Math3700 – normal subgps, quotients, 1st isomorphism thm

Recall: A subgp N is normal in G, written $N \trianglelefteq G$, if any of the following equivalent conditions hold:

- $gN = Ng \ \forall g \in G$
- $gNg^{-1} = N \ \forall g \in G$
- $gNg^{-1} \subseteq N \ \forall g \in G$ (i.e. $gng^{-1} \in N \ \forall g \in G, n \in N$)
- $N_G(N) = G$

normalizer $N_G H = \{g \in G \mid gH = Hg\}$ **Note**: $H \subseteq N_G(H)$

Examples

(1) $e \trianglelefteq G$ since $geg^{-1} = gg^{-1} = e \ \forall g \in G$

(2) $G \trianglelefteq G$ since $gg'g^{-1} \in G \ \forall g, g' \in G$

(3) $Z(G) \trianglelefteq G$ since $Z(G) = \{g \in G \mid gg' = g'g \ \forall g' \in G\}$, $g'Z(G) = Z(G)g' \ \forall g' \in G$

(4) If G is Abelian, $H \trianglelefteq G \ \forall H \leq G$ e.g. G cyclic

Example – $D_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$ s.t. $r^4 = e$, $rs = sr^{-1}$

Q. Which subgps of D_4 are normal?

1st try: $\langle s \rangle = \{e, s\}$

☹ $rsr^{-1} = r^2 s \notin \langle s \rangle$ $\Rightarrow$ $\langle s \rangle$ is not normal in D_4

2nd try: $\langle r \rangle = \{e, r, r^2, r^3\}$

Note $r^k \langle r \rangle r^{-k} = \langle r \rangle$ so just need to check s, sr, sr^2, sr^3:

- $s\langle r \rangle s^{-1} = s \langle r \rangle s$:

$$\left.\begin{aligned} &\hookrightarrow srs = s^2 r^{-1} = r^3 \in \langle r \rangle \\ &\hookrightarrow sr^2 s = s^2 r^{-2} = r^2 \in \langle r \rangle \\ &\hookrightarrow sr^3 s = s^2 r^{-3} = r \in \langle r \rangle \end{aligned}\right\} \Rightarrow s\langle r \rangle s = \langle r \rangle$$

- for any other sr^k,

$$\begin{aligned} sr^k \langle r \rangle (sr^k)^{-1} &= sr^k \langle r \rangle r^{-k} s^{-1} \\ &= s \langle r \rangle s \\ &= \langle r \rangle. \end{aligned}$$

☺ So $\langle r \rangle \trianglelefteq D_4$

Note: Any subgp of $\langle r \rangle$ is normal in D_4 b/c $\langle r \rangle$ is Abelian, $H \leq \langle r \rangle \Rightarrow H \trianglelefteq \langle r \rangle$ and then $H \trianglelefteq \langle r \rangle \trianglelefteq G \Rightarrow H \trianglelefteq G$

in particular, $\langle r^2 \rangle = Z(D_4) \trianglelefteq D_4$.

We also have $\langle s\rangle \cong \mathbb{Z}/2\mathbb{Z} \cong \langle r^2\rangle$, but $\langle s\rangle \ntrianglelefteq D_4$.

"Being normal is not about who you are, but your relation to the rest of the group."

Ex (important!) If $\phi: G \to G'$ is a gp hom, then $\ker\phi \trianglelefteq G$.

Pf/ WTS: $gkg^{-1} \in \ker\phi$ $\forall g \in G, k \in \ker\phi$. Plug in + use homomorphism property:

$$\begin{aligned}\phi(gkg^{-1}) &= \phi(g)\phi(k)\phi(g^{-1}) \\ &= \phi(g)e_{G'}\phi(g^{-1}) \\ &= \phi(g)\phi(g^{-1}) \\ &= \phi(gg^{-1}) \\ &= \phi(e) \\ &= e_{G'}.\end{aligned}$$

Hence $gkg^{-1} \in \ker\phi$ so $\ker\phi \trianglelefteq G$. □

Why care about normal subgps?

For any $H \leq G$, can consider the set of (left) cosets

$$G/H = \{g_1H, \dots, g_nH\}$$

Note: $n = |G:H| = \frac{|G|}{|H|}$ by Lagrange's Thm

where $g_i \in G$ are representatives s.t. $g_iH \cap g_jH = \emptyset$ for $i \neq j$.

e.g. $G = \mathbb{Z}$, $H = n\mathbb{Z} \leq \mathbb{Z}$

$$\mathbb{Z}/n\mathbb{Z} = \{0+\mathbb{Z}, 1+\mathbb{Z}, \dots, (n-1)+\mathbb{Z}\}$$

Thm G/H is a group $\iff H \trianglelefteq G$.

