M3100

Recitation 09/15

today: Relations + equivalence classes

Warm up: 1) Draw a picture of the reln on $\mathbb{R}$ given by $R = \{(a,b) \mid a,b \in \mathbb{R} \text{ and } a^2 < b\}$

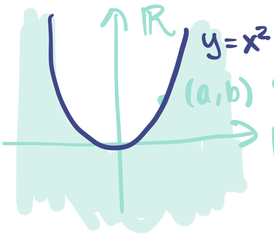
2) Let R be the reln on $\mathbb{R}$ w/ $a \sim b$ if $a - b \in \mathbb{Q}$.

reflexivity: $a \sim a$
symmetry: $a \sim b \Rightarrow b \sim a$
transitivity: $a \sim b$, $b \sim c \Rightarrow a \sim c$

(a) Convince yourself R is an equivalence reln

(b) Find 3 disjoint equivalence classes. $[a] = \{b \mid a \sim b\}$
$b \sim a$

1)



2) (a)
• $a - a = 0 \in \mathbb{Q}$ ✓

• $a - b \in \mathbb{Q}$

$b - a = -(a-b) \in \mathbb{Q}$ ✓

• $a - b,\ b - c \in \mathbb{Q}$

$a - c = \underbrace{(a-b)}_{\in \mathbb{Q}} + \underbrace{(b-c)}_{\in \mathbb{Q}} \in \mathbb{Q}$ ✓

(b) $[0] = \{b \in \mathbb{R} \mid 0 \sim b \iff b \sim 0 \iff b - 0 \in \mathbb{Q}\}$

