Prime ideals in the Burnside Tambara functor

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with Sam Ginnett

part of the 2019 CMRG at Reed College
prime ideals computed by Dress (1971)

prime Green ideals computed by Lewis (1980)

prime Tambara ideals computed by Nakaoka for cyclic $p$-groups (2014)

• C.-G. Ginnett for cyclic groups (2022)

*fix a finite group $G$ throughout*
The Burnside ring of $G$ is
\[ A(G) = \text{Gr}(\text{Fin}G) = \mathbb{Z}[\{G/H\}_{H \leq G}] / \sim \]
with $+$ = disjoint union and $\times$ = Cartesian product.

The prime ideals of the Burnside ring are
\[ \text{Spec}(A(G)) = \{ \ker \phi_q^H \mid H \leq G, q \text{ prime or 0} \} \]
where $\phi_q^H : A(G) \to \mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$
\[ X \mapsto |X^H| \]

Example:
\[ A(C_p) \cong \mathbb{Z}[t_p]/(t_p^2 - pt_p) \]
for $t_p = C_p/e$
and $1 = C_p/C_p$.

$\ker \phi_q^e = \langle t_p - p, q \rangle$
$\ker \phi_q^{C_p} = \langle t_p, q \rangle$
The Burnside Green functor of $G$ is given by

$$A_G = \{A(H)\}_{H \leq G}$$

with restrictions, transfers, and conjugation maps

$$A(C_p) \cong \mathbb{Z}[t_p]/(t_p^2 - pt_p)$$

$$A(e) \cong \mathbb{Z}$$

An ideal is $I = \{I(H)\}_{H \leq G}$ with $I(H) \subseteq A(H)$ an ideal so that $I$ is closed under $R$, $T$, and $c$

An ideal $P$ is prime if for any ideals $I$ and $J$, if $I \cdot J \subseteq P$ then $I \subseteq P$ or $J \subseteq P$

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1 $\mapsto t_p$

$$A(e) \cong \mathbb{Z}$$

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$T$

$1$ $\mapsto t_p$

$R$

$t_p \mapsto p$

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Spec$(A_G) = \{P(H, q)\}$ for $H \leq G$, $q$ prime or 0.
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\[ A(C_p) \cong \mathbb{Z}[t_p]/(t_p^2 - pt_p) \]

\[ A(e) \cong \mathbb{Z} \]

\[ T: 1 \mapsto t_p \]

\[ R: t_p \mapsto p \]

\[ N: \quad \quad \quad (t_p - p) \]

\[ \text{Spec}(A_G) = \{P(H, q)\} \text{ for } G = C_N, H \leq G, \ q \text{ prime or } 0 \]
Main Results

For any finite group $G$,

$$\{P(H, p) \mid H \leq G, p \text{ prime or } 0\} \subseteq \text{Spec}(A_G)$$

built from $\ker \phi_p^K$ s
Main Results

For any finite Abelian group $G$,

$$\{P(H, p) \mid H \leq G, p \text{ prime or 0}\} \subseteq \text{Spec}(A_G)$$

with certain containment relations.

- $P(H, 0) \subseteq P(H, p)$
- $P(K, 0) \subseteq P(H, 0)$ if $H \leq K$
- $P(K, p) \subseteq P(H, p)$ if $H \leq_K K$

look at “non-$p$ part”
Main Results

For $G = C_N$,

$$\{P(i, p) \mid i \mid n, p \text{ prime or 0}\} = \text{Spec}(A_G)$$

with certain containment relations.

- $P(i, 0) \subseteq P(i, p)$
- $P(j, 0) \subseteq P(i, 0)$ if $i \mid j$
- $P(j, p) \subseteq P(i, p)$ if $i \mid p \ j$

look at “non-p part”
Main Results

For $G = C_N$,

$$
\{P(i,p) \mid i \mid n, p \text{ prime or 0}\} = \text{Spec}(A_G)
$$

with certain containment relations.

- $P(i,0) \subseteq P(i,p)$
- $P(j,0) \subseteq P(i,0)$ if $i \mid j$
- $P(j,p) \subseteq P(i,p)$ if $i \mid j$

if $i = p^kj$ then

$$P(j,p) = P(i,p)$$

(for $p \neq 0$)
Main Results

Spec($A_{CN}$)

0 or prime

Spec($A_{C p^n}$)

Spec($A_{C_6}$)

- $P(i, 0) \subseteq P(i, p)$
- $P(j, 0) \subseteq P(i, 0)$ if $i \mid j$
- $P(j, p) \subseteq P(i, p)$ if $i \mid p \mid j$

The case $N = p^n$ is a theorem of Nakaoka and our proof relies on his result.

Conjecture: this result holds for all (Abelian?) groups $G$.
References

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- H. Nakaoka
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Thanks for listening!