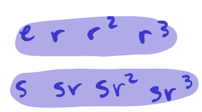
Ex. $\mathbb{Z}/n\mathbb{Z}$ is always a group since $n\mathbb{Z} \trianglelefteq \mathbb{Z}$ $\forall n$ (b/c $\mathbb{Z}$ is Abelian)

Defn. For $N \trianglelefteq G$, the group G/N is called the quotient of G by N.

Note $|G/N| = |G|/|N|$.

Ex. $D_4/\langle r\rangle = \{e\langle r\rangle, s\langle r\rangle\}$ is a group $\cong \mathbb{Z}/2\mathbb{Z}$

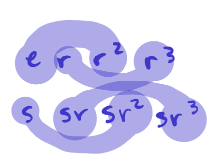
$|D_4 : \langle r\rangle| = 2$



$D_4/\langle r^2\rangle = \{e\langle r^2\rangle, r\langle r^2\rangle, s\langle r^2\rangle, sr\langle r^2\rangle\}$

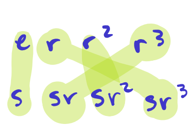
$|D_4 : \langle r^2\rangle| = 4$ $\quad \cong \mathbb{Z}/4\mathbb{Z}$ or K_4?

not cyclic...



Non-ex $D_4/\langle s\rangle = \{e\langle s\rangle, r\langle s\rangle, r^2\langle s\rangle, r^3\langle s\rangle\}$

$|D_4 : \langle s\rangle| = 4$



not a group: $r\langle s\rangle \cdot r\langle s\rangle = ?$ Not well-def'n'd!
$\overset{\shortparallel}{\{r, rs\}}$

$r\cdot r \in r^2\langle s\rangle \quad rs\cdot r = sr^{-1}r \in e\langle s\rangle$

$r\cdot rs \in r^2\langle s\rangle \quad rs\cdot rs = sr^{-1}rs \in e\langle s\rangle$

Rmk When proving things involving quotients G/H, checking well-def'n'd is important!!

mysterious Observation

lattice of subgps of D_4 containing $Z(D_4) = \langle r^2\rangle$

$$\begin{tikzcd} & D_4 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \langle s, r^2\rangle \arrow[dr, dash] & \langle r\rangle \arrow[d, dash] & \langle sr, r^2\rangle \arrow[dl, dash] \\ & \langle r^2\rangle & \end{tikzcd}$$

$$\begin{tikzcd} & K_4 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \langle(1,0)\rangle \arrow[dr, dash] & \langle(1,1)\rangle \arrow[d, dash] & \langle(0,1)\rangle \arrow[dl, dash] \\ & e & \end{tikzcd}$$

lattice of subgps of $D_4/\langle r^2\rangle \cong K_4$

Motivation: try to understand G by studying N and G/N ... using
- Isomorphism Theorems (today: 1^{st} one)
- universal property of quotient

The 1^{st} Isomorphism Theorem

– Let $\phi: G \to G'$ be a gp hom. Then

$$G/\ker\phi \cong \operatorname{im}\phi.$$

wow!

Note: If ϕ is surjective, $G/\ker\phi \cong G'$.

Universal Property of G/N

idea: $\left\{\text{gp homs } G \xrightarrow{\phi} G' \text{ s.t. } N \subseteq \ker\phi\right\} \Longleftrightarrow \left\{\text{gp homs } G/N \to H\right\}$

as a diagram:

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "q"'] & G' \\ G/N \arrow[ur, dotted] & \end{tikzcd}$$

if $\ker\phi \supseteq N$ then $\exists!$ gp hom $\psi: G/N \to G'$ s.t. $\phi = \psi q$

Examples using these

Ex. $D_4 \xrightarrow{\#\text{of } s} \mathbb{Z}/2\mathbb{Z}$ has $\ker = \{e, r, r^2, r^3\} = \langle r^4\rangle$

$\left.\begin{matrix} e \\ r \\ r^2 \\ r^3 \end{matrix}\right\} \to 0$

$\left.\begin{matrix} s \\ sr \\ sr^2 \\ sr^3 \end{matrix}\right\} \to 1$

$\overset{1^{st}\text{ iso}}{\Rightarrow}$ $D_4/\langle r^4\rangle \cong \mathbb{Z}/2\mathbb{Z}$

Ex. $D_4 \longrightarrow K_4$ has $\ker = \{e, r^2\} = \langle r^2 \rangle$

$$\begin{aligned} e &\mapsto (0,0) \\ r &\mapsto (1,0) \\ r^2 &\mapsto (0,0) \\ r^3 &\mapsto (1,0) \\ s &\mapsto (0,1) \\ sr &\mapsto (1,1) \\ sr^2 &\mapsto (0,1) \\ sr^3 &\mapsto (1,1) \end{aligned}$$

$\overset{1^{st}\text{ iso}}{\Longrightarrow} D_4/\langle r^2 \rangle \cong K_4$

Ex. (HW8 Exc 2(1)) What is $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z})/\langle (0,1) \rangle$ isomorphic to?