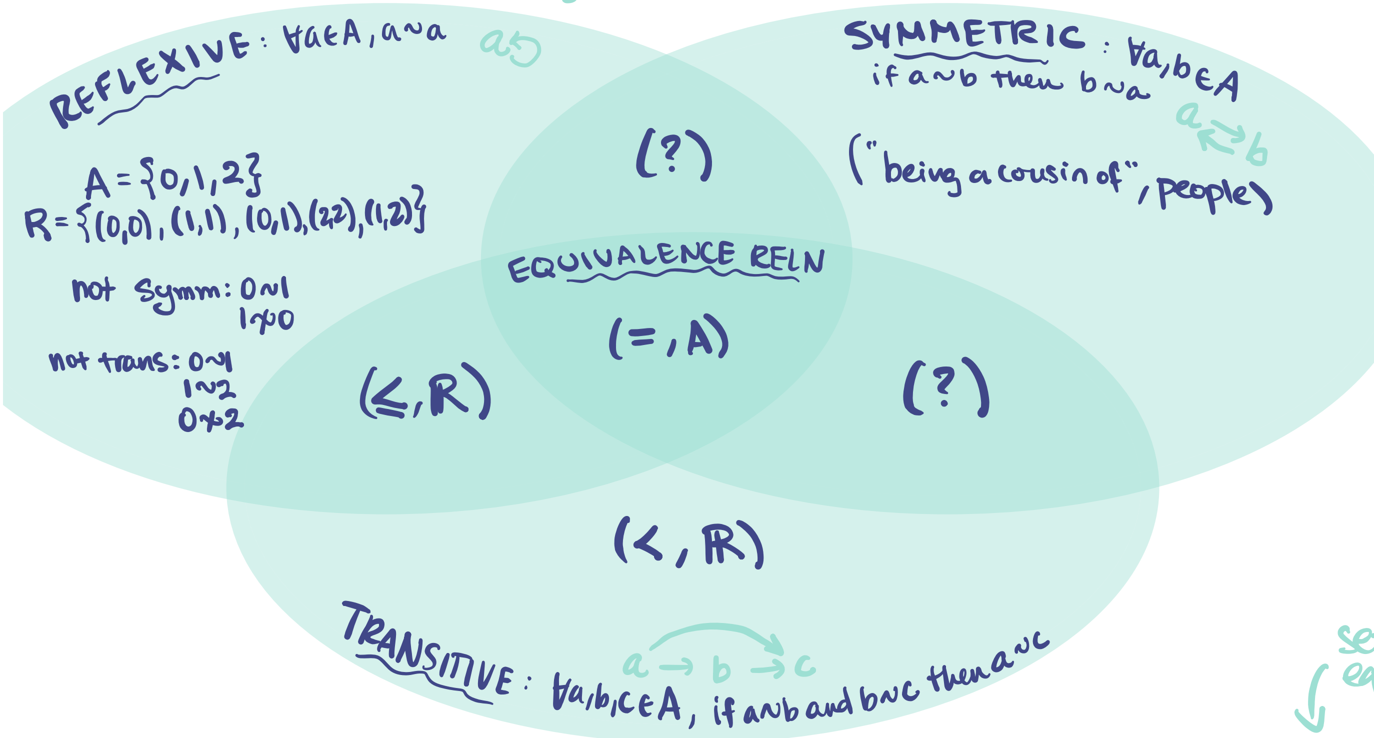
$= \mathbb{Q}$

$[\pi]$

$[\sqrt{2}]$ Pretty sure $\pi - \sqrt{2} \notin \mathbb{Q}$

Relations between sets A and B is a subset of $A \times B$ ← usually $B = A$

Write: $a \sim b$ or aRb (e.g. $a \leq b$)



Recall: The equivalence class of $a \in A$ is $[a] = \{b \in A \mid a \sim b\}$. The projection map $A \to A/\sim$ sends $a \mapsto [a]$.

set of equiv classes

Thm Let R be an equivalence reln. Then $\forall a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Pf/ If $[a] \cap [b] = \emptyset$, then done, so suppose $[a] \cap [b] \neq \emptyset$. (WTS $[a] = [b]$)

This means $\exists c \in A$ s.t. $c \in [a] \cap [b]$, i.e. $c \in [a]$ and $c \in [b]$.

($\subseteq$) Let $a' \in [a]$, so $a \sim a'$. Then:

	$a' \sim a$	by symmetry,
	$a' \sim c$	by transitivity, since $a \sim c$,
and	$c \sim b$	by symmetry,
so	$a' \sim b$	by transitivity,
	$b \sim a'$	by symmetry,

Hence $a' \in [b]$. Thus $[a] \subseteq [b]$.

($\supseteq$) Similar. (Exc or see notes). $\square$

Cor. If R is equiv. reln, get partition of $A = \bigcup_{i \in I} A_i$ for $A_i = [a_i]$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

Thm. $\{\text{Equivalence relns on } A\} \longleftrightarrow \{\text{partitions of } A\}$.

Pf/ ($\longleftarrow$) Suppose $A = \bigcup_{i \in I} A_i$ w/ $A_i \cap A_j = \emptyset$ for $i \neq j$. Define R s.t. aRb if $a, b \in A_i$. (WTS: R is (i) refl (ii) symm (iii) trans)

(i) For any $a \in A$, aRa since $a \in A_i$.
(ii) Suppose aRb, so $a, b \in A_i$, which means $b \sim a$.
(iii) Suppose aRb, bRc, so $a, b \in A_i$ and $b, c \in A_j$ for some $i, j \in I$. Since $b \in A_i \cap A_j$, $A_i \cap A_j \neq \emptyset \Rightarrow A_i = A_j$ and so $a, c \in A_i$, i.e. $a \sim c$. $\square$

Important Example:

$\hookrightarrow a \equiv b \pmod{n}$ "a is congruent to b mod n"

Let $n \in \mathbb{Z}_{>0}$ and consider the reln on $\mathbb{Z}$ given by $a \sim b$ if $a-b$ is a multiple of n.

Claim. This is an equivalence relation.

