M3700

Recitation Notes 09/10 Today: functions & relations

Warmup 1) Let A be a set w/ 3 elements and B a set w/ 2 elements.

(a) How many functions $A \to B$? $2 \times 2 \times 2 = 2^3 = 8$

(b) How many are injective? None

(c) How many are surjective? $8 - 2 = 6$

(d) How many are bijective? None

2) Let R be the relation on $\mathbb{R}$ where $a \sim_R b$ if $a - b \in \mathbb{Q}$

(a) Convince yourself R is an equivalence relation

(i) $a - a = 0$
(ii) $b - a = -(b-a)$
(iii) $a - c = (a-b) + (b-c)$

(b) Find 3 disjoint equivalence classes.

e.g. $[0] = \{a \in \mathbb{Q} \mid a - 0 \in \mathbb{Q}\} = \mathbb{Q}$

$[\pi]$

$[\sqrt{2}]$ (probably? $\pi - \sqrt{2} \notin \mathbb{Q}$)

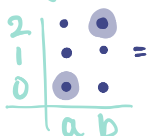
Review from Lecture

- fns: injective, surjective, bijective
- relns: equiv. relns, equiv. classes

Functions can be defined by ...

- subsets

$\{a,b\} \to \{0,1,2\}$



$= \{(a,0),(b,2)\}$

- multiple formulas

$f: \mathbb{R} \to \mathbb{R}$

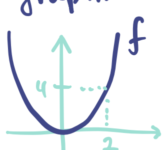
$f(x) = \begin{cases} x^2 & x \geq 0 \\ -(x^2) & x < 0 \end{cases}$

- descriptions

$f: \mathbb{N}_{>0} \to \mathbb{N}$

$f(n) := n^{\text{th}}$ prime

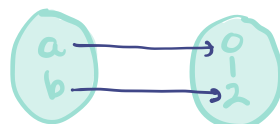
- graphs



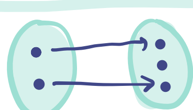
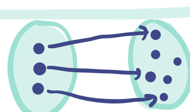
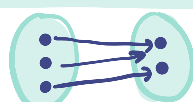
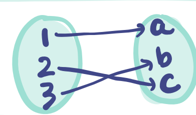
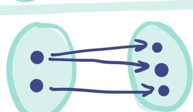
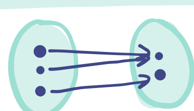
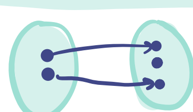
- table

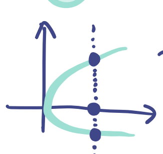
x	$f(x)$
a	0
b	2

- "bubbles w/ arrows"

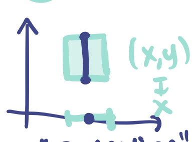


Properties of functions $f: X \to Y$

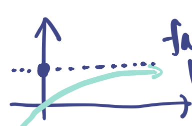
	well-defn'd	injective	surjective	bijective
defn	$\forall x, x' \in X$, if $x = x'$ then $f(x) = f(x')$	$\forall x, x' \in X$, if $f(x) = f(x')$ then $x = x'$	$\forall y \in Y\ \exists x \in X$ s.t. $f(x) = y$	surjective & injective
intuition	only one output	inclusion $\lvert X \rvert \leq \lvert Y \rvert$	onto $\lvert X \rvert \geq \lvert Y \rvert$	"equal" $\lvert X \rvert = \lvert Y \rvert$
example				
non-ex				everything in this table (except ↗↗)



fails VLT



"projection"



fails HLT

Example Consider $f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $(x,y) \mapsto x$ and $g: \mathbb{R} \to \mathbb{R}\times\mathbb{R}$, $x \mapsto (x,0)$

1) What are $f \circ g$ and $g \circ f$?

2) Which of $f, g, f\circ g, g\circ f$ are injective? Surjective?

Solution 1) The composition $f\circ g: \mathbb{R} \to \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $x \mapsto (x,0) \mapsto x$ is the identity $id_{\mathbb{R}}$.

The composition $g\circ f: \mathbb{R}^2 \to \mathbb{R} \to \mathbb{R}^2$, $(x,y) \mapsto x \mapsto (x,0)$ sends $(x,y) \in \mathbb{R}^2$ to $gf(x,y) = (x,0) \in \mathbb{R}^2$.

Note: Can write $g\circ f(x,y)$ or $gf(x,y)$ or $g(f(x,y))$.

