Math 3700 - 09/08 Recitation Notes

Warm up

→ a subset $A \subseteq B$ is proper if $A \neq B$.

1) examples of proper subsets of $\mathbb{Z}, \mathbb{R}, \mathbb{C}$
2) example of a set w/ no numbers in it
3) example of a fn from a sub set from (1) to (2).

1) $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \supseteq \{2,4,6,8\}, \mathbb{N}, \{0\}, \{2k \mid k \in \mathbb{N}\}$
$= \{x \mid x \text{ is even}, x \geq 0\}$

$\mathbb{R} \supseteq (0,1), \{\text{irrationals}\} = \mathbb{R} \setminus \mathbb{Q}, \{\sqrt{2}\}$

$\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R}\} \supseteq \{i\}, \mathbb{R} = \{a+0i \mid a \in \mathbb{R}\}$

2) $\{\text{dog}, \text{cat}\}, \{A,B,C,D\}$

3) $\{2,4,6,8\} \to \{A,B,C,D\}$

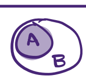
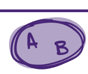
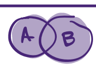
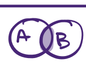
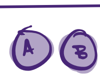
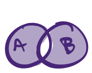
$2 \mapsto A$
$4 \mapsto B$
$6 \mapsto C$
$8 \mapsto D$

or $\{(2,A), (4,B), (6,C), (8,D)\}$

$\{2k \mid k \in \mathbb{N}\} \to \{\text{dog}, \text{cat}\}$

odds $\mapsto$ dog
evens $\mapsto$ cat

Main Concepts from Lecture: Sets - ZFC, operations on sets, powerset
functions - injectivity, surjectivity, bijectivity

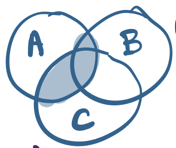
operation	notation	visual	meaning
Subset	$A \subseteq B$		$x \in A \Rightarrow x \in B$
equality	$A = B$		$A \subseteq B$ and $B \subseteq A$
union	$A \cup B$		$x \in A \cup B \Rightarrow x \in A$ or $x \in B$
intersection	$A \cap B$		$x \in A \cap B \Rightarrow x \in A$ and $x \in B$
Complement (need $A \subseteq B$)	$A \setminus B$ or B^c or $A-B$		$x \in A \setminus B \Rightarrow x \in A$ and $x \notin B$
disjoint union	$A \sqcup B$		$x \in A \sqcup B \Rightarrow x \in A$ "or" $x \in B$ ← even if $A \cap B \neq \emptyset$, distinguish betwn $x_A \in A$ and $x_B \in B$.
Symmetric difference	$A \triangle B$		$x \in A \triangle B \Rightarrow x \in A$ xor $x \in B$ ← exclusive or

e.g. $A = \{0\}, B = \{0\}$
$A \cup B = \{0\}$
but $A \sqcup B = \{0_A, 0_B\}$.

Example Let A, B, C be sets. Prove $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$. "distribution"

Proof/ ($\subseteq$) First we prove $A\cap(B\cup C) \subseteq (A\cap B)\cup(A\cap C)$. Let $x\in A\cap(B\cup C)$, so $x\in A$ and $x\in B\cup C$, so $x\in B$ or $x \in C$.

Visual



Case 1: Suppose $x\in B$. Then $x\in A\cap B$ so $x\in (A\cap B)\cup(A\cap C)$.

Case 2: Suppose $x\in C$. Then $x\in A\cap C$ so $x\in (A\cap B)\cup(A\cap C)$.

In either case, $x\in(A\cap B)\cup(A\cap C)$, hence $A\cap(B\cup C)\subseteq (A\cap B)\cup(A\cap C)$.

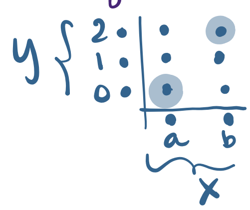
($\supseteq$) Now we show $(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$. Let $x\in(A\cap B)\cup(A\cap C)$, so $x\in A\cap B$ or $x\in A\cap C$. **Case 1**: Suppose $x\in A\cap B$. Then $x\in A$ and $x\in B$, so $x\in B\cup C$ as well and hence $x\in A\cap(B\cup C)$. **Case 2**: Suppose $x\in A\cap C$. So then $x\in A$ and $x\in C$, and thus $x\in B\cup C$. But then $x\in A\cap(B\cup C)$. In either case, $x\in A\cap(B\cup C)$ and so $(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$.

This proves $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$. $\square$

Extra practice: Show $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$.

Functions can be defined by

- subsets: e.g.



