative H-space. Another necessary condition for our candidate space is that its π_0 must be a group, as we know every loop space is grouplike. A natural question to ask is whether these conditions are sufficient. That is, *is every grouplike homotopy associative H-space equivalent to a loop space?* The answer is no, and there are many examples of such H-spaces which are not loop spaces.

Example 3.4. Whether or not S^n admits an H-space is a classically important question, due in part to its connection to Adam's solution to the Hopf invariant one problem [Ada60]. It turns out there is an H-space structure on the sphere S^n only when $n = 0, 1, 3, 7$. By identifying S^n as the unit sphere in one of the four normed division algebras ($\mathbb{R}, \mathbb{C}$, quaternions, and octonians), the product is the one induced by the algebra product. The sphere S^7 is an example of a grouplike H-space which is not a loop space (cf. [Sta63, §I.5]).

In order to be a loop space, a grouplike homotopy associative H-space needs more "higher homotopy coherent" structure. The explication and exploration of this coherence structure is largely due to J. Stasheff [Sta63], who cites the work of M. Sugawara as inspiration. The idea is that the homotopies yielding homotopy associativity must satisfy some homotopical relations themselves, and these homotopies must satisfy other homotopical relations, and so on. Spaces with this kind of higher coherence structure are called A_∞-spaces. To make all of this more precise, we will see how it manifests in loop spaces.

Our discussion in Section 2 demonstrates that the two ways of associating a product of three elements in a loop space are homotopic. For the sake of notation, we will abbreviate the product $\mu(x, y)$ by just xy. Hence homotopy associativity implies that there is a path from $(xy)z$ to $x(yz)$ for every triple $x, y, z \in X$. The next natural step is to look at the ways to associate products of four elements. There are five different ways to do this, and homotopy associativity gives paths between some of them:

$$\begin{tikzcd}
 & & (xy)(zw) \arrow[drr] & & \\
((xy)z)w \arrow[urr] \arrow[dr] & & & & x(y(zw)) \\
 & (x(yz))w \arrow[rr] & & x((yz)w) \arrow[ur] &
\end{tikzcd}$$

[4]Furthermore, reversing the parametrization of a path defines an inverse map $\Omega X \to \Omega X$, so in fact loop spaces are H-groups. Using Remark 3.2, this gives us another way to see that $\pi_0(\Omega X)$ is a group.

We see that there are two different paths between $((xy)z)w$ and $x(y(zw))$, one around the top of the pentagon and one around the bottom. In a loop space, these two paths are homotopic, which is to say there is a homotopy of the two homotopies between the 4-ary operations $((-,-)-)-$ and $-(-(-,-))$ which fills in the pentagon. Formally, this homotopy of homotopies is a map

$$(\Omega X)^4 \times I^2 \to \Omega X.$$

Fig. 4 illustrates how we get a map $I^2 \to \Omega X$ for a fixed tuple $(x, y, z, w) \in (\Omega X)^4$. After thinking about products of four elements, we can ask about products of five elements. Now, our coherence structure demands the existence of a homotopy $(\Omega X)^5 \times I^3 \to \Omega X$ between homotopies. Similarly for products of six elements, or seven, or any arbitrary n.

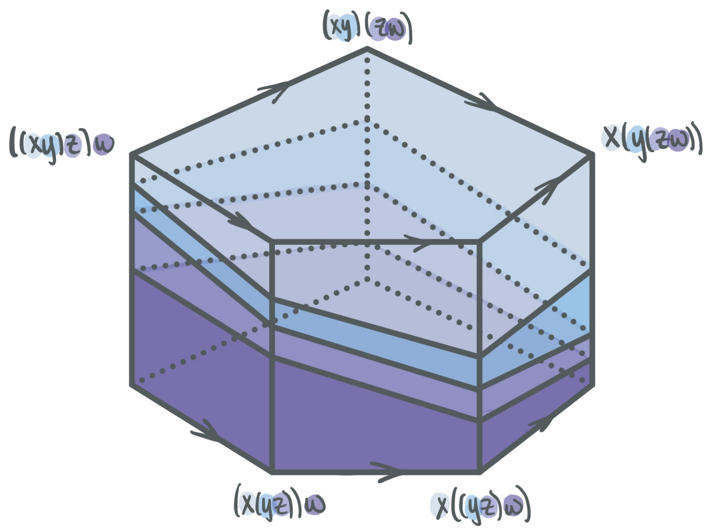


Figure 4: Homotopy coherence for a product of four elements $xyzw$**.** We know that the outside of the pentagonal box can be filled in according to the homotopy associativity diagram in Fig. 1. Homotopy coherence means we can "fill in" the rest of the box in a continuous way. Every point in the pentagon (which we can think of as I^2) gives us an explicit parametrization of the concatenation of x, y, z and w, and this parametrization varies smoothly as we move around in the pentagon.

An H-space where these homotopies between homotopies exist up to level n is called an A_n*-space*, and an H-space which is A_n for every n is called an A_∞-*space*. The definition of an A_n-structure on a space X given by Stasheff (see [Sta63, Definition I.1]) is a bit different than the "homotopy of homotopies" explanation we have given here, but the two ideas turn out to be essentially equivalent.

The main idea is to use the associahedra K_i (where $K_i \cong I^i$) and define a sequence of maps $M_i \colon K_{i-2} \times X^i \to X$ for $i \leq n$ which are defined appropriately on

$\partial K_{i-2} \times X^i$ in terms of M_j for $j < i$ and satisfy various coherence conditions.[5] These maps assemble into something Stasheff calls an A_n-form, which encapsulates our "homotopy of homotopies" idea. For instance, having an A_0-structure is equivalent to being an H-space, and having an A_1-structure is equivalent to being a homotopy associative H-space. A space has an A_∞-structure if it has an A_n-form for every n, and these spaces are called A_∞-spaces. The following theorem is implicit in [Sta63]:

Theorem 3.5 (Stasheff)**.** *A group-like H-space is equivalent to a loop space if and only if it is A_∞.*

The "only if" direction involves constructing a sort of classifying space for the A_∞-space X, defined as a quotient

$$BX = \coprod_{i \geq 0} K_{i+2} \times X^i / \sim$$

where $\sim$ involves the M_i maps. The result follows from showing X is equivalent to ΩBX. The idea of A_n- and A_∞-spaces can also be formally encoded and generalized by operads, which we will discuss presently. The recognition principle above is a special case of the recognition theorem (Theorem 3.18).

3.2 Operads

Operads provide a machinery for recording n-ary operations on a space X and the relationships between them. An operad consists of a collection of spaces $\{\mathcal{O}(n)\}_{n\geq 0}$ with maps between different levels, and if we can interpret $\mathcal{O}(n)$ as a space of n-ary functions on X, then we might learn something about the structure of X. The particular operads we will study here can detect the extent to which associativity and commutativity fail.

Definition 3.6. An *operad* $\mathcal{O}$ (over spaces) consists of the following data:

- For each $n \geq 0$, a space $\mathcal{O}(n) \in \mathbf{Top}$ such that $\mathcal{O}(0)$ is a single point $*$.
- Structure maps $\gamma\colon \mathcal{O}(n) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n) \to \mathcal{O}(k_1 + \cdots + k_n)$ such that the following associativity formula holds for all $f \in \mathcal{O}(n)$ and $g_i \in \mathcal{O}(k_i)$:
  - For any $h_1, \ldots, h_{k_1+\cdots+k_n}$,
  $$\gamma(\gamma(f; g_1, \ldots, g_n); h_1, \ldots, h_{k_1+\cdots+k_n}) = \gamma(f; F_1, \ldots, F_n)$$
  where $F_i = \gamma(g_i; h_{k_1+\cdots+k_{i-1}+1}, h_{k_1+\cdots+k_{i-1}+2}, \ldots, h_{k_1+\cdots+k_{i-1}+k_i})$ for $k_i > 0$ and $F_i = *$ if $k_i = 0$.

[5] It should be noted that in Stasheff's original work, he used a different indexing, taking K_i to be the $(i-2)$-associahedron so that the index i recorded the number of factors in the product. The indexing we use here is better adapted to the context of operads, however this means the reader comparing our notes to Stasheff's papers [Sta63] will have to adjust the indexing in some places by ± 2.

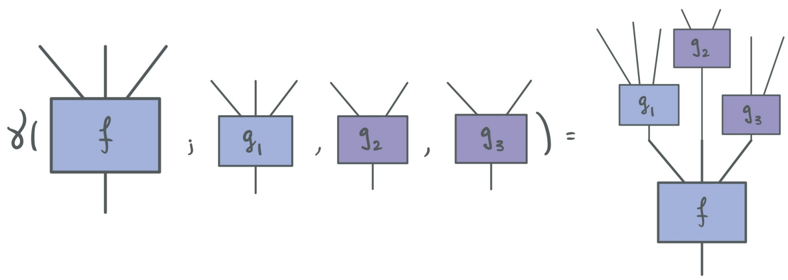
- An identity element $1 \in \mathcal{O}(1)$ such that
  - for all $k \geq 0$, $\gamma(1; -)\colon \mathcal{O}(k) \to \mathcal{O}(k)$ is the identity map;
  - for any $f \in \mathcal{O}(n)$,
$$\gamma(f; \underbrace{1, 1, \ldots, 1}_{n\text{-times}}) = f.$$
- A right action of the symmetric group Σ_n on $\mathcal{O}(n)$ for each $n \geq 0$ such that the following equivariance formulas are satisfied:
  - for all $f \in \mathcal{O}(n)$, $g_i \in \mathcal{O}(k_i)$ for $i = 1, \ldots, n$, and $\sigma \in \Sigma_n$,
$$\gamma(f\sigma; g_1, \ldots, g_n) = \gamma(f; g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)}) \cdot \sigma(k_1, \ldots, k_n)$$
    where $\sigma(k_1, \ldots, k_n) \in \Sigma_{k_1+\ldots k_n}$ permutes the n blocks given by the partition $k_1 + \cdots + k_n$ as σ would permute n letters (i.e. $\sigma(k_1, \ldots, k_n)$ treats the i^{th} block $k_i, k_i + 1, \ldots, k_i + k_{i+1} - 1$ as σ treats the i^{th} letter);
  - for $\tau_i \in \Sigma_{k_i}$, $i = 1, \ldots, n$,
$$\gamma(f\sigma; g_1\tau_1, \ldots, g_n\tau_n) = \gamma(f; g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)})(\tau_1 \oplus \cdots \oplus \tau_n)$$
    where $\tau_1 \oplus \cdots \oplus \tau_n$ denotes the image of $(\tau_1, \ldots, \tau_n)$ under the inclusion $\Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \hookrightarrow \Sigma_{k_1+\cdots+k_n}$.

The operad is called Σ-*free* if each Σ_n-action on $\mathcal{O}(n)$ is free. There is also a notion of a *non-*Σ *operad*, where the Σ_n-action is trivial for all $n \geq 0$.

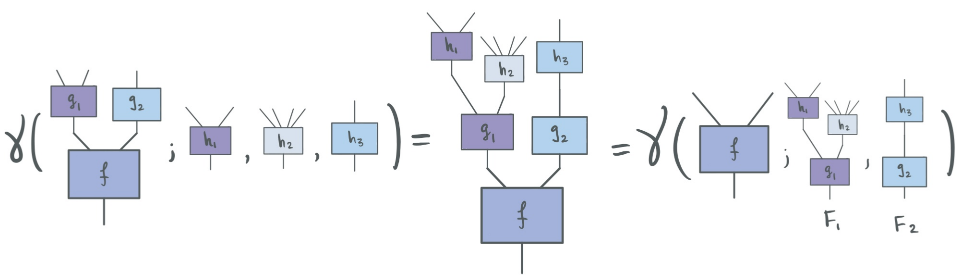
Remark 3.7. Strictly speaking, what we have defined above should be called a *topological, unital, symmetric operad*: topological because $\mathcal{O}$ takes values in spaces, unital because $\mathcal{O}(0)$ is required to be the unit in **Top**, and symmetric because there is a Σ-action. What we have called "non-Σ" operads are sometimes referred to as *non-symmetric* (or *planar*) *operads.* There is a forgetful-free adjunction between the categories of symmetric and non-symmetric operads.

