Math 3700 11/29 and 12/01

today: Group actions
Sylow theorems

Group Actions

idea: understand G by "what it does" to an object

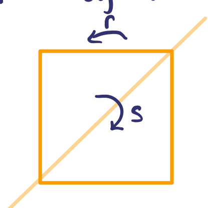
examples we've seen:

(1) elmts of Σ_n are (by defn) permutations of $\{1,\dots,n\}$

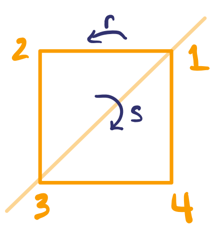
$(12):\{1,2,3\}\to\{1,2,3\}$

$(12)\cdot 1 = 2$
$(12)\cdot 2 = 1$
$(12)\cdot 3 = 3$

(2) elmts of D_4 are symmetries of a square



If we label the vertices of the square, can view $r=(1234)$ and $s=(24)$



Think of D_4 as acting on the vertices of the square $\{1,2,3,4\}$

$r\cdot 1 = 2$	$s\cdot 1 = 1$
$r\cdot 2 = 3$	$s\cdot 2 = 4$
$r\cdot 3 = 4$	$s\cdot 3 = 3$
$r\cdot 4 = 1$	$s\cdot 4 = 2$

Group actions abstract + make this rigorous:

Defn. An action of G on a set X is an assigment $G\xrightarrow{\alpha}\Sigma_X = \mathrm{Bij}(X)$
s.t. (i) $\alpha(e)=id$
(ii) $\alpha(g)\circ\alpha(h)=\alpha(gh)$. } i.e. α is a gp hom

Equivalently, an action is a map $*: G\times X\to X$ s.t.
easier to check { (i) $e*x=x \quad \forall x\in X$
(ii) $g*(h*x)=(gh)*x \quad \forall g,h\in G, x\in X$.

Notation $G \circlearrowright X$

Ex.

(1) Every gp acts on itself ($X=G$)

Define $*: G\times G \to G$ by gp operation. Check:

(i) $e*g = e\cdot g = g \quad \forall g\in G$

(ii) $g*(h*k) = g*(hk) = g(hk) = (gh)k = gh*k \quad \forall g,h,k\in G$

(1.5) $(\mathbb{R}^n,+)$ acts on itself by translations (geometric interpretation)

Given a vector $v\in\mathbb{R}^n$, define translation by v by $T_v: \mathbb{R}^n \to \mathbb{R}^n$.
$w \mapsto v+w$

Check: (i) $T_0(w) = 0+w = w \quad \forall w\in\mathbb{R}^n$

(ii) $T_{v_1}(T_{v_2}(w)) = v_1 + T_{v_2}(w) = v_1+(v_2+w)$
$= (v_1+v_2)+w$
$= T_{v_1+v_2}(w)$

properties of vector addition

(2) $GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by matrix/vector multiplication : $A*v = Av$

Check: (i) $Id*v = Id\, v = v$

(ii) $A*(B*v) = A(B*v) = A(Bv)$
$= (AB)v$
$= (AB)*v$

properties of mtx mult.

(3) handwavey: Laws of motion in physics should be the same across

- location
- time
- in every direction
- traveling in fixed direction @ fixed speed

$\rightsquigarrow$ can be described as $\underset{\substack{\uparrow \\ \text{10-dim'l}}}{G} \circlearrowright \mathbb{R}^4 = \{(x,y,z;t)\}$ spacetime

(4) If $H\leq G$ then ${}^{G}\circlearrowright G/H$ by $g*\hat{g}H = (g\hat{g})H$

Check: well-defn'd - if $\hat{g}_1H = \hat{g}_2H$ then WTS $g\hat{g}_1H = g\hat{g}_2H$

$\Updownarrow \qquad\qquad \Updownarrow$

$\hat{g}_2^{-1}\hat{g}_1\in H \qquad (g\hat{g}_2)^{-1}(g\hat{g}_1)\in H$

$(g\hat{g}_2)^{-1}(g\hat{g}_1) = \hat{g}_2^{-1}g^{-1}g\hat{g}_1 = \hat{g}_2^{-1}\hat{g}_1\in H$