Pf/ (reflexive) For any $a \in \mathbb{Z}$, $a \sim a$ since $a - a = 0 = 0 \cdot n$.
(symmetry) Suppose $a \sim b$, so $a - b = kn$ for some $k \in \mathbb{Z}$. Then

$$b - a = -(a-b) = -(kn) = (-k)n$$

so $b \sim a$.
(transitive) If $a \sim b$ and $b \sim c$, then $a - b = kn$ and $b - c = k'n$ for some $k, k' \in \mathbb{Z}$. Then

$$\begin{aligned} a - c &= a + 0 - c \\ &= a + (-b+b) - c \\ &= (a-b) + (b-c) = kn + k'n = (k+k')n, \end{aligned}$$

so $a \sim c$. $\square$

Defn. The equivalence classes are $\mathbb{Z}/\sim \;=: \mathbb{Z}/n\mathbb{Z}$. "$\mathbb{Z}$ mod $n\mathbb{Z}$"

Claim $|\mathbb{Z}/n\mathbb{Z}| = n$

Pf/ We will show there is a bijection $f: \mathbb{Z}/n\mathbb{Z} \to \{0, 1, \ldots, n-1\}$, which implies

$$|\mathbb{Z}/n\mathbb{Z}| = |\{0, 1, \ldots, n-1\}| = n.$$

Define $f[a] := $ remainder of a/n, i.e. $a = qn + r$

Well-defn'd: Suppose $[a] = [b]$. (WTS: $f[a] = f[b]$)
This means $a \sim b$, so $\underbrace{a - b = kn}_{a = kn + b}$ for some $k \in \mathbb{Z}$. If $b = q'n + r'$, then

$$a = kn + b = kn + q'n + r' = (k+q')n + r' \quad \text{so } f[a] = f[b].$$

Injective: Suppose $f[a] = f[b]$. (WTS $[a] = [b]$) Write $a = qn + r$ and $b = q'n + r'$.
So b/c $f[a] = f[b]$, this implies $r = r'$! Then

$$a - b = (qn + r) - (q'n + r') = qn + r - q'n - r' = qn - q'n = (q - q')n.$$

Hence $a \sim b$, i.e. $[a] = [b]$, so f is injective.

Surjective: Let $r \in \{0,1,\dots,n-1\}$. Then $f[r] = r$ since $r < n$, i.e. $r = 0 \cdot n + r$.
So this shows f is surjective.
Therefore f is a bijection hence $|\mathbb{Z}/n\mathbb{Z}| = n$. $\square$

e.g. clocks $(n=12)$
months $(n=12)$
meals $(n=3)$

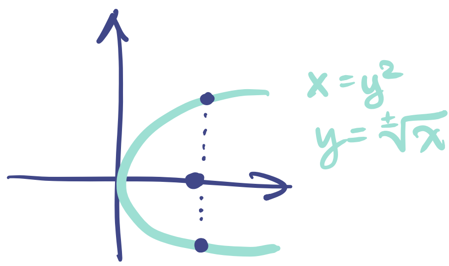
Example of not well-defined

$n=2$: $\mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \{1,2,3\}$

$\mathbb{Z} \longrightarrow \{1,2,3\}$

$0 \mapsto 1$
$1 \mapsto 1$
$2 \mapsto 2$
$3 \mapsto 3$
$4 \mapsto 1$
$5 \mapsto 2$
$\vdots \quad \vdots$

$f[0] = 1$
$f[2] = 2$
$[0] = [2]$



Practice

(ignore)

Part I. 1) Find examples of $f: \mathbb{R} \to \mathbb{R}$ s.t. f is
(a) bijective
(b) not injective nor surjective
(c) injective but not surjective
(d) surjective but not injective

2) (from Lecture) $f: X \to Y$ is bijective $\Leftrightarrow$ it has an inverse $g: Y \to X$

Minute Sheet

- how was the pace today?
- what's something you found interesting/confusing from today?
- what was most helpful today?

