M3700 Notes from Recitation 9/6

Warm up 1) examples of subsets of

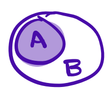
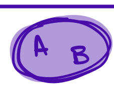
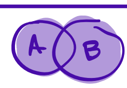
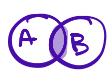
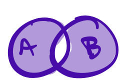
Countable
- $\mathbb{N} = \{0,1,2,3,\dots\} \supseteq \{1,2,3\},\ \{2k \mid k \in \mathbb{N}\} = \{x \mid x \in \mathbb{N} \text{ is even}\}$
- $\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\} \supseteq \{-1,0,100\},\ \mathbb{N}$
- $\mathbb{Q} = \{\frac{a}{b} \mid a,b \in \mathbb{Z}\} \supseteq \{\frac{1}{b} \mid b \in \mathbb{Z}\},\ \mathbb{Z},\ \mathbb{N}$

Uncountable
- $\mathbb{R} = \overline{\mathbb{Q}}$ ← "closure" or "completion" $\supseteq \{\sqrt{2}\},\ \{\text{irrationals}\} = \mathbb{R} \setminus \mathbb{Q},\ [0,1] = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$
- $\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R},\ i = \sqrt{-1}\} \supseteq \{ni \mid n \in \mathbb{N}\},\ \{a+bi \mid a^2+b^2 = 1\}$

2) example of set w/ no numbers in it {Tuesday, Wednesday}, {A, B, C}, {x | x is blue}

3) example of something which is <u>not</u> a set number? matrix? me???

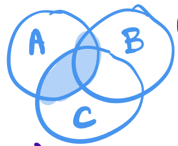
Main Concepts from Lecture: Sets, ZFC (axiom of choice), power sets, operations on sets

operation	notation	visual	meaning
subset	$A \subseteq B$		$x \in A \Rightarrow x \in B$
equality	$A = B$		$A \subseteq B$ and $B \subseteq A$
union	$A \cup B$		$x \in A \cup B \Rightarrow x \in A$ or $x \in B$
intersection	$A \cap B$		$x \in A \cap B \Rightarrow x \in A$ and $x \in B$
complement (need $A \subseteq B$)	$A \setminus B$ or B^c or $A - B$		$x \in A \setminus B \Rightarrow x \in A$ and $x \notin B$
disjoint union	$A \sqcup B$		$x \in A \sqcup B \Rightarrow x \in A$ "or" $x \in B$ ← even if $A \cap B \neq \emptyset$, distinguish betwn $x_A \in A$ and $x_B \in B$.
symmetric difference	$A \triangle B$		$x \in A \triangle B \Rightarrow x \in A$ xor $x \in B$ ← exclusive or

Example Let A, B, C be sets. Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof/ $(\subseteq)$ First we prove $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$, so $x \in A$ or $x \in B \cup C$, so $x \in B$ or $x \in C$.

Visual



Case 1: Suppose $x \in B$. Then $x \in A \cap B$ so $x \in (A \cap B) \cup (A \cap C)$.

Case 2: Suppose $x \in C$. Then $x \in A \cap C$ so $x \in (A \cap B) \cup (A \cap C)$.

In either case, $x \in (A \cap B) \cup (A \cap C)$, hence $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

$(\supseteq)$ Now we show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Let $x \in (A \cap B) \cup (A \cap C)$, so $x \in A \cap B$ or $x \in A \cap C$. Case 1: Suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$, so $x \in B \cup C$ as well and hence $x \in A \cap (B \cup C)$. Case 2: Suppose $x \in A \cap C$. So then $x \in A$ and $x \in C$, and thus $x \in B \cup C$. But then $x \in A \cap (B \cup C)$. In either case, $x \in A \cap (B \cup C)$ and so $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

This proves $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. $\square$

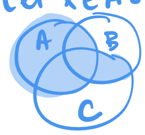
Practice Problems

Part I:
1) Show $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for sets A, B, C.
2) Show $B \setminus (B \setminus A) = A$ for sets $A \subseteq B$.
3) Show $C \setminus (B \setminus A) = (C \setminus B) \cup A$ for sets $A \subseteq B \subseteq C$.