Other generalizations of operads include *colored operads* (or *symmetric multicategories*) —an operad as we have defined it is a colored operad with one color (or multicategory with one object)— and ∞-*operads*, both of which are covered in Lurie's Higher Algebra [Lur17, §2.1].

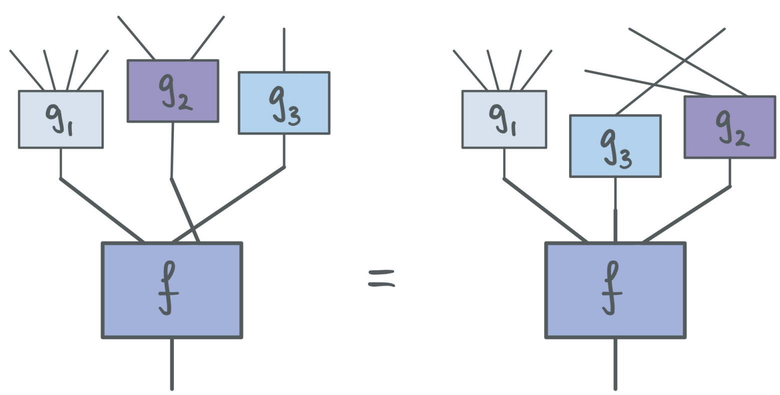
This definition is more easily understood through pictures. Following [Bae02], we can visualize an element of $\mathcal{O}(n)$ as an abstract n-ary operation, a little box with n "input wires" and one "output wire." The structure maps then tell us how to compose these abstract operations. To form the operation $\gamma(f; g_1, \ldots, g_k)$, we can imagine attaching the output wire of g_i to the i^{th} input wire of f for each $i = 1, \ldots, k$.



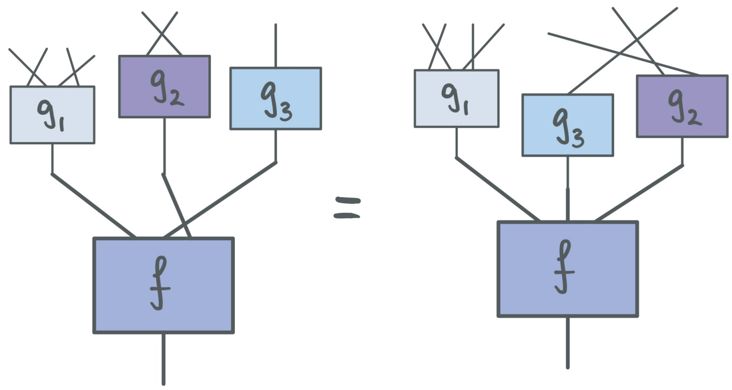
The associativity formula means that it should not matter whether we first attach the h_j to the g_i then attach those to f, or whether we first attach the g_i to f, and then attach the h_j. For example, we could get to $f(g_1(h_1, h_2), g_2(h_3))$ in two different ways: we could first compose the g_i's in the f, $f(g_1(-,-), g_2(-))$ and then plug in the h's; or, we could first plug in the h's, obtaining $F_1 = g_1(h_1, h_2)$ and $F_2 = g_2(h_3)$, and then plug *those* into f to get $f(F_1, F_2)$.



The Σ_n-action can be interpreted as permuting the input wires of f, crossing them over each other. The equivariance formulas ensure that operations we want to think of as the same are actually identified under this action. The first equivariance formula says instead of permuting the inputs of f and then attaching the g_i, we could instead permute the g_i, attach them to f, and then permute the inputs of *that* operation.



The second equivariance formula says that if we permute the inputs of f *and* permute the inputs of the g_i before attaching them, we could instead just permute the g_i, attach them, and then permute the inputs of that operation.



These pictures should serve to convince the reader that the coherence conditions in the definition of operads "do what they should." Having defined these objects, we are contractually obligated to tell you about the morphisms between them.

Definition 3.8. A *morphism* of operads $\phi\colon \mathcal{O} \to \mathcal{O}'$ is a collection of continuous Σ_n-equivariant maps $\phi_n\colon \mathcal{O}(n) \to \mathcal{O}'(n)$ such that $\phi_1(1) = 1$ and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}(n) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n) & \xrightarrow{\gamma} & \mathcal{O}(k_1 + \cdots + k_n) \\ {\scriptstyle \phi_n \times \phi_{k_1} \times \cdots \times \phi_{k_n}}\big\downarrow & & \big\downarrow{\scriptstyle \phi_{k_1+\cdots+k_n}} \\ \mathcal{O}'(n) \times \mathcal{O}'(k_1) \times \cdots \times \mathcal{O}'(k_n) & \xrightarrow[\gamma']{} & \mathcal{O}'(k_1 + \cdots + k_n). \end{array}$$

Remark 3.9. Operads and their morphisms can be defined more generally, replacing **Top** with any symmetric monoidal category (and making the appropriate symbol replacements, such as replacing $\times$ with $\otimes$, etc.). This ability to transplant the machinery of operads to other categories is quite useful, particularly for mathematical physics (see [MSS00, Chapter 5] for an extensive discussion of some examples).

One of the most powerful and important uses of operads (at least for our purposes) is their ability to act upon a space. An action of an operad on a space X is a collection of maps $\mathcal{O}(n) \times X^n \to X$ for each n which play nice with the structure maps and the Σ-action. This allows us to interpret elements of $\mathcal{O}(n)$ as n-ary operations on X, since for each $f \in \mathcal{O}(n)$ we get a map $f\colon X^n \to X$. Formally, an action of $\mathcal{O}$ on X is expressed as a morphism of $\mathcal{O}$ to a certain operad defined in terms of X.

Definition 3.10. For any based space X, the *endomorphism operad of* X is an operad End_X given by the following data:

- For $n \geq 0$, $\mathrm{End}_X(n) = \mathrm{Map}_*(X^n, X)$ is the space of based maps $X^n \to X$. We take X^0 to be a point $*$ and $\mathrm{End}_X(0)$ to be the basepoint inclusion $* \to X$.
- The structure maps γ are given by $\gamma(f; g_1, \cdots, g_n) = f(g_1 \times \cdots \times g_n)$ for $f \in \mathrm{End}_X(n)$ and $g_i \in \mathrm{End}_X(k_i)$.
- The identity element $1 \in \mathrm{End}_X(1)$ is just the identity map $\mathrm{id}\colon X \to X$.
- The Σ-action is given by $f\sigma\colon x \mapsto f(\sigma \cdot x)$ for $f \in \mathrm{End}_X(n)$, where Σ_n acts on X^n by permuting the factors, $\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma^1(1)}, \ldots, x_{\sigma^{-1}(n)})$.

We say that an operad $\mathcal{O}$ *acts* (or *operates*) on a space X if there is a morphism of operads $\mathcal{O} \to \mathrm{End}_X$. In this case, we say X is an *$\mathcal{O}$-algebra* or an *algebra over $\mathcal{O}$*.

Example 3.11. The simplest example of a non-Σ operad is the *associative operad* $\mathcal{A}$, defined by setting $\mathcal{A}(n) = *$ for all n. An action of $\mathcal{A}$ on X is a map $* \to \mathrm{End}_X(n)$ for each n, i.e. it picks out a single n-ary operation $X^n \to X$ for each n. The fact that the morphism $\mathcal{A} \to \mathrm{End}_X$ has to preserve the operadic structure implies that each of these chosen n-ary operations can actually be defined in terms of the single binary operation picked out by $\mathcal{A}(2) = * \to \mathrm{End}_X(2)$. For instance, if we call this operation μ, we see that $\gamma(\mu; 1, \mu) = \mu(-, \mu(-, -))$ and $\gamma(\mu; \mu, 1) = \mu(\mu(-, -), -)$ are both in the image of $\mathcal{A}(3) = * \to \mathrm{End}_X(3)$, which is a point, implying $\mu(-, \mu(-, -)) = \mu(\mu(-, -), -)$. Strict associativity of an n-fold product is similarly achieved. Additionally, note that the structure maps force the basepoint to act as the unit for μ. This means X is an $\mathcal{A}$-algebra if and only if it has a unital, associative multiplication. Thus in **Top**, the $\mathcal{A}$-algebras are precisely the topological monoids.

As this example demonstrates, being an algebra over a particular operad can characterize an entire class of spaces. The next question to ask is whether there is an operad which exactly picks out loop spaces. In light of our discussion in Subsection 3.1, it is perhaps not surprising that the answer is yes. In fact, the maps $M_i\colon K_i \times X^i \to X$ assemble into an action of the non-Σ *associahedra operad* $\mathcal{K}$ on X, with $\mathcal{K}(i) = K_i$, illustrated for $i = 0, 1, 2, 3$ in Fig. 5.

A crucial feature of this operad is that each of its spaces is contractible, $\mathcal{K}(i) \simeq *$. As we saw in Example 3.11, the fact that the n^{th} level of the operad $\mathcal{A}(n)$ is just a point means that there is "only one" n-ary operation on an algebra X (i.e. the multiplication on X is strictly associative). If the n^{th} level of our operad is *not* equal to $*$, so we are dealing with "more than one" n-ary operation on X, then contractibility is the next best thing we could ask for, from a homotopy theoretic point of view. Put a different way, the best kind of associativity short of *strict* associativity is that of an A_∞-space.

Definition 3.12. An *A_∞-operad* is a non-Σ operad $\mathcal{O}$ such that $\mathcal{O}(n)$ is contractible for all n. An A_∞-algebra (an algebra over an A_∞-operad) is called an A_∞-space.

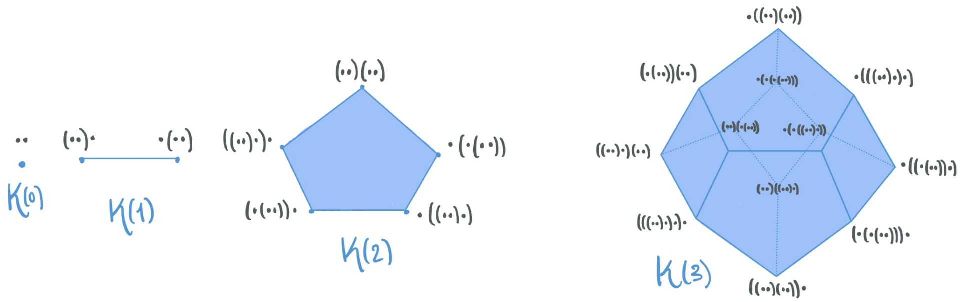


Figure 5: Some spaces of $\mathcal{K}$. The vertices of the n-associahedron correspond to parenthesizations of n letters. There is an edge between two vertices that are related by changing only one pair of parentheses, and a face among vertices that are related by changing two pairs of parentheses, and so on.

It turns out a space is A_∞ if and only if it has an action of $\mathcal{K}$, so this "new" definition of A_∞-spaces coincides with the one we saw previously. Theorem 3.5 can thus be restated in terms of A_∞-algebras:

Theorem 3.13. *A grouplike space is equivalent to a loop space if and only if it is an A_∞-algebra (equivalently, a $\mathcal{K}$-algebra).*

Any $\mathcal{A}$-algebra is naturally an $\mathcal{A}_\infty$-algebra, just by precomposing with the trivial map $\mathcal{K}(n) \to \mathcal{A}(n) = *$. But we know A_∞-spaces are not the whole story. What about A_n-spaces for finite n, how do these fit into the picture? The maps of Stasheff still assemble into an "action" of $\mathcal{K}(i)$, but only for $i \leq n$. For example, homotopy associativity is equivalent to just the condition of $\mathcal{K}(1)$. By properly truncating the operad $\mathcal{K}$ at level n, we get another operad $\mathcal{K}_n$ whose algebras are precisely the A_n-spaces. Thus we can write down a hierarchy of homotopy associativity, as done in [Bel17]:

$$\begin{array}{c}\text{Topological}\\ \text{monoids}\end{array} \Rightarrow A_\infty\text{-spaces} \Rightarrow \cdots \Rightarrow A_n\text{-spaces} \Rightarrow \cdots \Rightarrow A_1\text{-spaces} \Rightarrow H\text{-spaces}.$$

3.3 E_∞-Operads and E_∞-Spaces

E_n-operads play an analogous role for commutativity that the A_n-operads play for associativity. In fact, E_n-operads are just the Σ-free counterparts of A_n-operads, i.e. the spaces of an E_n-operad are contractible up to level n with a free Σ-action. If we think of each point in this space as determining a n-ary operation on X, the free Σ-action can be thought of as permuting the variables. Just as we have a non-Σ operad $\mathcal{A}$ for strict associativity, so do we have a Σ-free operad for strict commutativity.