Part II. 1) Let $|A| = n$. How many distinct relations are there on A?

Bonus: How many reflexive, symm, transitive, etc?

2) (from HW2) Let $\sim$ be the relation on $\mathbb{R}\setminus\{0\}$ given by $x \sim y$ if $xy > 0$. Describe the corresponding partition.

Part III. 1) Let $A = \{$differentiable functions $\mathbb{R} \to \mathbb{R}\}$. Define R by $f \sim_R g$ if $f(0) = g(0)$.
(a) Prove R is an equivalence relation
(b) Let $S = A/\sim_R$ and define $F: S \to \mathbb{R}$ by $F[f] = f(0)$.
Prove F is well-defn'd + bijective.

2) Let $A = \{$[redacted] lines in the plane $\mathbb{R}^2\}$. Prove:
(a) "is parallel to" is an equiv. reln.
(b) "is perpendicular to" is not.
(c) Show slope: $A/\text{parallel} \longrightarrow \mathbb{R}$ is well-defined + bijective.

Bonus Can you construct $\mathbb{Z}$ from a reln on $\mathbb{N} \times \mathbb{N}$?
How about $\mathbb{Q}$ from a reln on $\mathbb{Z} \times \mathbb{Z}\setminus\{0\}$?

Define: $(a,b) R (c,d)$ if $a+d = b+c$.
"$a-b = c-d$"

1. R is equiv reln
2. $\mathbb{N}\times\mathbb{N}/R \xrightarrow{\text{bij}} \{\dots,-2,-1,0,1,2,\dots\}$

1. (reflexive) WTS: $(a,b)R(a,b)$

We have $a+b = b+a$ b/c addition is commutative, hence R is reflexive.

(symm) Suppose $(a,b)R(c,d)$, so $a+d = b+c$. (WTS: $(c,d)R(a,b) \Leftrightarrow c+b = d+a$)

We know: $b+c = a+d$ by symmetry of $=$,
$c+b = d+a$ by comm. of $+$.

(trans) If $(a,b)R(c,d)$ and $(c,d)R(e,f)$, then $a+d = b+c$ and $c+f = d+e$. Then

$$(a+d)+(c+f) = (b+c)+(d+e)$$

$$\cancel{(d+c)}+(a+f) = \cancel{(d+c)}+(b+e) \quad \text{by comm + assoc. of } +$$

$$a+f = b+e \quad \text{by cancellation.}$$

② **Note**: if $a \geq b$, then $(a,b) R (a-b, 0)$
if $a < b$, then $(a,b) R (0, b-a)$

So $[(a,b)]$ looks like $[(0,n)]$ or $[(n,0)]$. Define

$$\mathbb{N}\times\mathbb{N}/R \to \{\ldots,-1,0,1,\ldots\}$$
$$[(n,0)] \mapsto n$$
$$[(0,n)] \mapsto -n. \quad \square$$

If $|A|=n$, show $\#\{\text{relations on } A\} = ?$

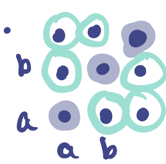
$$\updownarrow$$

$$\#\{\text{Subset of } A\times A\} = |\mathcal{P}(A\times A)| = 2^{|A\times A|} = 2^{n\times n} = 2^{n^2}.$$

Recall: If $|X| = m$, then $|\mathcal{P}(X)| = 2^m$

reflexive $(a,a) \in$ Subset

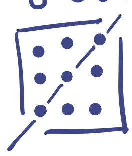
e.g.



2^{n^2-n}

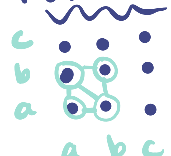
$\llcorner\ \underbrace{|A\times A|}_{n^2} - \underbrace{|A|}_{n}$

symmetric



$2^{\left(\frac{n^2+n}{2}\right)}$

transitive



$b \sim a \quad a \sim b$