2) We will show that:

(a) f is surjective & not injective

(b) g is injective & not surjective

(c) $f\circ g$ is bijective

(d) $g\circ f$ is neither.

(a) We first show f is surjective. Let $x \in \mathbb{R}$. Then we can find an element of $\mathbb{R}^2$, e.g. $(x,0)$ such that $f(x,0) = x$. Hence f is surjective. However f is not injective since $f(x,0) = x = f(x,1)$, but $(x,0) \neq (x,1)$ in $\mathbb{R}^2$.

(b) To see that g is injective, suppose $g(x) = g(x')$. This means $(x,0) = (x',0)$ so $x' = x$. On the other hand, g is not surjective since, e.g. $(1,1) \notin g(\mathbb{R})$.

(c) Since $f\circ g = id_{\mathbb{R}}$, it is bijective since the identity is always a bijection.
exercise if you don't believe

(d) We can find counterexamples for both injectivity + surjectivity of $g\circ f$: for example, $g\circ f$ is not injective since $gf(x,0) = (x,0) = gf(x,1)$, and $g\circ f$ is not surjective since $(1,1) \notin gf(\mathbb{R}^2)$. $\square$

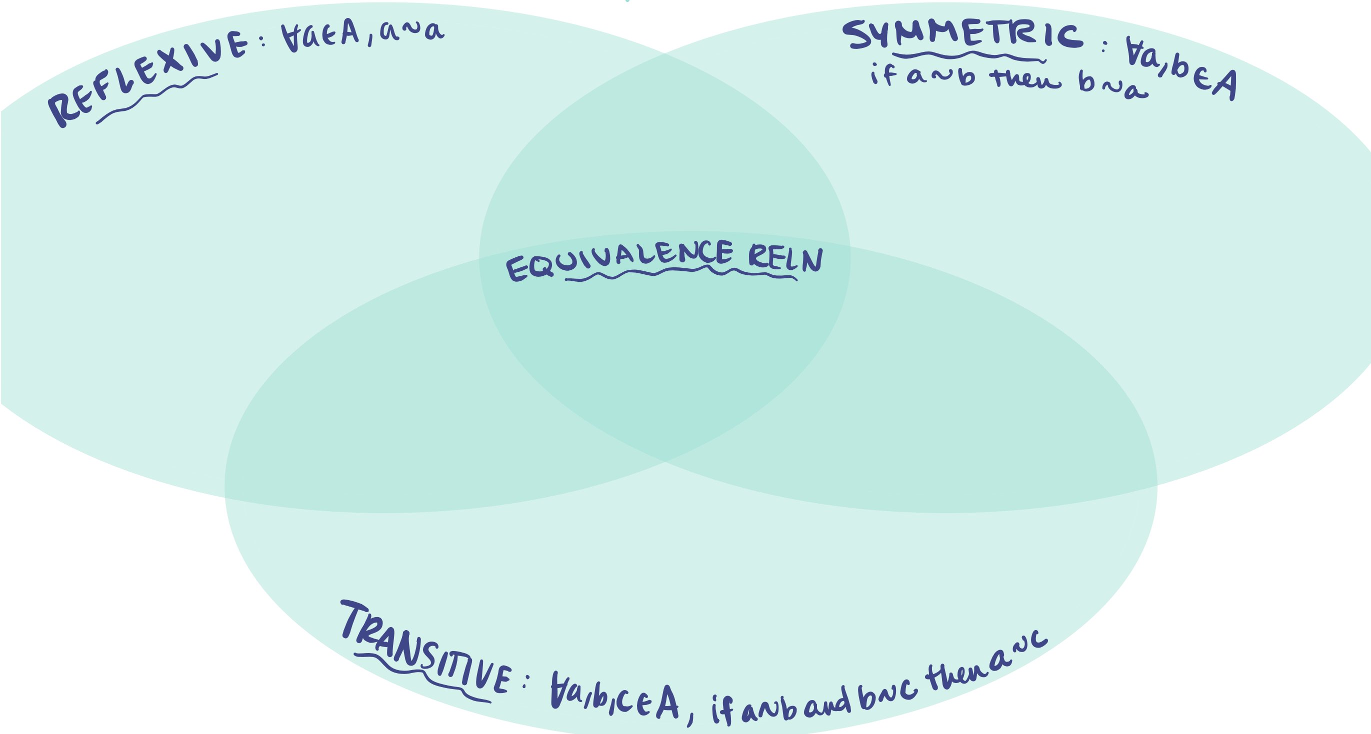
Prop (1) The composition of injective functions is injective.
(2) The composition of surjective functions is surjective. $\Big\} \Rightarrow$ Composition of bij fns is bij.