Example 3.14. The simplest example of a Σ-free operad is the *commutative operad* $\mathcal{C}$, where $\mathcal{C}(n) = *$ for all n. As for the associative operad (Example 3.11), an action $\mathcal{C}$ picks out an n-ary operation on X for each n. Just as before, each of these n-ary operations is determined by the choice of binary operation μ. (In other words, an $\mathcal{C}$-algebra is also an $\mathcal{A}$-algebra.) The free Σ-action and the structure maps imply that, for example, $(a,b) \mapsto \mu(a,b)$ and $(a,b) \mapsto \mu(b,a)$ must be the same map. In other words, products of two elements must commute. Commutativity of products of any n elements is similarly implied. This means X is a $\mathcal{C}$-algebra if and only if it is has a unital, associative, and commutative multiplication. In **Top**, the $\mathcal{C}$-algebras are precisely the Abelian topological monoids.

Strict commutativity is forced by the fact that the space of n-ary operations is just a point for each n. If the space of n-ary operations is not a point (so our multiplication is not strictly commutative), the next best thing we could ask for is that it have the homotopy type of a point. This is the idea of E_∞-operads.

Definition 3.15. An *E_∞-operad* is a Σ-free operad whose spaces are contractible, and algebras over such an operad are called *E_∞-spaces.*

Just like with A_∞, we can "truncate" E_∞-operads to form E_n-operads, whose spaces are contractible up to level n. We then get a hierarchy of commutativity:

$$\begin{array}{c}\text{Abelian}\\ \text{topological}\\ \text{monoids}\end{array} \Rightarrow E_\infty\text{-spaces} \Rightarrow \cdots \Rightarrow E_n\text{-spaces} \Rightarrow \cdots \Rightarrow E_1\text{-spaces}.$$

We can ask whether there is an analogy of $\mathcal{K}$ for commutativity, that is, one operad $\mathcal{O}$ such that a space is an E_∞-space if and only if it is an algebra over $\mathcal{O}$, or an E_n-space if and only if it is an algebra over the truncation $\{\mathcal{O}(i)\}_{i \leq n}$. It turns out there is such an operad $\mathcal{C}_\infty$, whose n^{th} truncation $\mathcal{C}_n$ is called the *n-cubitos operad* in Spanish. (In English, this is the *little n-cubes operad*, but the author likes the Spanish translation used in [Oso11]).

Definition 3.16. Let J^n denote the interior of the n-cube I^n. An *n-cubito* is a linear embedding $c\colon J^n \to J^n$, such that $c = c_1 \times \dots c_n$ where each $c_i\colon J \to J$ is a linear function. The *n-cubitos operad* is given by the following data:

- For $m \geq 0$, $\mathcal{C}_n(m)$ consists of m-tuples $\langle c_1, \dots, c_m \rangle$ of n-cubitos c_i whose images are pairwise disjoint. We can regard $\langle c_1, \dots, c_m \rangle$ as a map $\underbrace{J^n \sqcup \cdots \sqcup J^n}_{m\text{-times}} \to J^n$, and we topologize $\mathcal{C}_n(m)$ as a subspace of $\mathrm{Map}(\sqcup_m J^n, J^n)$.[6] We think of $C_n(0) = \langle\ \rangle$ as the unique "embedding" of the empty set into J^n.

[6] Equivalently, we can topologize $\mathcal{C}_n(m)$ as a subspace of J^{2nm}, where an n-cubito $\langle c_1, \dots, c_m \rangle$ is identified with the point $(c_1(\alpha), c_1(\beta), \dots, c_m(\alpha), c_m(\beta)) \in J^{2nm}$ for $\alpha = (1/4, \dots, 1/4), \beta = (3/4, \dots, 3/4) \in J^{nm}$. See [May72, Lemma 4.2] for a full proof of this fact.

- The structure maps are defined as $\gamma(c; d_1, \ldots, d_k) = c(d_1 \sqcup \cdots \sqcup d_k)$ for $c \in C_n(m)$ and $d_i \in \mathcal{C}_n(k_i)$. This resulting map embeds $k_1 + \cdots + k_m$ pairwise disjoint n-cubitos into J^n.

- The identity element $1 \in \mathcal{C}_n(1)$ is the identity map $\mathrm{id}\colon J^n \to J^n$.

- The Σ-action permutes the n-cubitos, with $\langle c_1, \ldots, c_m \rangle \sigma = \langle c_{\sigma(1)}, \ldots, c_{\sigma(m)} \rangle$ for $\sigma \in \Sigma_m$. Since the images of the c_i are required to be disjoint, this action is free.

The idea here is that specifying a linear map $c_n\colon J^n \to J^n$ is equivalent to specifying the "little cube" which is the range of c_n inside of J^n. An element $\langle c_1, \ldots, c_m \rangle$ will look like m little n-cubes nicely embedded inside a bigger, ambient n-cube. The structure maps "glue in" each of the d_i to one of the n-cubitos in c; this is illustrated for $n = 2$ in Fig. 6.

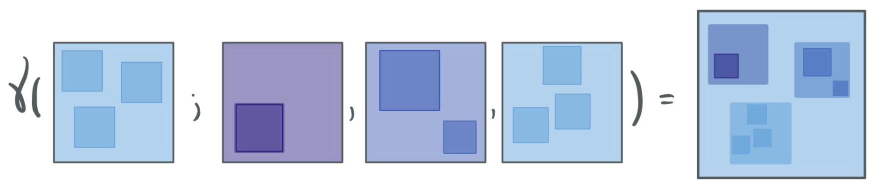


Figure 6: Example of structure maps for 2-cubitos. Given $c \in \mathcal{C}_2(3)$ and three 2-cubitos $d_1 \in \mathcal{C}_2(1), d_2 \in \mathcal{C}_2(2)$, and $d_3 \in \mathcal{C}_2(3)$, form the "composition" $\gamma(c; d_1, d_2, d_3)$ by gluing each of the d_i to one of the little cubes in c. By forgetting about the ambient d_i cube, and only remembering the 2-cubitos within each d_i, we get an element of $\mathcal{C}_2(6)$.

Observe that we can turn an n-cubito c into an $(n+1)$-cubito $c \times 1$ by taking the product with $1\colon J \to J$. In fact, we get an inclusion $\mathcal{C}_n(m) \to \mathcal{C}_{n+1}(m)$ given by $\langle c_1, \ldots, c_m \rangle \mapsto \langle c_1 \times 1, \ldots, c_m \times 1 \rangle$, and these maps assemble into an inclusion of operads $\mathcal{C}_n \to \mathcal{C}_{n+1}$. Hence we can take the colimit $\mathrm{colim}_n\, \mathcal{C}_n(m) =: \mathcal{C}_\infty(m)$ for each m, and these spaces assemble into the ∞-*cubitos operad* $\mathcal{C}_\infty$. Note that this operad is Σ-free, as each $\mathcal{C}_n$ is. It turns out that a space is E_∞ if and only if it admits an action of ∞-cubitos, so $\mathcal{C}_\infty$ is precisely the operad we will use to characterize infinite loop spaces. The following theorem is [May72, Theorem 4.8]:

Theorem 3.17. *For $1 \leq n \leq \infty$, $\mathcal{C}_n$ is an E_n-operad.*

Idea of proof. We need to show that $\mathcal{C}_n(m)$ is contractible for $m \leq n$. The crux of the argument is showing that $\mathcal{C}_n(m)$ is Σ_m-equivariantly homotopy equivalent to the (labeled) m^{th} configuration space of J^n (m-tuples of points in J^n whose coordinates are pairwise distinct). This is done by constructing an explicit deformation retraction from $\mathcal{C}_n(m)$ to this configuration space, given by $g_n\colon \langle c_1, \ldots, c_m \rangle \mapsto (c_1(1/2), \ldots, c_m(1/2))$. Known results about the labeled configuration spaces of

$\mathbb{R}^n$ (equivalently, J^n) imply that $\pi_i(C_n(m)) = 0$ for $i \leq n$. For C_∞, taking $g_\infty = \operatorname{colim}_n g_n$ gives a Σ_m-invariant homotopy equivalence from $C_\infty(m)$ to the m^{th} configuration space of $J^\infty \cong \mathbb{R}^\infty$. Since $\mathbb{R}^\infty$ is contractible, it follows that C_∞ is an E_∞-operad. □

Our primary motivation for this definition is to reinterpret iterated loop spaces as $\mathcal{C}_n$-algebras. Thus we should understand how $\mathcal{C}_n$ acts on an n-fold loop space $\Omega^n X$. To define an action of $\mathcal{C}_n$ on $\Omega^n X$, we need to give a collection of Σ_n-equivariant maps $\phi_k \colon \mathcal{C}_n(k) \to \mathrm{Map}_*((\Omega^n X)^k, \Omega^n X)$ which preserve identity and play nice with the structure maps. For this purpose, it will be easiest to think of elements of $\Omega^n X = \mathrm{Map}_*(S^n, X)$ as maps $I^n \to X$ which take the boundary ∂I^n to the basepoint $x_0 \in X$.

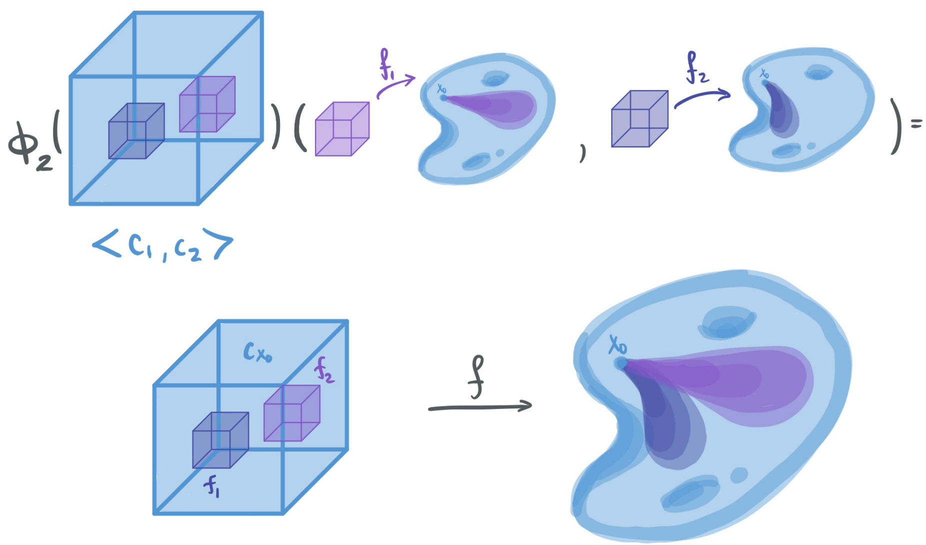


Figure 7: An example of the action of $\mathcal{C}_n(2)$ on $\Omega^n X$. An element $\langle c_1, c_2\rangle \in \mathcal{C}_n(2)$ determines a map $\phi_2\langle c_1, c_2\rangle \colon (\Omega^n X)^2 \to \Omega^n X$ which sends two elements $f_1, f_2 \in \Omega^n X$ to their "concatenation" $f \in \Omega^n X$.