(i) $e*\hat{g}H = (e\hat{g})H = \hat{g}H$

(ii) $g_1*(g_2*\hat{g}H) = g_1*(g_2\hat{g})H = (g_1(g_2\hat{g}))H = ((g_1g_2)\hat{g})H = (g_1g_2)*\hat{g}H.$

(5) Suppose $X \circlearrowleft G$ and let Y be any set. Consider $\mathrm{Hom}(X,Y)$ fns $X \to Y$; can we give it a G-action? together

want: $G \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Y)$

$(g, f: X \to Y) \longmapsto (g * f): X \to Y$

$$(g*f)(x) = ? \quad f(g*x)?$$

Check: (i) $(e*f)(x) = f(x) \Rightarrow e*f = f$ ✓

(ii) $g*(h*f)(x) = g*f(h*x) = f(g*(h*x))$

$((gh)*f)(x) = f((gh)*x) \quad \overset{?}{=}$

there's a mistake here!

$f(h*x) \in Y$ so $g*f(h*x)$ doesn't make sense! $(g*f)(x) \neq g*f(x)$ meaningless

$$g*(h*f)(x) = (h*f)(g\cdot x)$$
$$= f(h\cdot(g\cdot x))$$
$$= f(hg\cdot x) \neq f(gh\cdot x) = ((gh)*f)(x)$$

To fix: define $(g*f)(x) = f(g*x)$.

Q. If $G \circlearrowleft X$, what can we say about G? about X? open-ended

More specific Qs (1) Given X, what kinds of G act non-trivially on it?

(2) Given G, what kinds of X have non-trivial G-action?

(1)

Example Suppose $X = \{☺, 😐, ☹\}$

- How many actions of $\mathbb{Z}/5$ on X?

  Suppose $\alpha: \mathbb{Z}/5 \to \Sigma_X$ ($\cong \Sigma_3$) is a gp hom. Since $\mathbb{Z}/5 = \langle 1 \rangle$, α is determined by $\alpha(1)$ since $\alpha(k) = \alpha(1)^k$

  By the 1^{st} iso thm, $|\alpha(1)|$ divides $|\mathbb{Z}/5| = 5$ and $|\Sigma_X| = |X|! = 6$

  $G/\ker(\alpha) \cong \mathrm{im}(\alpha) \leq \Sigma_X$

  $\Rightarrow |\alpha(1)| = 1$ so $\alpha(1) = \mathrm{id}$ so α is the trivial action.

- How many actions of $\mathbb{Z}/15$ on X?

  Similarly, must have $|\alpha(1)| \mid 15, 6 \Rightarrow |\alpha(1)| = 1$ or 3

  - if $|\alpha(1)| = 1$ then α is trivial action
  - if $|\alpha(1)| = 3$ then $\alpha(1) = (123)$ or (132)

Thus we have 3 distinct actions.

<u>In general</u>: If $\gcd(|G|, |X|) = 1$, then only G-action on X is trivial

Pf / exercise. Hint: 1^{st} iso thm + Lagrange's Thm.

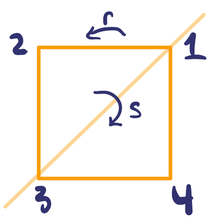
(2) Orbits + Stabilizers

<u>Defn</u> Let $G \circlearrowright X$. The <u>orbit</u> of $x \in X$ is $\mathcal{O}_x = \{g \cdot x \mid g \in G\} \subseteq X$

The <u>stabilizer</u> of $x \in X$ is $\mathrm{Stab}_x = \{g \in G \mid g \cdot x = x\} \leq G$. (exc.)