Pf/ Suppose $f: X \to Y$ and $g: Y \to Z$ are injective. We want to show $gf: X \to Z$ is injective. If $g(f(x)) = g(f(x'))$, then $f(x) = f(x')$ by injectivity of g. But then injectivity of f implies $x = x'$. $\square$

(2) Now suppose $f: X \to Y$ and $g: Y \to Z$ are surjective. Given $z \in Z$, we want to find $x \in X$ so that $gf(x) = z$. By surjectivity of g, there is $y \in Y$ s.t. $g(y) = z$. By surjectivity of f, $y = f(x)$ for some $x \in X$. Then $g(f(x)) = g(y) = z$ for this x. $\square$

Relations between sets A and B is a subset of $A \times B$ ⟵ often $B = A$

Write: aRb or $a \sim b$ if (a,b) in this subset



Recall: The equivalence class of $a \in A$ is $[a] = \{ b \in A \mid a \sim b \}$.

Thm Let R be an equivalence reln. Then $\forall a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Pf/ If $[a] \cap [b] = \emptyset$, then the desired claim holds, so suppose $[a] \cap [b] \neq \emptyset$. This means $\exists c \in A$ so that $c \in [a]$ and $c \in [b]$. We will use this c to show $[a] = [b]$.

$(\subseteq)$ Let $a' \in [a]$. This means $a \sim a'$. Then:

$a' \sim a$ by symmetry,

$a' \sim c$ by transitivity, since $a \sim c$

$a' \sim b$ by transitivity, since $b \sim c$ (by symmetry)

and hence $a' \in [b]$. $(\supseteq)$ Similarly, if $b' \in [b]$, then $b \sim b'$ and so

$b' \sim b$ by symm.

$b' \sim c$ by trans. & $b \sim c$

$b' \sim a$ by trans. & $c \sim a$

so $b' \in [a]$. Hence $[a] = [b]$. $\square$

Cor. If R is equiv. reln, get partition of $A = \bigcup_{i \in I} A_i$ for $A_i = [a_i]$ and $A_i \cap A_j = \emptyset$.

Thm. $\{\text{Equivalence relns on } A\} \longleftrightarrow \{\text{partitions of } A\}$.

Pf/ We have seen equiv. reln $\rightsquigarrow$ partition. Now suppose $A = \bigcup_{i\in I} A_i$ w/ $A_i \cap A_j = \emptyset$ for $i \neq j$.

Define R s.t. aRb if $a,b \in A_i$. WTS: (i) R is reflexive (ii) symmetric (iii) transitive.

(i) For any $a \in A$, aRa since $a \in A_i$.

(ii) Suppose aRb. Then $a,b \in A_i$ also means bRa.

(iii) Suppose aRb and bRc, so $a,b \in A_i$ and $b,c \in A_j$ for some i,j. Then $b \in A_i \cap A_j$, so we must have $i=j$. Thus $a,c \in A_i$ so aRc. $\square$

Important Example:

Let $n \in \mathbb{Z}_{>0}$ and consider the reln on $\mathbb{Z}$ given by $a \sim b$ if $a-b$ is a multiple of n. ↳ "congruent mod n" $a \equiv b \pmod n$

Claim. This is an equivalence relation.

Pf/ (reflexive) For any $a \in \mathbb{Z}$, $a \sim a$ since $a-a = 0 = 0 \cdot n$.

(symmetric) Suppose $a \sim b$, so $a-b = kn$ for some $k \in \mathbb{Z}$. Then

$$b-a = -(a-b) = -kn \quad \text{so} \quad b \sim a.$$

(transitive) If $a \sim b$ and $b \sim c$, then $a-b = kn$ and $b-c = k'n$ for some $k,k' \in \mathbb{Z}$. Then

$$\begin{aligned} a-c &= a+b-b-c && \text{since } b-b=0 \\ &= a-b+b-c && \text{since + is commutative} \\ &= kn+k'n \\ &= (k+k')n && \text{by distribution.} \end{aligned}$$

$\square$

Defn. The equivalence classes are $\mathbb{Z}/\!\sim\ =: \mathbb{Z}/n\mathbb{Z}$. "$\mathbb{Z}$ mod $n\mathbb{Z}$"

Claim $|\mathbb{Z}/n\mathbb{Z}| = n$

Pf/ We will show there is a bijection $f: \mathbb{Z}/n\mathbb{Z} \to \{0,1,\dots,n-1\}$, which implies

$$|\mathbb{Z}/n\mathbb{Z}| = |\{0,1,\dots,n-1\}| = n.$$

Define $f[a]$ = remainder of a/n.