Given an element $\langle c_1, \ldots, c_k\rangle \in \mathcal{C}_n(k)$ and k maps of $f_i \colon (I^n, \partial I^n) \to (X, x_0) \in \Omega^n X$, we need to construct an element of $\Omega^n X$ which is somehow the "multiplication" of the f_i. The idea is that this output element $f \colon (I^n, \partial I^n) \to (X, x_0)$ will agree with f_i on the n-cubito c_i, viewed as living in the domain I^n of f, and will be the constant map at x_0 on the complement of all the c_i. This explanation is hopefully clarified by the picture in Fig. 7. The symbolic definition of f is

$$f(t) = \begin{cases} f_i(c_i^{-1}(t)) & t \in \operatorname{im} c_i; \\ x_0 & \text{otherwise.} \end{cases}$$

The reader can verify (or take it on faith) that this construction preserves identity and behaves appropriately with respect to the structure maps and Σ_n-action. Hence we get a morphism of operads $\mathcal{C}_n \to \mathrm{End}_{\Omega^n X}$, implying every n-fold loop space is an E_n-space.

Fig. 7 also illustrates how $\mathcal{C}_n$ encodes homotopy commutativity (for $n \geq 2$) since there is a homotopy between $\phi_2\langle c_1, c_2\rangle(f_1, f_2)$ and $\phi_2\langle c_1, c_2\rangle(f_2, f_1)$ given by expanding, shrinking, and sliding the n-cubitos c_1 and c_2 around each other. This "proof by picture" is similar to the one often given to shown that the higher homotopy groups are Abelian. This proof does *not* work in the case $n = 1$, since the action of $\mathcal{C}_1$ on ΩX is just the usual concatenation of loops, which we know is not commutative. As is implied by this observation, $\mathcal{C}_1$ has the structure of an A_∞-operad, so every E_1-space is an A_∞-space. The converse is true as well, meaning $E_1 = A_\infty$.

We have hopefully convinced the reader that every n-fold loop space is a $\mathcal{C}_n$-algebra and hence an E_n-space. As is perhaps expected at this point, being a grouplike $\mathcal{C}_n$-algebra is equivalent to being an iterated loop space. This is the recognition theorem of May [May72]:

Theorem 3.18 (May)**.** *For $1 \leq n \leq \infty$, a (connected) grouplike space is E_n (equivalently, a $\mathcal{C}_n$-algebra) if and only if it is weakly equivalent to an n-fold loop space.*

We have already discussed the "if" part of the theorem, and the next subsection addresses the "only if" part. But before concluding this subsection, we want to remark that much of this discussion could be done over a different symmetric monoidal category other than **Top**. That is, we can still define A_n- and E_n-operads over some other category (e.g. spectra), and ask what the A_n- and E_n-objects are in this context. To the author's knowledge, there is no recognition principle for any category other than spaces.

3.4 Monads and Deloopings

To prove the full recognition theorem, May connects operads to slightly simpler gadgets known as *monads*. The main idea is that every operad $\mathcal{O}$ can be associated to a monad T such that the $\mathcal{O}$-algebras are precisely the algebras over T. Before unpacking this construction, we recall some necessary definitions from [May72]. A more general treatment of monads and operads can also be found in [May02].

Definition 3.19. A *monad* on a category $\mathscr{C}$ is a functor $T\colon \mathscr{C} \to \mathscr{C}$ together with natural transformations $\mu\colon T^2 \Rightarrow T$ and $\eta\colon 1 \Rightarrow T$ such that the following diagrams commute:

$$\begin{tikzcd} T \arrow[r, Rightarrow, "T\eta"] \arrow[dr, Rightarrow] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "\eta_T"'] \arrow[dl, Rightarrow] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd}$$

A *morphism of monads* $\psi\colon (T; \mu, \eta) \to (T'; \mu', \eta')$ is a natural transformation of functors $\psi\colon T \to T'$ such that the following diagrams commute for all objects $X \in \mathscr{C}$:

$$\begin{tikzcd} & X \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\ TX \arrow[rr, "\psi"'] & & TX \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "\psi^2"] \arrow[d, "\mu"'] & (T')^2X \arrow[d, "\mu'"] \\ TX \arrow[r, "\psi"'] & T'X \end{tikzcd}$$

In slick math terms, a monad is the monoid object in the category of endofunctors of $\mathscr{C}$. In this spirit, the natural transformation μ is called the *multiplication* of the monad and η is called the *unit*. We can now define the algebras over these monoid objects.

Definition 3.20. An *algebra over a monad* $T\colon \mathscr{C} \to \mathscr{C}$ is an object $X \in \mathrm{Ob}\,\mathscr{C}$ together with a *structure map* $TX \xrightarrow{a} X$ such that the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "a"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "\mu_X"] \arrow[d, "Ta"'] & TX \arrow[d, "a"] \\ TX \arrow[r, "a"'] & X \end{tikzcd}$$

The structure map a can be thought of as a left action of T on X, which essentially makes X into left module over the monoid T. The diagrams above are sometimes called the *unit triangle* and *multiplication square* for X.

A *morphism of* T*-algebras* $f\colon (X,a) \to (X',a')$ is a morphism $f\colon X \to X'$ such that the following diagram commutes:

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "a"'] & TX' \arrow[d, "a'"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

We denote the category of T-algebras and their morphisms by $Alg_T(\mathscr{C})$.

For any $X \in \mathrm{Ob}\,\mathscr{C}$, the object TX is a T-algebra whose structure is given by setting $a = \mu_X$. The monad axioms ensure the necessary diagrams commute. Such an algebra is called a *free algebra*, and there is a forgetful-free adjunction $F\colon \mathscr{C} \leftrightarrows Alg_T(\mathscr{C})\colon U$. This free construction lets us view T as a functor $\mathscr{C} \to Alg_T(\mathscr{C})$, and in this way, a monad T is completely determined by its algebras. More generally, any adjunction $L\colon \mathscr{C} \leftrightarrows \mathscr{D} : R$ gives rise to the monad $(RL; R\varepsilon, \eta)$, where $\eta\colon 1 \Rightarrow RL$ and $\varepsilon\colon LR \Rightarrow 1$ are the unit and counit of the adjunction, respectively.

Example 3.21. Of particular interest for our purposes is the monad arising from the suspension-loop adjunction $\Sigma \dashv \Omega$. Recall that $\Sigma X \cong S^1 \wedge X$ for $X \in \mathbf{Top}_*$, so we can write an element of ΣX as an equivalence class $[t,x]$ for $t \in S^1$ and $x \in X$. The unit η of the adjunction maps X to $\Omega\Sigma X$ by sending a point $x \in X$ to the loop $t \mapsto [x,t]$; this is also the unit of the monad $\Omega\Sigma$. The counit ε maps

$$\Sigma\Omega X \xrightarrow{\varepsilon_X} X$$
$$[\gamma, t] \mapsto \gamma(t)$$

for $\gamma \in \Omega X$ and $t \in S^1$. Thus the multiplication $\mu = \Omega\varepsilon$ of the monad is given by components $\Omega\Sigma\Omega X \xrightarrow{\mu_X} \Omega X$ which send $\gamma\colon S^1 \to \Sigma\Omega X$ to the loop $t \mapsto \varepsilon_X(\gamma(t))$. In general, the adjunction $\Sigma^n \dashv \Omega^n$ gives rise to a monad $\Omega^n\Sigma^n$ for $1 \leq n \leq \infty$.

We can connect monads to operads by associating a monad to an arbitrary operad $\mathcal{O}$. Monads that arise in this way form a particularly convenient collection to work with.

Example 3.22. Given an operad $\mathcal{O}$, we can construct a monad $(T_{\mathcal{O}}; \mu, \eta)$ associated to $\mathcal{O}$. Let $T_{\mathcal{O}}\colon \mathbf{Top}_* \to \mathbf{Top}_*$ be given by

$$T_{\mathcal{O}}X = \bigsqcup_{n\geq 0} \mathcal{O}(n) \times X^n / \sim,$$

where $\sim$ is generated by:

- $(f\sigma, x) \sim (f, \sigma x)$ for $f \in \mathcal{O}(n)$ and $f \in X^n$, where σ acts on X^n by permuting the coordinates;
- Define $\sigma_i\colon \mathcal{O}(n) \to \mathcal{O}(n-1)$ by

$$\sigma_i f = \gamma(f; 1, \ldots, 1, *, 1, \ldots, 1),$$

  where $* \in \mathcal{O}(0)$ appears in the $(i+1)^{th}$ spot. Declare $(\sigma_i f, x) \sim (f, s_i x)$ where $s_i x = (x_1, \ldots, x_i, *, x_{i+1}, \ldots, x_n) \in X^{n+1}$.

The first relation says permuting the variables in f is the same as permuting the coordinates in X^n. The second relation says that including the basepoint $*$ is well-behaved, in the sense that $f(-, \ldots, *, \ldots, -) \sim f_i$ where $f_i\colon X^{n-1} \xrightarrow{x_i = *} X^n \xrightarrow{f} X$. Given $f \in \mathcal{O}(n)$ and $x \in X^n$, we let $[f, x]$ denote the image of (f, x) in $T_{\mathcal{O}}X$. If $\phi\colon X \to Y$ in $\mathbf{Top}_*$ then

$$\begin{aligned} T_{\mathcal{O}}\phi\colon T_{\mathcal{O}}X &\to T_{\mathcal{O}}Y \\ [f, x] &\mapsto [f, \phi^n(x)]; \end{aligned}$$

one can verify that this makes $T_{\mathcal{O}}$ into an endofunctor on $\mathbf{Top}_*$.

The monad structure maps $\mu\colon T_{\mathcal{O}}^2 X \to T_{\mathcal{O}}X$ and $\eta\colon X \to T_{\mathcal{O}}X$ are given by

- $\mu[f, [g_1, x_1], \ldots, [g_n, x_n]] = [\gamma(f; g_1, \ldots, g_n), x_1, \ldots, x_n]$ for $f \in \mathcal{O}(n)$, $g_i \in \mathcal{O}(k_i)$, and $x_i \in X^{k_i}$;
- $\eta(x) = [1, x]$ for $x \in X$.

The structure built into the operad $\mathcal{O}$ ensure that μ and η are well-defined and satisfy the necessary coherence diagrams.

The assignment of an operad to its associated monad is functorial, meaning that if $\psi\colon \mathcal{O} \to \mathcal{O}'$ is a morphism of operads, then there is a morphism of the associated monads $\tilde{\phi}\colon T_{\mathcal{O}} \to T_{\mathcal{O}'}$ given by

$$\begin{aligned}\tilde{\psi}_X\colon T_{\mathcal{O}}X &\to T_{\mathcal{O}'}X\\ [f,x] &\mapsto [\psi_n(f),x]\end{aligned}$$

for $X \in \mathbf{Top}_*$, $f \in \mathcal{O}(n)$, and $x \in X^n$. The topology of $T_{\mathcal{O}}X$ and other properties of the assignment $\mathcal{O} \mapsto T_{\mathcal{O}}$ are discussed in [May72, §2], the most crucial of which is the following proposition (Proposition 2.8 in [May72]):

Proposition 3.1. *There is an isomorphism of categories between the category of $\mathcal{O}$-algebras and the category of $T_{\mathcal{O}}$-algebras $T_{\mathcal{O}}[\mathbf{Top}_*]$.*

The idea of the proof is that the data of a $T_{\mathcal{O}}$-algebra structure on a space X is equivalent to the data of a morphism $\mathcal{O} \to \mathrm{End}_X$. In a bit more detail, a left action $a\colon T_{\mathcal{O}}X \to X$ determines and is determined by a collection of maps $\theta_n\colon \mathcal{O}(n)\times X^n \to X$ satisfying certain relations. It turns out that these relations are equivalent to those required for the θ_n to assemble into a morphism of operads $\theta\colon \mathcal{O} \to \mathrm{End}_X$. Moreover, $T_{\mathcal{O}}X$ can be viewed as the free $T_{\mathcal{O}}$-algebra (or $\mathcal{O}$-algebra) generated by X, in the sense that the monad $T_{\mathcal{O}}$ is the one which arises in the forgetful-free adjunction $U\colon T_{\mathcal{O}}[\mathbf{Top}_*] \leftrightarrows \mathbf{Top}_* : F$ where $U(X,a) = X$ and $F(X) = (T_{\mathcal{O}}X, \mu)$. The upshot is that we can now work entirely in the context of monads and their algebras.