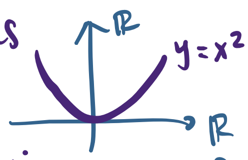
$= \{(a,0),(b,2)\}\subseteq X\times Y$

- formulas:

e.g. $f:\mathbb{R}\to\mathbb{R}$
$f(x) = x^2$

or multiple formulas: $f(x) = \begin{cases} x^2 & x\geq 0 \\ -(x^2) & x\leq 0\end{cases}$

- graphs

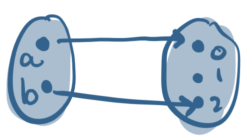


- descriptions: e.g. $f:\mathbb{N}\to\mathbb{N}$, $f(n) := n^{\text{th}}$ prime

- table:

x	$f(x)$
a	0
b	2

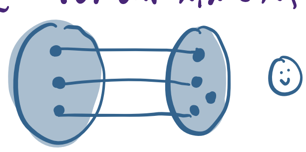
- "bubbles w/ arrows":

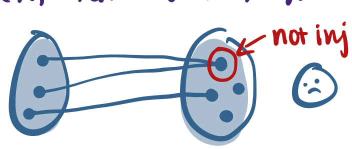


We can use "visual" approaches to understand injectivity ("inj"), surjectivity ("surj"):

Let $f: X\to Y$.

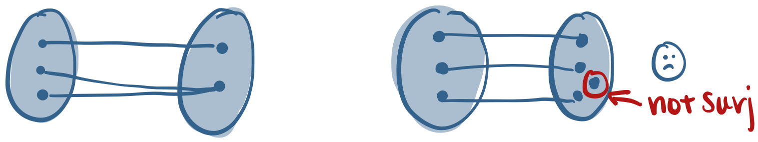
Injective: For all $x, x'\in X$, if $f(x) = f(x')$ then $x = x'$





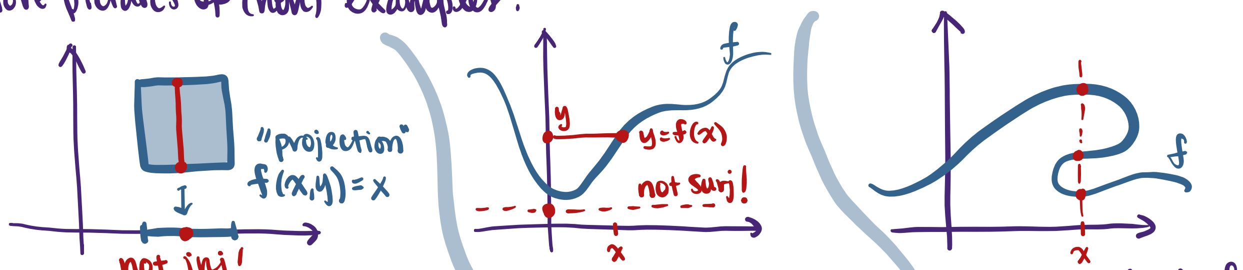
Note $|X|\leq|Y|$

Surjective: For all $y \in Y$, there is $x \in X$ s.t. $f(x) = y$.



Note $|X| \geq |Y|$

More pictures of (non)-examples:



injectivity $\leftrightsquigarrow$ not losing info ("inclusion")

Surjectivity $\leftrightsquigarrow$ horizontal line test

well-defined $\leftrightsquigarrow$ vertical line test

Example Consider $f : \underset{\mathbb{R}^2}{\mathbb{R} \times \mathbb{R}} \to \mathbb{R}$, $(x,y) \mapsto x$ and $g : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $x \mapsto (x,0)$.

1) What are $f \circ g$ and $g \circ f$?

2) Which of $f, g, f\circ g, g \circ f$ are injective? Surjective?

> **Note**: function composition of $f : X \to Y$ and $g : Y \to Z$ can be written:
> $g \circ f$, gf , $g(f)$
> or even $X \xrightarrow{f} Y \xrightarrow{g} Z$

Solns 1) We have $f \circ g : \mathbb{R} \to \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $x \mapsto (x,0) \mapsto x$ so $f \circ g = id_{\mathbb{R}}$.

Similarly, $g \circ f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $(x,y) \mapsto x \mapsto (x,0)$ so $g \circ f(x,y) = (x,0)$.

2) I claim f is surjective but not injective. **Proof**: (Surj) Let $x \in \mathbb{R}$. Then f takes $(x,0)$ to x, so f is surjective. (Not inj) Moreover f is not injective since $f(x,0) = x = f(x,1)$ but $(x,0) \neq (x,1)$.

On the other hand, g is injective but not surjective. **Proof**: (inj) If $g(x) = g(x')$, then $(x,0) = (x',0)$, but this implies $x = x'$. (Not surj) However g is not surjective because, e.g. $(x,1) \notin g(\mathbb{R})$ for any $x \in \mathbb{R}$.

Claim $f \circ g$ is a bijection **Proof** The identity on a set is always a bijection. (Prove this if you want!)

Claim $g \circ f$ is neither injective nor surjective. **Proof** It is not inj. b/c $g\circ f(x,0) = g \circ f(x,1)$ and not surj b/c $(x,1) \notin (g\circ f)(\mathbb{R}^2)$.

Practice Part I. 1) Prove $\{3m + 4n \mid m,n \in \mathbb{Z}\} = \mathbb{Z}$

2) (from HW1) Prove or find counterex: $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$

3) Show $C \setminus (B \setminus A) = (C \setminus B) \cup A$ for sets $A \subseteq B \subseteq C$.