or G_x

<u>Ex</u> $D_4 \circlearrowright$ 



$\mathcal{O}_1 = \{1, r \cdot 1 = 2, r^2 \cdot 1 = 3, r^3 \cdot 1 = 4\} = X$

$\mathcal{O}_2 = \{2, 3, 4, 1\} = X$

in fact $\mathcal{O}_3 = \mathcal{O}_4 = X$ also

$\mathrm{Stab}_1 = \{e, s\} = \langle s \rangle = \mathrm{Stab}_3$

$\mathrm{Stab}_2 = \{e, r^{-1}sr\} = \langle r^{-1}sr \rangle = \mathrm{Stab}_4$

this is a <u>transitive</u> action: $\mathcal{O}_x = X$ $\forall x \in X$

<u>Ex</u> $GL_2(\mathbb{R}) \circlearrowright \mathbb{R}^2$ by $A * v = Av$

$\mathcal{O}_{(0,0)} = \{A\binom{0}{0} \mid A \in GL_2(\mathbb{R})\}$
$= \{(0,0)\}$ "fixed pt" of the action

$\mathrm{Stab}_{(0,0)} = \{A \in GL_2(\mathbb{R}) \mid A\binom{0}{0} = (0,0)\}$
$= GL_2(\mathbb{R})$

$\mathcal{O}_{(1,0)} = \{A\binom{1}{0} \mid A \in GL_2(\mathbb{R})\}$
$= \{(a_{11}, a_{21}) \mid A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in GL_2(\mathbb{R})\}$
$= \mathbb{R}^2 \setminus \{(0,0)\}$

$\mathrm{Stab}_{(1,0)} = \{A \mid A\binom{1}{0} = (1,0)\}$
$= \{A \mid (a_{11}, a_{21}) = (1,0)\}$
$= \{\begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix} \mid y \neq 0\}$

<u>Important Theorems</u> Let $X \circlearrowleft G$

(a) the orbits partition X

$\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$ or $\mathcal{O}_x = \mathcal{O}_y$

(b) $\exists$ bijection $\mathcal{O}_x \to G/\mathrm{Stab}_x$ and in particular

$|\mathcal{O}_x| = |G : \mathrm{Stab}_x|$. "orbit stabilizer thm"

(c) let $X = \mathcal{O}_{x_1} \sqcup \mathcal{O}_{x_2} \sqcup \cdots \sqcup \mathcal{O}_{x_n}$, then $|X| = \sum_{i=1}^{n} |\mathcal{O}_{x_i}| = \sum_{i=1}^{n} |G : Stab_{x_i}|$

Ex D_4 

(a) 1 2 / 3 4 trivial partition

(b) $\mathcal{O}_1 = \{1,2,3,4\}$

$Stab_1 = \{e, s\} = \langle s \rangle$ $\rightsquigarrow$ $G/\langle s\rangle = \{e\langle s\rangle, r\langle s\rangle, r^2\langle s\rangle, r^3\langle s\rangle\}$

bij $\updownarrow$

$\mathcal{O}_1 = \{1,2,3,4\}$

(c) not interesting b/c 1 orbit

Application of Gp Actions:

Sylow's Theorems (quickly)

Let G be a group and p a prime.

Defn - G is a p-group if $|G| = p^k$ for some $k \geq 1$

A subgroup H (in any gp) is a p-subgp if $|H| = p^k$

If $|G| = p^\alpha n$ where $p \nmid n$, then a subgp $H \leq G$ w/ order p^α is a Sylow p-subgp

Notation - $Syl_p(G) := \{$ Sylow p-subgps of $G \}$

$n_p(G) = |Syl_p(G)|$

Thms (1) $Syl_p(G) \neq \emptyset$

(2) if $P_1, P_2 \in Syl_p(G)$, then $\exists g \in G$ s.t. $P_1 = g P_2 g^{-1}$

(3) if $H \leq G$ is a p-subgp, then $\exists P \in Syl_p(G)$ s.t. $H \leq P$.

(4) $n_p(G) \equiv 1 \pmod p$.