Of course, we are interested in the n-cubitos operad $\mathcal{C}_n$ and its associated monad $C_n := T_{\mathcal{C}_n}$. The Approximation Theorem (covered in [May72, §5–6]) states that the monad C_n is "the same" as the monad $\Omega^n\Sigma^n$ obtained from the adjunction $\Sigma^n \dashv \Omega^n$.

Theorem 3.23 (The Approximation Theorem)**.** *Let η_n denote the unit of $\Sigma^n \dashv \Omega^n$ and θ_n denote the C_n-algebra structure map induced by the action of $\mathcal{C}_n$ on $\Omega^n(\Sigma^n X)$. Consider the composite*

$$\alpha_n\colon C_nX \xrightarrow{C_n\eta_n} C_n\Omega^n\Sigma^n X \xrightarrow{\theta_n} \Omega^n\Sigma^n X.$$

For all n, the map α_n is an equivalence.

In other words, an E_n-space will be a $\Omega^n\Sigma^n$-algebra, so it now suffices to consider the latter sort of objects in proving the "only if" direction of the recognition theorem.

The next idea in the proof of the recognition theorem is to use the two-sided bar construction ([May72, §9]). This construction and its variants have many applications, both inside and outside of algebraic topology, but we will limit our exposition to understanding how the two-sided bar construction gives an n-fold delooping of a $\Omega^n\Sigma^n$-algebra.

Definition 3.24. [May72, Construction 9.6] Given a monad $(T;\mu,\eta)$ over **Top**, a (left) T-algebra $(X,a)\in Alg_T(\mathbf{Top})$, and a "right T-algebra"[7] (S,λ), the *two-sided simplicial bar construction* $B_*(S,T,X)$ is the simplicial space with

$$B_k(S,T,X)=ST^kX.$$

The face maps d_i apply either the multiplication $\mu\colon T^2\to T$ or the action maps $a\colon TX\to X$, $\lambda ST\to S$, depending on the index i. Specifically,

$$\begin{aligned} d_0&\colon ST^kX\to\lambda(ST)T^{k-1}X, \\ d_i&\colon ST^kX\to ST^{i-1}\mu(T^2)T^{k-i-1}X \qquad\qquad 1\le i\le k-1, \\ d_n&\colon ST^kX\to ST^{k-1}\mu(TX). \end{aligned}$$

The degeneracy map s_j inserts T at the $(j+1)$th spot

$$s_j\colon ST^kX\to ST^{k+1}X$$

for $0\le j\le k$. So for example s_0 inserts a T right after the S so it becomes the zeroth T in the composition T^{k+1}. The *two-sided bar construction* $B(S,T,X)$ is the geometric realization of $B_*(S,T,X)$.

In particular, given an E_n-space X, the Approximation Theorem tells us that X is a (left) $\Omega^n\Sigma^n$-algebra. Since $\Omega^n\Sigma^n$ is also a right $\Omega^n\Sigma^n$-algebra (cf. [May72, Example 9.5.1]), we can consider $B_*(\Omega^n\Sigma^n,\Omega^n\Sigma^n,X)$. It turns out that, upon taking geometric realization, $B(\Omega^n\Sigma^n,\Omega^n\Sigma^n,X)$ is homotopy equivalent to X. One can then show that $B_*(\Omega^n\Sigma^n,\Omega^n\Sigma^n,X)$ is homotopy equivalent to $\Omega^nB_*(\Sigma^n,\Omega^n\Sigma^n,X)$ as simplicial objects. The final step is showing that Ω^n commutes with geometric realization for certain simplicial spaces so that there is a chain of equivalences

$$X\xrightarrow{\simeq}B(\Omega^n\Sigma^n,\Omega^n\Sigma^n,X)\xrightarrow{\simeq}\Omega^nB(\Sigma^n,\Omega^n\Sigma^n,X).$$

Along with proving the "only if" direction of the recognition theorem (Theorem 3.18), this argument also gives us an explicit delooping of X. In the case $n=\infty$, this process is an example of an infinite loop machine which we will return to in Section 5.

4 Segal's Γ-spaces

The theory developed by Segal [Seg74] provides another way to characterize infinite loop spaces. Rather than using algebras over an operad, Segal uses a special kind of functor called a Γ-space to capture the structure of infinite loop spaces. These

[7] What we mean here is a functor $S\colon\mathbf{Top}\to\mathbf{Top}$ with a natural transformation $\lambda\colon ST\Rightarrow S$ which satisfies various commutativity diagrams, basically identical to the ones for T-algebras. We can think of λ as a right-action of T on the functor S, which justifies the nickname "right T-algebra;" May calls such objects *T-functors* in **Top**.

Γ-spaces can be thought of as generalizing Abelian topological monoids, and more generally the idea of a Γ-structure can be transported to other categories so that resulting Γ-objects generalize the commutative monoidal objects of that category. This theory was originally intended to apply only to infinite loop spaces, but other variants have since developed to treat n-fold loop spaces (cf. [Ada78, §2.5]). More details about Γ-spaces and their counterparts can be found in Segal's original work [Seg74] or other sources such as [Fre12, Mat12a, Mat12b]. Segal's approach is similar to May's in spirit in the sense that it produces a particular Ω-spectrum (involving the classifying space construction) when given certain data, and this spectrum comes with a specific delooping. It is a theorem of May and Tomason [MT78] that May's and Segal's delooping processes are equivalent, meaning they will produce equivalent spectra when fed the same data. In fact, any appropriately defined method of turning spaces into Ω-spectra (called an "infinite loop space machine") will be equivalent to Segal's method.

4.1 Γ-spaces

Segal defines Γ-spaces as a contravariant functor on a particular category Γ, although this category can be a bit difficult to understand. The category Γ can be thought of as having finite pointed sets as objects with partially defined maps between them. That is, a morphism $S \to T$ in Γ is a subset $S' \subseteq S$ containing the basepoint of S and a map $S' \to T$. The composition of two morphisms is the usual composition defined wherever it makes sense.

It is much easier to make sense of the opposite category Γ^{op}. In fact, Γ^{op} is just $\mathbf{Fin}_*$, the category of finite sets and based maps between them; this observation is originally due to D. Anderson [And71]. We will continue to use the notation Γ and Γ^{op} to remain true to Segal's original definitions.

Remark 4.1. Every object of Γ^{op} is isomorphic to

$$n^+ = \{0, 1, \ldots, n\}$$

for some $n \in \mathbb{Z}_{\geq 0}$, where we think of 0 as the basepoint. The object 0^+ makes Γ^{op} into a pointed category. We will sometimes make our definitions solely in terms of these objects n^+ or even talk about them as "the" objects of Γ^{op}. We will make extensive use of this skeleton of Γ^{op} (equivalently, $\mathbf{Fin}_*$), which we denote $\mathscr{F}$, in Section 5 when we discuss categories of operators.

Definition 4.2. A Γ-*space* is a functor $A \colon \Gamma^{\mathrm{op}} \to \mathbf{Top}_*$ such that $A(0^+)$ is contractible. A Γ-space A is *special* if for all pointed finite sets S, T, the map

$$A(S \vee T) \to A(S) \times A(T)$$

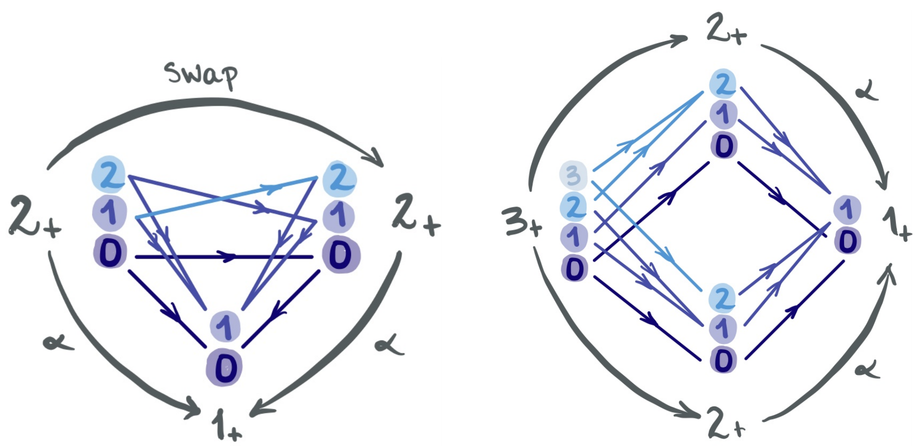
is a weak homotopy equivalence of pointed spaces.

The map $A(S \vee T) \to A(S) \times A(T)$ is the one induced by the collapse maps $S \vee T \to S$ and $S \vee T \to T$. By Remark 4.1, a special Γ-space is determined by its value on 1^+, in the sense that $A(n^+)$ is canonically equivalent to $A(1^+)^n$. In this sense, $A(1^+)$ can be thought of as the "underlying space" which determines a special Γ-space A, and $A(n^+)$ can be thought of as n-tuples of elements in $A(1^+)$.

Example 4.3. There is an associated special Γ-space for every Abelian toplogical monoid M. Forgetting the monoid structure, we are left with a pointed topological space $(M, 0)$, where 0 is the unit. Define $A_M\colon \Gamma^{\mathrm{op}} \to \mathbf{Top}_*$ on objects by $S \mapsto \mathbf{Top}_*(S, M)$. Note that A_M is even a bit nicer than an arbitrary special Γ-space, in that the map $A_M(S \vee T) \to A_M(S) \times A_M(T)$ is actually an isomorphism. In particular, there is a canonical isomorphism $A_M(n^+) \cong M^n$, and so we can recover the monoid M as $A_M(1^+)$. Given a map $f\colon S \to T$ in Γ^{op}, we define $f_* := A_M(f)\colon \mathbf{Top}_*(S, M) \to \mathbf{Top}_*(T, M)$ by "integrating" a given $\phi \in \mathbf{Top}_*(S, M)$,

$$f_*(\phi)\colon t \mapsto \begin{cases} 0 & t = *; \\ \sum_{s \in \phi^{-1}(t)} \phi(s) & t \neq *. \end{cases}$$

Note that A_M sends the map $\alpha\colon 2^+ \to 1^+$ with $\alpha(2) = \alpha(1) = 1$ to the addition operation $M^2 \to M$. Functoriality of A_M requires that this addition is strictly commutative and associative, as can be seen by applying A_M to the diagrams:



Thus every Abelian topological monoid yields a special Γ-space. For certain special Γ-spaces, the converse is true as well. In particular, if $A(S \vee T) \to A(S) \times A(T)$ is an isomorphism for every $S, T \in \Gamma^{\mathrm{op}}$, then $A(1^+)$ is an Abelian topological monoid whose associated special Γ-space is again A. Such a Γ-space is sometimes called a *strong* Γ-space.

The reader who is familiar with simplicial objects has undoubtedly noticed the similarities between simplicial spaces and Γ-spaces. This resemblance can be codified by defining a functor between Γ^{op} and Δ^{op}. Recall that Δ is the category of nonempty finite ordered sets and order-preserving maps. Any object of Δ is isomorphic to

$$[n] = \{0 < 1 < \cdots < n\}$$

for some $n \geq 0$. We can define a functor $\kappa\colon \Delta^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$ which allows us to turn Γ-spaces into simplicial spaces.