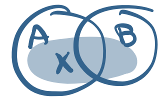
Solns / Hints

1) For $\subseteq$, write $x = 1 \cdot x = (4-3)\cdot x = 4x - 3x = 3(-x) + 4x$.

2) ($\supseteq$ is true) Let $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$. This means $x \in \mathcal{P}(A)$ or $x \in \mathcal{P}(B)$, i.e. $x \subseteq A$ or $x \subseteq B$.

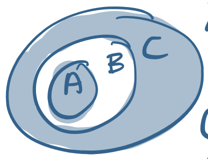
In either case, $X \subseteq A \cup B$ so $X \in \mathcal{P}(A \cup B)$. ($\supseteq$ is false) Let $X \in \mathcal{P}(A \cup B)$, so $X \subseteq A \cup B$. Is it true $X \subseteq A$ or $X \subseteq B$? Nope. (For instance, take $X = A \cup B$.) Explicit example:

$$A = \{0,1,2\} \quad B = \{2,3\}$$
$$X = \{1,2,3\}$$



3) ($\subseteq$) Let $x \in C \setminus (B \setminus A)$. Then $x \in C$ and $x \notin B \setminus A$, i.e. $x \in A$ or $x \notin B$. If $x \in A$, then $x \in A \cup (C \setminus B)$. If $x \notin B$, then $x \in C$ implies $x \in C \setminus B$. In either case, $x \in (C \setminus B) \cup A$ so $C \setminus (B \setminus A) \subseteq (C \setminus B) \cup A$.

Visual:



($\supseteq$) Let $x \in (C \setminus B) \cup A$, so $x \in A$ or $x \in C \setminus B$.

① First suppose $x \in A$. Since $A \subseteq C$, this implies $x \in C$ as well. Moreover, $x \notin B \setminus A$ since $x \in A$. Thus $x \in C \setminus (B \setminus A)$.

② Now suppose $x \in C \setminus B$, so $x \in C$ and $x \notin B$. Since $x \notin B$, we must have $x \notin B \setminus A$ (since $B \setminus A \subseteq B$). Thus $x \in C \setminus (B \setminus A)$. This finishes the proof. $\square$

Part II

1) Find examples of $f: \mathbb{R} \to \mathbb{R}$ s.t.
   (a) f is bijective
   (b) f is not injective or surjective
   (c) f is injective but not surjective
   (d) f is surjective but not injective

2) (from HW1) Prove or find counterex.: $f(P \cup Q) = f(P) \cup f(Q)$ for $f: A \to B$ and $P, Q \subseteq A$.

3) (from Lecture) Prove $f: X \to Y$ is bijective $\Leftrightarrow$ it has an inverse $g: Y \to X$

4) Suppose $g \circ f$ is injective. (a) Show f is injective
   (b) find an example s.t. g not injective but $g \circ f$ is.

Solns / Hints

1) For example... (a) $f = \mathrm{id}$ or $f(x) = -x$

(b) $f(x) = 0$

(c) $f(x) = \begin{cases} x+1 & x \geq 0 \\ x & x < 0 \end{cases}$ (if you want a continuous example, something like: )

(d) $f(x) = \begin{cases} x-1 & x \geq 0 \\ x & x < 0 \end{cases}$ (continuous example: $\tan(x)$' )

2) ($\subseteq$) Let $b \in f(P \cup Q)$, so $b = f(a)$ for some $a \in P \cup Q$. Then $a \in P$ or $a \in Q$, so $b \in f(P)$ or $b \in f(Q)$, respectively. This means $b \in f(P) \cup f(Q)$.

($\supseteq$) Similar! I'll leave this for you.

3) ($\Rightarrow$). Suppose f is bijective, so it is inj. and surj. We will use this to construct $g: Y \to X$. Given $y \in Y$, surjectivity of f tells us we can find $x \in X$ so that $f(x) = y$. Set $g(y) = x$ for this x. But how do we know g is well-defined? That is, how do we know there isn't more than one x to choose from? This follows from injectivity of f, because if there is $x' \in X$ so that $f(x') = y$, then $f(x') = f(x)$ so in fact $x' = x$. Finally, by construction,

$g(f(x)) = g(y) = x$ and $f(g(y)) = f(x) = y$, so g is an inverse for f.

($\Leftarrow$) Suppose g is an inverse for f. We will show f is inj + surj.

(inj) Suppose $f(x) = f(x')$. Applying g to both sides, we get $x = gf(x) = gf(x') = x'$.

(surj) Let $y \in Y$. Then $y = f(g(y))$, so setting $x = g(y)$, we have found $x \in X$ so that $f(x) = y$. $\square$

4) (a) This is hidden in the proof of II.3 above :)

(b) The example we saw earlier works (w/ symbols f and g switched — this is why I got confused)

$$f : \mathbb{R} \to \mathbb{R}^2 \quad \text{and} \quad g : \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto (x,0) \qquad\qquad (x,y) \mapsto x$$

Then $g \circ f : \mathbb{R} \to \mathbb{R}$ is the identity (hence injective) but g isn't injective.