Thought process: $\langle (0,1) \rangle = \{(0,1), (0,2), \dots, (0,6)\} \cong e \times \mathbb{Z}/6\mathbb{Z}$ so probably $\mathbb{Z}/4\mathbb{Z}$

<u>Claim</u>. $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z})/\langle (0,1) \rangle \cong \mathbb{Z}/4\mathbb{Z}$

Pf/ Use 1^{st} iso thm (please!) Let $\phi: \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ be projection onto the first factor. Then ϕ is surjective and

$$\ker \phi = \{(0,0), (0,1), \dots, (0,5)\} = \langle (0,1) \rangle \trianglelefteq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}.$$

By the 1^{st} isomorphism thm, $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z})/\langle (0,1) \rangle \cong \mathbb{Z}/4\mathbb{Z}$.

BONUS

Ex. Show $SL(n, \mathbb{R}) \trianglelefteq GL(n, \mathbb{R})$.

Recall $SL(n,\mathbb{R}) = \{(n \times n)\text{-matrices w/ } \det = 1\} \leq GL(n, \mathbb{R}) = \{(n\times n)\text{-matrices w/ } \det \neq 0\}$.

<u>Idea</u>: Construct $\phi: GL(n,\mathbb{R}) \to G$ for some G s.t. $\ker\phi = SL(n,\mathbb{R})$

The determinant $\det: GL(n,\mathbb{R}) \to \mathbb{R}^\times$ is a group hom (since $\det(AB) = \det(A)\det(B)$)

And $\ker(\det) = \det^{-1}(1) = SL(n,\mathbb{R})$. Moreover, $\det$ is surjective since for any $x \in \mathbb{R}$,

$$\det \begin{bmatrix} x & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix} = x.$$

Thus $SL(n,\mathbb{R}) \trianglelefteq GL(n,\mathbb{R})$ and $GL(n,\mathbb{R})/SL(n,\mathbb{R}) \cong \mathbb{R}^\times$ by the 1^{st} iso thm.

Ex. Is it possible to find a gp hom $\phi: D_4 \to G$ for some G s.t. $\ker\phi = \langle s \rangle$?

<u>No!</u> If yes, then $D_4/\langle s \rangle \cong \operatorname{im}\phi$ would be a group, but this implies $\langle s \rangle \trianglelefteq D_4$ $\perp$

Ex. (HW8 Exc 4(3)) Find all homs $\mathbb{Q}/\mathbb{Z} \to \mathbb{Z}$

By univ. property, if $f: \mathbb{Q}/\mathbb{Z} \to \mathbb{Z}$ is a homom then $\exists g: \mathbb{Q} \to \mathbb{Z}$ s.t.
$$\begin{tikzcd} \mathbb{Q} \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z} \\ \mathbb{Q}/\mathbb{Z} \arrow[ur, "f"'] & \end{tikzcd}$$

What can we say about g? Suppose $g(1) = k \in \mathbb{Z}$. Then

$$\begin{aligned} k = g(1) &= g(\tfrac{1}{2}) + g(\tfrac{1}{2}) = 2g(\tfrac{1}{2}) \\ &= g(\tfrac{1}{3}) + g(\tfrac{1}{3}) + g(\tfrac{1}{3}) = 3g(\tfrac{1}{3}) \\ &\;\vdots \end{aligned}$$

$= n g(\frac{1}{n}) \ \forall n \geq 1$

$\Rightarrow n | k \ \forall n \geq 1 \Rightarrow k = 0$. Similarly, $g(x) = 0 \ \forall x \in \mathbb{Q}$ so $g \equiv 0$. Hence $f \equiv 0$ as well.

Conclusion : There is one gp hom $\mathbb{Q}/\mathbb{Z} \to \mathbb{Z}$: the trivial one.

~ break ~

HW time

minute sheet :
- what's something you found confusing?
- what's something you found interesting?
- anything else you want me to know?

Exc 3

(1) Let A be Abelian and $T(A) \subseteq A$ the elmts of finite order
   (i) Show $T(A) \leq A$
   (ii) Show $A/T(A)$ is torsion free — G is t.f. if $T(G) = e$
   (iii) Let $H \leq A$. Show A is torsion $\Longleftrightarrow$ H and A/H are torsion — G is torsion if $T(G) = G$

Exc 4

(2) Study $\mathbb{Q}/\mathbb{Z}$:
   (i) Show $\mathbb{Q}/\mathbb{Z}$ contains exactly one cyclic subgp of order n, $\forall n \geq 1$.
   (ii) Find all homoms $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$
   (iii) Find all homs $\mathbb{Q}/\mathbb{Z} \to \mathbb{Z}$

Exc 5

(3) Let $V \leq S_4$ be given by $V = \{e, (14)(23), (13)(24), (12)(34)\}$
   (i) Show $V \trianglelefteq S_4$
   (ii) Show $S_4/V \cong S_3$.