Well-defined: Suppose $[a]=[b]$. We WTS $f[a]=f[b]$, i.e. a and b have the same remainder when divided by n. Since $[a]=[b]$, we know $a-b=kn$ for some n. If $b = qn+r$, then $a = kn+b = kn+qn+r = (k+q)n+r$, so the claim holds.

Injective: Suppose $f[a]=f[b]$. We want to show $a-b=kn$ for some k. Write

$$a = qn+r \quad \text{and} \quad b = q'n+r'.$$

Since $f[a]=f[b]$, we know $r=r'$, so $a-b = qn+r-q'n-r' = qn-q'n = (q-q')n$.
Hence $[a]=[b]$, so f is injective.

Surjective: Let $r \in \{0, \ldots, n-1\}$. Then $f[r] = r$ since $r < n$ and so $\frac{r}{n} = 0$ remainder r.

This shows f is a bijection, hence $|\mathbb{Z}/n\mathbb{Z}| = n$. $\square$

e.g.
clocks ($n=12$),
months ($n=12$),
meals ($n=3$)...

Practice

Part I. 1) Find examples of $f: \mathbb{R} \to \mathbb{R}$ s.t. f is
- (a) bijective
- (b) not injective nor surjective
- (c) injective but not surjective
- (d) surjective but not injective

2) (from Lecture) $f: X \to Y$ is bijective $\Leftrightarrow$ it has an 'inverse' $g: Y \to X$

this is very useful!

Part II. 1) Let $|A| = n$. How many distinct relations are there on A?

Bonus: How many reflexive, symm, transitive, etc?

2) (from HW2) Let $\sim$ be the relation on $\mathbb{R}\setminus\{0\}$ given by $x \sim y$ if $xy > 0$. Describe the corresponding partition.

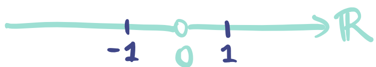
Part III. 1) Let $A = \{$differentiable functions $\mathbb{R} \to \mathbb{R}\}$. Define R by $f \sim_R g$ if $f(0) = g(0)$.
- (a) Prove R is an equivalence relation
- (b) Let $S = A/\sim_R$ and define $F: S \to \mathbb{R}$ by $F[f] = f(0)$. Prove F is well-defn'd + bijective.

2) Let $A = \{$parallel lines in the plane $\mathbb{R}^2\}$. Prove:
- (a) "is parallel to" is an equiv. reln.
- (b) "is perpendicular to" is not.
- (c) Show slope: $A/\text{parallel} \longrightarrow \mathbb{R}$ is well-defined + bijective.

Bonus Construction of $\mathbb{Z}$ from $\mathbb{N}$: Define a relation on $\mathbb{N} \times \mathbb{N}$ by $(a,b) R (c,d)$ if $a+d = b+c$.
- (a) Show this is an equivalence relation.
- (b) Define $\mathbb{Z} = \mathbb{N} \times \mathbb{N}/\sim$. Prove this is bijective to $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Extra bonus: Can you do something similar to construct $\mathbb{Q}$ from a reln on $\mathbb{Z} \times \mathbb{Z}\setminus\{0\}$?

II.2 (sketch)



Note that

$[1] = \{x \in \mathbb{R} \mid 1 \cdot x > 0\} = \mathbb{R}_{>0} = (0, \infty)$

and $[-1] = \{x \in \mathbb{R} \mid -1 \cdot x > 0\} = \mathbb{R}_{<0} = (-\infty, 0)$.

This partitions $\mathbb{R}\setminus\{0\}$ into $(-\infty, 0) \cup (0, \infty) = [-1] \cup [1]$.

III.1(b) Let $f, g \in A$ and note that

$$F[f] = F[g] \Longleftrightarrow f(0) = g(0) \Longleftrightarrow f \sim g.$$

This shows F is well-defined and injective. For surjectivity, let $a \in \mathbb{R}$ and set $c_a(x) \equiv a$ to be the constant fn at a. Then $F[c_a] = c_a(0) = a$.

Hence F is a bijection. Note: this implies $|S| = |\mathbb{R}|$ is uncountable!