Definition 4.4. The functor $\kappa\colon \Delta^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$ sends a finite ordered set S to the pointed (unordered) set $(S, \min S)$. In particular, $\kappa[n] = n^+$. Defining κ on morphisms is a bit trickier, and we will only define it on the representative objects $[n]$, although the general case is basically the same (see [Fre12, 19.23]). Suppose we have an order-preserving map $f\colon [n] \to [m]$. Given $k \in [m]$, define $\hat{k}$ to be the minimum of $\{j \in f[n] : j \geq k\}$, unless $k = 0$ or $\{j \in f[n] : j \geq k\} = \varnothing$, in which case set $\hat{k} = 0$. Now define $\kappa f\colon m^+ \to n^+$ by

$$\kappa f(k) = \begin{cases} 0 & \hat{k} = 0; \\ \min f^{-1}(\hat{k}) & \text{otherwise.} \end{cases}$$

It is instructive to understand how κ behaves on the face and degeneracy maps which generate the morphisms of Δ^{op}. Recall that for each $n \geq 0$ there are $n+1$ injective *coface* maps $d^i\colon [n-1] \to [n]$, where the superscript indicates which object is not contained in the image, and $n+1$ surjective *codegeneracy* maps $s^j\colon [n+1] \to [n]$, where now the superscript indicates which object in the image is mapped onto twice. Explicitly, these maps are given by

$$d^i(k) = \begin{cases} k & k < i; \\ k+1 & k \geq i, \end{cases} \qquad \text{and} \qquad s^j(k) = \begin{cases} k & k \leq j; \\ k-1 & k > j, \end{cases}$$

for $0 \leq i, j \leq n$. The opposite category Δ^{op} has corresponding *face* maps d_i and *degeneracy* maps s_j. The reader can verify (or just believe the author) that

$$\kappa d^i(k) = \begin{cases} k & k \leq i; \\ k-1 & k > i, \end{cases} \qquad \text{and} \qquad \kappa s^j(k) = \begin{cases} k & k \leq j; \\ k+1 & k > j, \end{cases}$$

for $0 \leq j \leq n$ and $0 \leq i < n$. For $i = n$, the map looks slightly different: $\kappa d^n(k) = k$ for $k \neq n$ and $\kappa d^n(n) = 0$. The similarities in the formulas between the two displays are striking. We see that κd^i (for $i \neq n$) is the degeneracy map s_i, although strictly speaking the former is a map of unordered pointed sets $n_+ \to (n-1)_+$ while the latter is a map of ordered sets $[n] \to [n-1]$. Similarly, $\kappa s^j\colon n_+ \to (n+1)_+$ is the face map $d_{j+1}\colon [n+1] \to [n]$.

If we compose κ with the special Γ-space A_M associated to an Abelian topological monoid M, the induced face and degeneracy maps are given by

$$\begin{aligned} (\kappa d^i)_*\colon M^n &\to M^{n-1} \\ (m_1, \ldots, m_n) &\mapsto (m_1, \ldots, m_i + m_{i+1}, \ldots, m_n) \end{aligned}$$

(for $i \neq n$; the n^{th} map $(\kappa d^n)_*$ just forgets the last entry m_n) and

$$\begin{aligned}(\kappa s^j)_* \colon M^n &\to M^{n+1}\\ (m_1, \ldots, m_n) &\mapsto (m_1, \ldots, m_{i-1}, 0, m_i, \ldots, m_n),\end{aligned}$$

which are the familiar face and degeneracy maps on the classifying space of M (viewed as a category with one object whose set of morphisms is M). This observation begs us to define an analogous classifying space construction for Γ-spaces.

Definition 4.5. The *realization* of a Γ-space A is the geometric realization of its associated simplicial space $A \circ \kappa$, denoted simply $|A|$. The *classifying space* of A is another Γ-space $BA \colon \Gamma^{\mathrm{op}} \to \mathbf{Top}_*$. Given $S \in \Gamma^{\mathrm{op}}$, we may consider the Γ-space $T \mapsto A(S \times T)$. The classifying space of A sends S to the realization of this Γ-space, $BA(S) = |A(S \times -)|$.

It can be verified that $A(S \times -)$ is actually a Γ-space, which is special if A is special (see [Fre12, Lemma 19.32]). Note that there is a canonical homeomorphism $BA(1^+) \cong |A|$ and $BA(0^+)$ is the basepoint of $|A|$.

Remark 4.6. We can define Γ-objects more generally by replacing $\mathbf{Top}_*$ with some other pointed category with finite products. Just as Γ-spaces can be see as a generalization of Abelian topological monoids, so do these Γ-objects generalize the monoid objects in $\mathscr{C}$. For example, taking $\mathscr{C} = \mathbf{Set}_*$ we can talk about (special) Γ-sets, or taking $\mathscr{C} = \mathbf{Cat}$ (with distinguished object the one object category with a single identity morphism) we can talk about (special) Γ-categories. Both of these examples will be relevant in our upcoming discussion. In particular, both of these categories admit a classifying space functor, and so we can similarly define realizations and classifying spaces of their Γ-objects.

We can now iterate the classifying space construction for a Γ-space A, getting a sequence of Γ-spaces $A, BA, B^2A, \ldots$ along with a sequence of pointed topological spaces $A(1^+), BA(1^+), B^2A(1^+), \ldots$. This second sequence of topological spaces assembles into a spectrum, which is in fact an Ω-spectrum if the commutative monoid $\pi_0(A(1^+))$ is an Abelian group. This motivates the following definition of a *very* special Γ-space.

Definition 4.7. A *very special* Γ*-space* A is a special Γ-space such that $\pi_0(A(1^+))$ is a group.

Remark 4.8. In general, if A is a special Γ-space, the composition

$$\Gamma^{\mathrm{op}} \xrightarrow{A} \mathbf{Top}_* \xrightarrow{\pi_0} \mathbf{Set}_*$$

is a special Γ-set, and this special Γ-set structure ensures $\pi_0(A(1^+))$ is a commutative monoid. However, as discussed in [Seg74, §4], we can always replace a special Γ-space A with a very special Γ-space A' such that $\pi_0(A'(1^+))$ is the *group completion* (Definition 3.3) of the Abelian monoid $\pi_0(A(1^+))$ and there is a weak equivalence of spectra $BA \to BA'$.

The upshot is that if we have a very special Γ-space A, then we get an Ω-spectrum $A(1^+), BA(1^+), B^2A(1^+)\ldots$ and we know from Subsection 2.1 that this is equivalent to an infinite loop space.

Theorem 4.9. *If A is a very special Γ-space, then $A(1^+)$ is an infinite loop space, with a delooping $A(1^+) \xrightarrow{\simeq} \Omega^k B^k A(1^+)$.*

Of course, every infinite loop space $\Omega^\infty X$ can be turned into a special Γ-space which is determined by $1^+ \mapsto \Omega^\infty X$. Thus being a very special Γ-space is equivalent to being an infinite loop space. Note that the "very" assumption in the theorem above is not necessary, since we can replace a special Γ-space with an equivalent very special one, as described in Remark 4.8.

4.2 Examples

Theorem 4.9 can be applied to give a fascinating slew of examples. For instance, if we start out with a discrete Abelian group M, the associated Γ-space from Example 4.3 turns out to be a very special Γ-space, and the associated Ω-spectrum is the Eilenberg-Mac Lane spectrum for M which we saw in Example 2.6. Arguably one of the most powerful applications of the theorem is its ability to associate an infinite loop space to a symmetric monoidal category. This is more or less because every symmetric monoidal category gives rise to a special Γ-category.[8]

Theorem 4.10. *[Segal] Let $\mathscr{C}$ be a symmetric monoidal category, and let $\iota\mathscr{C}$ denote the subcategory of isomorphisms of $\mathscr{C}$. Then there is an Ω-spectrum whose underlying infinite loop space is equivalent to the group completion of the classifying space $B(\iota\mathscr{C})$.*

The idea here is to construct a special Γ-category for $\mathscr{C}$, that is, a functor $\hat{\mathscr{C}}\colon \Gamma^{\mathrm{op}} \to \mathbf{Cat}$, such that $\hat{\mathscr{C}}(n^+)$ is equivalent to $(\iota\mathscr{C})^n$. The objects of the category $\hat{\mathscr{C}}(n^+)$ are certain functors from the power set of n^+ into $\mathscr{C}$, and the morphisms are natural isomorphisms between these functors. By post-composing with the classifying space functor $B\colon \mathbf{Cat} \to \mathbf{Top}_*$, we get a special Γ-space $B\hat{\mathscr{C}}$ such that $B\hat{\mathscr{C}}(1^+)$ is equivalent to the classifying space of $\iota\mathscr{C}$. We do not lose any homotopical information by considering $\iota\mathscr{C}$, since the equivalence of categories $\mathscr{C} \simeq \iota\mathscr{C}$ induces an equivalence of classifying spaces $B\mathscr{C} \simeq B(\iota\mathscr{C})$.

The special Γ-space $B\hat{\mathscr{C}}$ gives us a spectrum $B\hat{\mathscr{C}}(1^+), B^2\hat{\mathscr{C}}(1^+), \ldots$ and after possibly group-completing (see Remark 4.8), we get an Ω-spectrum which we denote $\mathbf{B}\mathscr{C}$. The spaces in this Ω-spectrum give us a delooping of the group completion of the classifying space $B(\iota\mathscr{C})$.

[8]If we impose an extra condition on the Γ-objects, we actually get an equivalence of categories between the Γ-objects of $\mathscr{C}$ and the commutative monoidal objects of $\mathscr{C}$. Namely, we need to ask for *strong* Γ-objects, i.e. those for which the map $A(S \sqcup T) \to A(S) \times A(T)$ is actually an isomorphism instead of just a weak equivalence. So for example, there is an equivalence between strong Γ-categories and strict symmetric monoidal categories.

Theorem 4.10 can be expanded to take *topological categories* as input, and in fact Segal states the theorem this way. Topological category here means a category internal to **Top** (i.e. the objects and morphisms are both topological spaces, and the structure maps such as composition are continuous). Note that any small category can be viewed as a topological category with the discrete topology on the collections of objects and morphisms. In the following examples, we assume our categories are discrete unless we make explicit mention of the topological structure.

Example 4.11 (The Barratt-Priddy-Quillen Theorem)**.** Take $\mathscr{C}$ to be $\mathbf{Fin}_*$, with a symmetric monoidal structure given by disjoint union and $\varnothing$. Since we are taking classifying spaces in any case, we may as well consider $\mathscr{F}$, the skeleton of $\mathbf{Fin}_*$ with objects n^+ for $n \geq 0$ and morphisms basepoint-preserving maps. Then

$$B\hat{\mathscr{F}}(1^+) = \coprod_{n \geq 0} B\Sigma_n,$$

since the isomorphisms in $\mathscr{F}$ are precisely the (basepoint-preserving) permutations of n^+, for each $n \geq 0$. We claim that $\mathbf{B}\mathscr{F}$ (the Ω-spectrum associated to $B\hat{\mathscr{F}}$) is the sphere spectrum $\mathbb{S}$.

By the Yoneda Lemma, it suffices to show

$$\mathrm{Hom}_{\mathbf{Sp}}(\mathbf{B}\mathscr{F}, E) \simeq \mathrm{Hom}_{\mathbf{Sp}}(\mathbb{S}, E)$$

for any spectrum E. To simplify the problem a bit, first observe that $\mathrm{Hom}_{\mathbf{Sp}}(\mathbb{S}, E)$ is just $\pi_0(E)$. The second observation is that $\mathbf{B}$ admits a right adjoint $\mathbf{A}$. This functor $\mathbf{A}$ turns spectra into Γ-spaces by mapping a spectrum E to the Γ-space $\mathbf{A}E$ with

$$\mathbf{A}E(n^+) = \mathrm{Hom}_{\mathbf{Sp}}(\mathbb{S}^n, E).$$

So the "underlying space" of this Γ-space is $\mathbf{A}E(1^+) = \pi_0(E)$. Segal shows that $\mathbf{B} \dashv \mathbf{A}$ is an adjunction between **HoSp** (the category of spectra and homotopy classes of maps) and the homotopy category of Γ-spaces with level-wise acyclic Hurewicz fibrations inverted [Seg74, §3]. Let $\mathscr{A}$ denote the latter category. By adjunction and the construction of $\mathbf{A}$, it thus suffices to show

$$\mathrm{Hom}_{\mathscr{A}}(B\hat{\mathscr{F}}, \mathbf{A}E) \cong \pi_0(\mathbf{A}E(1^+)).$$

Segal proves this directly by exhibiting a natural isomorphism between the two spaces. We first define a map $\mathrm{Hom}_{\mathscr{A}}(B\hat{\mathscr{F}}, \mathbf{A}E) \to \pi_0(\mathbf{A}E(1^+))$. Given a morphism $B\hat{\mathscr{F}} \to \mathbf{A}E$ in $\mathscr{A}$, we map that morphism to the element of $\pi_0(\mathbf{A}E(1^+))$ which is hit by the single point $B\Sigma_1 \in B\hat{\mathscr{F}}(1^+)$.