Solutions

1) $(\subseteq)$ Let $x \in A \cup (B \cap C)$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, hence $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B$ and $x \in C$. But then $x \in A \cup B$ and $x \in A \cup C$. So therefore $x \in (A \cup B) \cap (A \cup C)$. This shows $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Visual:

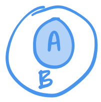


$(\supseteq)$ Let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cap (B \cup C)$ so we're done. Now suppose $x \notin A$. Since $x \in A \cup B$, we must have $x \in B$. Similarly since $x \in A \cup C$, we must have $x \in C$. Hence $x \in B \cap C$ so $x \in A \cup (B \cap C)$. This shows $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Since we have shown both set inclusions, the two sets are equal. $\square$

2) $(\subseteq)$ Let $x \in B \setminus (B \setminus A)$, so $x \in B$ and $x \notin B \setminus A$. To say $x \notin B \setminus A$ is to say $x \in A$ or $x \notin B$. But we know $x \in B$, so it must be that $x \in A$.

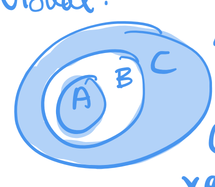
Visual: $(\supseteq)$ Let $x \in A$, so $x \notin B \setminus A$. Since $A \subseteq B$, $x \in A$ implies $x \in B$. Thus $x \in B \setminus (B \setminus A)$.



We have shown $B \setminus (B \setminus A) \subseteq A$ and $A \subseteq B \setminus (B \setminus A)$, hence they are equal. $\square$

3) $(\subseteq)$ Let $x \in C \setminus (B \setminus A)$. Then $x \in C$ and $x \notin B \setminus A$, i.e. $x \in A$ or $x \notin B$. If $x \in A$, then $x \in A \cup (C \setminus B)$. If $x \notin B$, then $x \in C$ implies $x \in C \setminus B$. In either case, $x \in (C \setminus B) \cup A$ so $C \setminus (B \setminus A) \subseteq (C \setminus B) \cup A$.

Visual:



$(\supseteq)$ Let $x \in (C \setminus B) \cup A$, so $x \in A$ or $x \in C \setminus B$.

① First suppose $x \in A$. Since $A \subseteq C$, this implies $x \in C$ as well. Moreover, $x \notin B \setminus A$ since $x \in A$. Thus $x \in C \setminus (B \setminus A)$.

② Now suppose $x \in C \setminus B$, so $x \in C$ and $x \notin B$. Since $x \notin B$, we must have

$x \notin B \setminus A$ (since $B \setminus A \subseteq B$). Thus $x \in C \setminus (B \setminus A)$. This finishes the proof. $\square$

Part II

1) (from lecture) If X is a finite set, $|X| = n$, show $|\mathcal{P}(X)| = 2^{|X|} = 2^n$.

2) (from HW1) Prove or find a counterexample: $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ for sets A, B.

Hints

1) Prove by induction or a counting argument: for each $x \in X$, can assign it to 0 or 1 based on whether you include it in the subset.

Bonus Q: What if X is infinite?

2) We always have $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ (why?) What about the other inclusion?

Part III

1) Prove $\{x^3 \mid x \in \mathbb{R}\} = \mathbb{R}$

2) Prove $\{3m + 4n \mid m, n \in \mathbb{Z}\} = \mathbb{Z}$

3) Prove $\{x \mid x \text{ can be written as a sum of 3 consecutive integers}\} = \{3n \mid n \in \mathbb{Z}\}$

Main Ideas

1) For any $x \in \mathbb{R}$, we have $x^3 = x \cdot x \cdot x \in \mathbb{R}$. This shows $\subseteq$. For $\supseteq$, given $y \in \mathbb{R}$, set $x = \sqrt[3]{y}$. Then $x^3 = (\sqrt[3]{y})^3 = y$ so $y \in \{x^3 \mid x \in \mathbb{R}\}$.

2) For $\subseteq$, write $x = 1 \cdot x = (4-3)x = 4x - 3x = 3(-x) + 4x$.

3) Both directions use a "clever 0":

$$3n = n + n + n = n + n + n + 0 = n + n + n + (1 - 1) = (n-1) + n + (n+1).$$

Bonus Q: For each $r \in \mathbb{R}$, set $A_r = \{r\}$ and $B_r = [0, |r|]$. (closed interval)

Convince yourself (visually or w/ proof) that

$$\bigcap_{r \in \mathbb{R}} A_r = \emptyset \qquad \bigcup_{r \in \mathbb{R}} A_r = \mathbb{R}$$

$$\bigcap_{r \in \mathbb{R}} B_r = \{0\} \qquad \bigcup_{r \in \mathbb{R}} B_r = [0, \infty)$$

infinite intersections — infinite unions