Important Consequences:

- $n_p(G) = |G : N_G(P)|$ so $n_p(G) \mid |G|/p^\alpha$
- $P \in Syl_p(G)$ is normal $\Longleftrightarrow$ $n_p(G) = 1$
- If G is Abelian, any subgp is normal, so $n_p(G) = 1$
  use Thm of f.g. Ab gps to say more...
- If $P \in Syl_p(G)$ and $Q \in Syl_q(G)$ for $p \neq q$, then $P \cap Q = e$

Example Problems

(1) Exc 3(1) pt 1 There is only one gp of order 1001

For any n we have $\mathbb{Z}/n\mathbb{Z}$, so there's at least one. Suppose $|G| = 1001$; we will show $G \cong \mathbb{Z}/1001\mathbb{Z}$.

Note $|G| = 1001 = 7 \cdot 11 \cdot 13$. Apply Sylow Thms to $n=7$:

$$\underbrace{n_7}\ \Big|\ 11 \cdot 13 \quad \text{and} \quad n_7 \equiv 1 \pmod 7$$

$\Rightarrow n_7 = 11$ or 13 or 1 $\qquad$ $\left.\begin{matrix} 11 \equiv_7 4 \\ 13 \equiv_7 6 \end{matrix}\right\} \Rightarrow n_7 = 1$

Similarly argue $n_{11} = n_{13} = 1$. Let P_7, P_{11}, P_{13} be the Sylow subgps. Since $P_7 \cap P_{11} = e$, $P_7 P_{11} \cong P_7 \times P_{11} \cong \mathbb{Z}/7 \times \mathbb{Z}/11 \cong \mathbb{Z}/77$

Similarly, $P_{13} \cap P_7 P_{11} = e$ so $P_7 P_{11} P_{13} \cong \mathbb{Z}/77 \times \mathbb{Z}/13 \cong \mathbb{Z}/1001\mathbb{Z}$.

But $P_7 P_{11} P_{13} \leq G$ has the same size as G, hence $G = P_7 P_{11} P_{13} \cong \mathbb{Z}/1001\mathbb{Z}$.

(2) Exc 3(2) pt 1 Show G is not simple if $|G| = 80$

idea: Show $n_p = 1$ for some p $\qquad$ └ if $N \trianglelefteq G$ then $N = e$ or $N = G$

Since $|G| = 80 = 2^4 \times 5$, the Sylow Thms tell us $n_5 \mid 2^4$ and $n_5 \equiv 1 \pmod 5$ so

$n_5 = 1$
$n_5 = 2 \equiv_5 2$
$n_5 = 4 \equiv_5 4$
$n_5 = 8 \equiv_5 3$
$n_5 = 16 \equiv_5 1$

if $n_5 = 1$, done, so assume $n_5 = 16$. By 1st Sylow, each of these 16 subgps has order 5 (hence $\cong \mathbb{Z}/5$), and are all distinct so intersect only at e. Thus we get $16 \cdot 4 = 64$ elmts of order 5 in G. This leaves $80 - 64 = 16$ elmts of G.

Now, $n_2 \mid 5$ and $n_2 \equiv 1 \pmod 2$ $\Rightarrow n_2 = 1$ or 5. But if $n_2 = 5$, then we have 5 Sylow subgps each of size $2^4 = 16$, which isn't possible (G is too small)

So $n_2 = 1$ and we're done.

~ break ~ $\qquad$ Minute sheet: - what was most helpful today
- what would you like to see more of? less of?

HW time

Exc 2) Consider $T = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \begin{matrix} a,b,d \in \mathbb{R} \\ ad \neq 0 \end{matrix} \right\}$ under mtx mult.

(1) Show $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ dy \end{bmatrix}$ defines $T \circlearrowright \mathbb{R}^2$

(2) Find the orbits + stabilizers

(3) Show $U = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \middle| \, b \in \mathbb{R} \right\} \trianglelefteq T$ and identify T/U

Exc 4) Let $|G| = pq$ for $p > q$ prime

(1) Show G has a unique Sylow p-gp P which is normal in G

(2) Find $|\mathrm{Aut}(P)|$

(3) Let $H \in \mathrm{Syl}_q(G)$. Show if $p \not\equiv 1 \pmod q$ then ${}^{H}\circlearrowright N$ by conjugation is trivial

(4) Conclude G is cyclic if $p \not\equiv 1 \pmod q$.