Constructing the inverse to this map is a bit more complicated. Let a be a point of $\mathbf{A}E(1^+)$, and let F_n denote the homotopy fiber of the map $\mathbf{A}E(n^+) \to \mathbf{A}E(1^+)^n$ at $(a, \ldots, a)$. Then the F_n are contractible and assemble into a functor F from the subcategory of $\mathscr{F}$ whose morphisms are injections to **Top**. The idea is to introduce an auxiliary category $\mathscr{S}_F$ whose objects are pairs $(n^+, x \in F_n)$ and morphisms $(m^+, y) \to (n^+, x)$ are injections $\phi\colon m^+ \to n^+$ in $\mathscr{F}$ such that $F\phi(y) = x$. We can then form the associated Γ-space $B\hat{\mathscr{S}}_F$, . The forgetful map $B\hat{\mathscr{S}}_F \to B\hat{\mathscr{F}}$ induced by $(n^+, x \in F_n) \mapsto n^+$ is actually an isomorphism in $\mathscr{A}$ (by the contractibility of the F_n).

The important takeaway from all of this is that, when given $a \in \mathbf{A}E(1^+)$, it suffices to exhibit a map $B\hat{\mathscr{S}}_F \to \mathbf{A}E$ in $\mathscr{A}$. The specific construction of the Γ-space $B\hat{\mathscr{S}}_F$ means that this map can be given levelwise: Recall that $B\hat{\mathscr{S}}_F(k^+)$ is just the classifying space of the category $\hat{\mathscr{S}}_F(k^+)$, so it will do us good to understand the latter. An object of the category $\hat{\mathscr{S}}_F(n^+)$ can be thought of as a pair (ϕ, x) where $\phi\colon m^+ \to n^+$ is a surjective morphism in Γ^{op} such that $\phi^{-1}(0) = \{0\}$ and $x \in F_m$. There is a morphism $(\phi\colon m^+ \to n^+, x) \to (\psi\colon k^+ \to n^+, y)$ only when $m = k$ and $x = y$, in which case a morphism is given by a bijection $m^+ \to m^+$, i.e. an element of Σ_m. We can map each object $(\phi\colon m^+ \to n^+, x \in F_m)$ into $\mathbf{A}E(n^+)$ via the composition

$$F_m \to \mathbf{A}E(m^+) \xrightarrow{\phi^*} \mathbf{A}E(n^+).$$

Moreover, if there is a morphism $(\phi, x) \to (\psi, x)$, then the two objects have the same image in $\mathbf{A}E(n^+)$, so we get a well-defined map $B\hat{\mathscr{S}}_F(n^+) \to \mathbf{A}E(n^+)$. These levelwise maps assemble into a morphism $B\hat{\mathscr{S}}_F \to \mathbf{A}E$ in $\mathscr{A}$. After showing this map is the same for any point in the same connected component of $\mathbf{A}E(1^+)$ as a, we are done.

Example 4.12 (Topological K-theory)**.** Now take $\mathscr{C}$ to be $\mathbf{Vect}(\mathbb{C})$, the category of finite-dimensional complex vector spaces. The direct sum $\oplus$ and the trivial vector space (0) give $\mathbf{Vect}(\mathbb{C})$ a symmetric monoidal structure. As before, we can consider the skeleton $\mathscr{V}$ of $\mathbf{Vect}(\mathbb{C})$ whose objects are $\mathbb{C}^n$ for $n \geq 0$. Then

$$B\hat{\mathscr{V}}(1^+) = \coprod_{n \geq 0} BGL_n(\mathbb{C}),$$

where we must take the topology on $GL_n(\mathbb{C})$ into account. Moreover, the Gram-Schmidt process gives a deformation retraction of $GL_n(\mathbb{C})$ onto $U(n)$, the group of $n \times n$ unitary matrices. It turns out that the group completion of $B\hat{\mathscr{V}}(1^+)$ is equivalent to $\mathbb{Z} \times BU$, and the delooping machinery of Segal (or May) then shows that the Ω-spectrum $\mathbf{B}\mathscr{V}$ is equivalent to the connective topological K-theory spectrum.

Example 4.13 (Algebraic K-theory)**.** Let R be a ring and take $\mathscr{C}$ to be $\mathbf{Mod}^{\mathrm{proj}}_{\mathrm{f.g.}}(R)$, the category of finitely generated projective modules over R. The symmetric monoidal structure on $\mathbf{Mod}^{\mathrm{proj}}_{\mathrm{f.g.}}(R)$ is given by direct sum and the trivial R-module. The work of Quillen [Qui73, Gra76] shows that the algebraic K-theory space of R is the group completion of $B(\iota\mathbf{Mod}^{\mathrm{proj}}_{\mathrm{f.g.}}(R))$, and delooping allows for the construction of the algebraic K-theory spectrum of R.

Hopefully these few examples demonstrate the incredible power of May's and Segal's machinery. For more details and other examples, we encourage the interested reader to look at [Seg74, §2] and [Ada78, §2.6].

5 Infinite Loop Space Machines and the Uniqueness Theorem

We have seen two different methods for characterizing and delooping infinite loop spaces, one using operads and one using Γ-spaces. In particular, both approaches come with a way to produce Ω-spectra when fed certain data (an E_∞-space or a special Γ-space, along with a way to group complete to a grouplike E_∞-space or a very special Γ-space), and we might wonder how these two methods compare. Certainly, any space which admits a delooping by May's method will admit a delooping by Segal's method, and vice versa, but we can ask whether these two deloopings will be "the same" in that they will give equivalent Ω-spectra. The uniqueness theorem of May and Thomason [MT78] tells us this is the case, assuring us that any two loop space machines will produce equivalent spectra when fed the same data. This section will be spent unpacking exactly what that means, although we will leave most of the details to the original paper.

5.1 Categories of Operators

Roughly, an infinite loop space machine is any functor which constructs spectra out of simpler space-level data. May and Thomason capture this idea of "space-level data" via the definition of a category of operators, which is a topological category closely connected to $\mathbf{Fin}_*$ (the category of finite pointed sets and pointed maps between them).

Definition 5.1. Let $\mathscr{F}$ be the category whose objects are the finite based sets n^+ for $n \geq 0$ and whose morphisms are the basepoint-preserving maps between them. In other words, $\mathscr{F}$ is a skeleton of $\mathbf{Fin}_*$. Let Π be the wide subcategory of $\mathscr{F}$ whose morphisms are those maps $\phi\colon m^+ \to n^+$ such that $\phi^{-1}(j)$ has at most one element for any non-basepoint $j \in n^+$.

The idea is that a category of operators $\mathscr{G}$ will sit between Π and $\mathscr{F}$ and hold information about different operations. The objects of $\mathscr{G}$ look like the natural numbers, and the morphisms $\mathscr{G}(m^+, n^+)$ can be thought of as a space of operations with m inputs and n outputs. The subcategory Π consists of the "elementary operations" which are injective away from the basepoint 0; this can be interpreted to mean that the operations in Π do not combine distinct variables.

Definition 5.2. A *category of operators* (over **Top**) is a category $\mathscr{G}$ enriched over **Top** with object space $\mathbb{N}$ (as a discrete space), along with (continuous) functors

$$\Pi \xrightarrow{\iota} \mathscr{G} \xrightarrow{\varepsilon} \mathscr{F}$$

such that both ι and ε are the identity on objects and $\varepsilon \circ \iota$ is the inclusion $\Pi \subseteq \mathscr{F}$. A *morphism* between categories of operators is a (continuous) functor $\nu\colon \mathscr{G} \to \mathscr{G}'$ such that $\nu(n^+) = n^+$ and the following diagram commutes:

$$\begin{tikzcd} & \mathscr{G} \arrow[dd, "\nu"] \arrow[dr, "\varepsilon"] & \\ \Pi \arrow[ur, "\iota"] \arrow[dr, "\iota'"'] & & \mathscr{F} \\ & \mathscr{G}' \arrow[ur, "\varepsilon'"'] & \end{tikzcd}$$

A morphism ν is an *equivalence* if each map $\mathscr{G}(m^+, n^+) \to \mathscr{G}'(m^+, n^+)$ is a weak equivalence of topological spaces.

In [MT78], the authors also impose some minor, technical conditions on $\mathscr{G}$ (see, for example, Addendum 1.7). A crucial motivation for this definition is that we can now talk about $\mathscr{G}$-spaces, and these objects will generalize both Γ-spaces and algebras over an operad.

Definition 5.3. A *$\mathscr{G}$-space* is a functor $X\colon \mathscr{G} \to \mathbf{Top}_*$, written $n \mapsto X_n$ on objects, which satisfies the following conditions:

- The functor X is enriched in $\mathbf{Top}_*$, which in particular means that $X_0 \simeq *$ and the adjoint maps $\mathscr{G}(m^+, n^+) \times X_m \to X_n$ are continuous.
- For $n > 1$, there is an equivalence $X\delta\colon X_n \xrightarrow{\simeq} X_1^n$, where $\delta = (\delta_1, \ldots, \delta_n)$ for
$$\delta_i\colon n^+ \to 1^+$$
with $\delta_i(j)$ equal to 1 if $j = i$ and 0 otherwise.
- If $\phi\colon m^+ \to n^+$ is an injection in Π, then $X\phi\colon X_m \to X_n$ is a Σ_ϕ-equivariant cofibration, where Σ_ϕ is the group of permutations $\sigma\colon n^+ \to n^+$ such that $\sigma\phi = \phi$.

A morphism of $\mathscr{G}$-spaces is just a natural transformation. A morphism $\eta\colon X \to X'$ is an *equivalence* of $\mathscr{G}$-spaces if each $\eta_n\colon X_n \to X'_n$ is a weak equivalence.

Remark 5.4. Like many things we have discussed in this write-up, these definitions can be generalized. We can enrich a category of operators over some category other than $\mathbf{Top}$, such as the category of G-spaces for some finite group G. This approach gives rise to the study of equivariant infinite loop space theory, as in [MMO17]. For more general discussion of categories of operators, see [May18].

The "operations" recorded as morphisms in $\mathscr{G}$ will dictate the structure on $\mathscr{G}$-spaces. To illustrate this idea, we consider the following scenario: Given a topological space $X \in \mathbf{Top}$, we can try to form a $\mathscr{G}$-space whose n^{th} level is X^n (where it is understood that $X^0 = *$). If $\mathscr{G} = \Pi$, then this works for any $X \in \mathbf{Top}$, as the conscientious reader can check. However, if $\mathscr{G}$ contains the morphisms illustrated in the diagrams of Example 4.3, then functoriality implies X actually needs to be an Abelian monoid. In this sense, $\mathscr{G}$-spaces are just Π-spaces with additional structure, and the "closer" $\mathscr{G}$ is to $\mathscr{F}$, the closer $\mathscr{G}$-spaces are to having a commutative monoid structure.

It is not too difficult to imagine how both Γ-spaces and operads might fit into this framework. Indeed, Γ-spaces are just $\mathscr{G}$-spaces if we take $\mathscr{G} = \mathscr{F}$. The less obvious connection is the one between categories of operators and operads. The following example constructs a category of operators $\mathcal{O}^{\otimes}$ associated to an operad $\mathcal{O}$, such that the $\mathcal{O}^{\otimes}$-spaces are precisely the $\mathcal{O}$-algebras.

Example 5.5. Let $\mathcal{O}$ be an operad. Its associated category of operators $\mathcal{O}^{\otimes}$ has

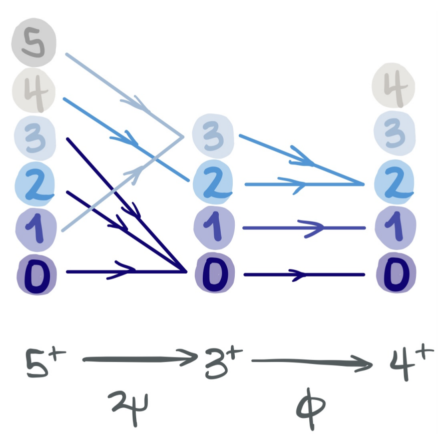
$$\mathcal{O}^{\otimes}(m^+, n^+) = \coprod_{\phi \in \mathscr{F}(m^+, n^+)} \prod_{1 \leq j \leq n} \mathcal{O}(|\phi^{-1}(j)|),$$

and $\mathcal{O}^{\otimes}(m^+, 0^+)$ is a point $*$ indexed by the unique map $m^+ \to 0^+$ in $\mathscr{F}$. An element of $\mathcal{O}^{\otimes}(m^+, n^+)$ is a morphism $\phi\colon m^+ \to n^+$ in $\mathscr{F}$ and an n-tuple of elements $(c_1, \ldots, c_n)$ where $c_j \in \mathcal{O}(|\phi^{-1}(j)|)$, and we write this element as $(\phi; c_1, \ldots, c_n)$. For $(\phi; c_1, \ldots, c_n) \in \mathcal{O}^{\otimes}(m^+, n^+)$ and $(\psi; d_1, \ldots, d_m) \in \mathcal{O}^{\otimes}(k^+, m^+)$, define their composition to be

$$(\phi; c_1, \ldots, c_n) \circ (\psi; d_1, \ldots, d_m) = (\phi \circ \psi; \gamma(c_1; \times_{\phi(i)=1} d_i)\sigma_1, \ldots, \gamma(c_n; \times_{\phi(i)=n} d_i)\sigma_n),$$

where the d_i are ordered according to the natural natural ordering on their indices (as a subset of m^+) and σ_j is a certain permutation of $|\phi \circ \psi^{-1}(j)|$ letters. In particular, σ_j reorders the elements of $\phi \circ \psi^{-1}(j)$ (viewed as a subset of k^+) so that the elements of $\psi^{-1}(i) \subseteq \phi^{-1}(j)$ precede all the elements of $\psi^{-1}(i') \subseteq \phi^{-1}(j)$ if $i < i'$ in k^+.

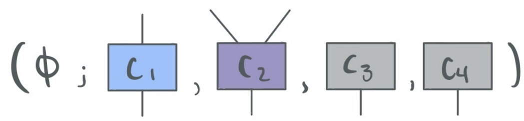
We can clarify the definition of composition through an example, using pictures. Suppose we have the following two morphisms $\phi\colon 3^+ \to 4^+$ and $\psi\colon 5^+ \to 3^+$ in $\mathscr{F}$:



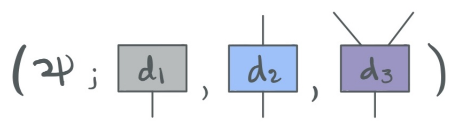
In $\mathcal{O}^{\otimes}$, these two morphisms give us elements of the form $(\phi; c_1, c_2, c_3, c_4)$ and $(\psi; d_1, d_2, d_3)$, where the c_i and d_j are chosen based on the sizes of $\phi^{-1}(i)$ and $\psi^{-1}(j)$, respectively. For example, since

$$\phi^{-1}(i) = \begin{cases} 0 & i = 3, 4; \\ 1 & i = 1; \\ 2 & i = 2, \end{cases}$$

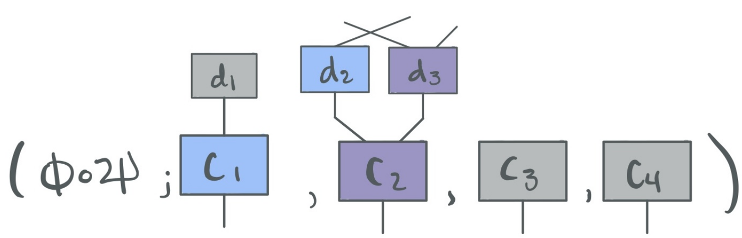
we want to pick $c_1 \in \mathcal{O}(1)$, $c_2 \in \mathcal{O}(2)$, and $c_3 = c_4 \in \mathcal{O}(0) = *$, so then $(\phi; c_1, c_2, c_3, c_4)$ will look like this:



Similarly, $(\psi; d_1, d_2, d_3)$ will look something like this:



We know how to compose $\phi \circ \psi \colon 5^+ \to 4^+$ in $\mathscr{F}$, as in the first picture. The more complicated part is how to stitch the c_i and d_j together. Before going through the process of this composition, we will show the result:



The most interesting thing is happening in the second slot, so we will focus our attention there. By definition, the second slot is $\gamma(c_2; \times_{\phi(i)=2} d_i)\sigma_2$. Since $\phi^{-1}(2) = \{2, 3\}$, this is $\gamma(c_2; d_2, d_3)\sigma_2$, where σ_2 is telling us how to permute the three inputs of $\gamma(c_2; d_2, d_3)$. Specifically, since $\psi^{-1}(2) = \{4\}$ and $\psi^{-1}(3) = \{1, 5\}$, $\sigma_2 \in \Sigma_3$ is the permutation (23). This final step gives us the element of $\mathcal{O}(3)$ that is pictured above.

The functors $\Pi \xrightarrow{\iota} \mathcal{O}^{\otimes} \xrightarrow{\varepsilon} \mathscr{F}$ are defined to be the identity on objects. On morphisms, ι is given by

$$\iota(\phi) = (\phi; d_1, \ldots, d_n) \qquad \text{for } d_j = \begin{cases} * \in \mathcal{O}(0) & |\phi^{-1}(j)| = 0, \\ 1 \in \mathcal{O}(1) & |\phi^{-1}(j)| = 1; \end{cases}$$

for $\phi \in \Pi(m^+, n^+)$. The functor ε is defined on morphisms by the obvious choice

$$\varepsilon(\psi; c_1, \ldots, c_n) = \psi$$

for $(\psi; c_1, \ldots, c_n) \in \mathcal{O}^{\otimes}(m^+, n^+)$. Note that ε is an equivalence if each $\mathcal{O}(j)$ is contractible.

The category $\mathcal{O}^{\otimes}$ is constructed specifically so that the $\mathcal{O}^{\otimes}$-spaces are precisely the $\mathcal{O}$-algebras ([MT78, Lemma 4.2]). This gives us a way to compare the operadic approach to infinite loop spaces with Segal's approach. In order to make this comparison precise, we need a way to transport $\mathscr{G}$-spaces across different categories of operators.

Given a map $\nu \colon \mathscr{G} \to \mathscr{G}'$, a $\mathscr{G}'$-space Y can be pulled back to a $\mathscr{G}$-space $\nu^* Y$. The converse operation, discussed in [Seg74, Appendix B] and [MT78, §1], turns a $\mathscr{G}$-space X into a $\mathscr{G}'$-space $\nu_* X$, with the following important result:

Theorem 5.6. *Let $\nu \colon \mathscr{G} \to \mathscr{G}'$ be an equivalence of categories of operators. Let X be a $\mathscr{G}$-space and Y be a $\mathscr{G}'$-space. Then there are natural equivalences of $\mathscr{G}$-spaces*

$$\nu^* \nu_* X \leftarrow 1_* X \to X$$

where 1_* is induced by the identity on $\mathscr{G}$. There is also a natural equivalence of $\mathscr{G}'$-spaces $\nu_* \nu^* Y \mapsto Y$.

This theorem tells us that $\mathscr{G}$-spaces are essentially the same thing as $\mathscr{F}$-spaces whenever $\mathscr{G}$ is equivalent to $\mathscr{F}$. This basically tells us May's and Segal's inputs to infinite loop spaces are equivalent, since an E_∞-operad has contractible spaces and therefore its associated category of operators is equivalent to $\mathscr{F}$. The next (and final!) subsection covers the equivalence in more detail, as well as that of the output spectra, using the machinery we have developed thus far.

5.2 The Uniqueness Theorem

In order to state the uniqueness theorem for infinite loop space machines, we need to know what an infinite loop space machine *is*. We want both operads and Γ-spaces to give rise to these machines, and we saw in the previous subsection how categories of operators provide an umbrella for both types of gadgets. This motivates the definition of infinite loop space machines in terms of categories of operators and spaces over them. Moreover, if we want to hope for any sort of uniqueness theorem, Theorem 5.6 tells us that infinite loop space machines should be related to categories of operators $\mathscr{G}$ which are equivalent to $\mathscr{F}$.

Definition 5.7. Let $\mathscr{G}$ be a category of operators equivalent to $\mathscr{F}$. An *infinite loop space machine* on $\mathscr{G}$-spaces is defined as a functor E from $\mathscr{G}$-spaces to connective spectra, written $EX = \{E_n X\}$ for a $\mathscr{G}$-space X, along with a natural group completion $X_1 \mapsto E_0 X$.

Such an object gives a canonical way to turn $\mathscr{G}$-spaces into infinite loop spaces (since infinite loop spaces arise as grouplike spaces which are the 0^{th} term of an Ω-spectrum, as we saw in Subsection 2.1), which justifies the name "infinite loop space machine." We can reinterpret Segal's method as an infinite loop space machine on $\mathscr{F}$-spaces, denoted by S. Specifically, given a $\mathscr{F}$-space (i.e. a Γ-space) X, SX is the spectrum introduced prior to Theorem 4.9:

$$S_n X = B^n X(1^+)$$

where $B^n X$ is the n^{th}-iterated classifying space of the Γ-space X, as defined in Definition 4.5.

Similarly, we can explicitly see how the operadic machinery yields an infinite loop space. Let $\mathscr{C}_\infty := \mathscr{G}_{\mathcal{C}_\infty}$ denote the category of operators associated to the ∞-cubitos operad (Definition 3.16). Then any $\mathscr{C}_\infty$-space X is in fact a $\mathcal{C}_\infty$-algebra, which we associate to the spectrum MX with

$$M_n X = \Omega^n B(\Omega^\infty \Sigma^\infty, \Omega^\infty \Sigma^\infty, X)$$

where $B(\Omega^\infty \Sigma^\infty, \Omega^\infty \Sigma^\infty, X)$ is the two-sided bar construction discussed at the end of Subsection 3.4. See [MT78, §6] for more details.

Since $\mathscr{C}_\infty$ is equivalent to $\mathscr{F}$, Theorem 5.6 tells us that Γ-spaces are equivalent to $\mathcal{C}_\infty$-algebras (i.e. E_∞-spaces). We want to know that the spectra resulting from the infinite loop space machinery are also equivalent. That is, if we feed the functors S and M the same[9] data X, we want to get equivalent spectra. More generally, the uniqueness theorem proves that an arbitrary infinite loop space machine E will produce equivalent spectra to S when the two machines are given equivalent data.

The idea is to use Theorem 5.6 to turn an infinite loop space machine E on $\mathscr{G}$-spaces into an infinite loop space machine on $\mathscr{F}$-spaces. Specifically, the $\mathscr{F}$-space version of E is just the "pushforward" $E\varepsilon^*$, where $E\varepsilon^*(X) = E(\varepsilon^* X)$ for any $\mathscr{F}$-space X. Here $\varepsilon\colon \mathscr{G} \to \mathscr{F}$ is the equivalence built into the data of $\mathscr{G}$. The precise statement of the uniqueness theorem is as follows:

Theorem 5.8 (May and Thomason)**.** *Given any loop space machine E on $\mathscr{G}$-spaces, there is a natural equivalence of spectra between $E\varepsilon^*(X)$ and SX for any $\mathscr{F}$-space X.*

Conversely, we could transport Segal's machine from $\mathscr{F}$-spaces to $\mathscr{G}$-spaces, and restate the theorem in the context of $\mathscr{G}$. The $\mathscr{G}$-space version of Segal's machine is of course the pullback $S\varepsilon_*$, with $S\varepsilon_*(Y) = S(\varepsilon_* Y)$ for any $\mathscr{G}$-space Y. A corollary of the uniqueness theorem and Theorem 5.6 is that there is a natural equivalence of spectra between EY and $S\varepsilon_*(Y)$ for any $\mathscr{G}$-space Y. In other words, the machines E and S are completely equivalent.

[9] What we mean by "the same" data X is that X is unchanged when transported across $\mathscr{F}$ and $\mathscr{C}_\infty$ as in Theorem 5.6.

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