JMM 2024

Prime ideals in the Burnside Tambara functor

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part of the 2019 CMRG at Reed College

Burnside ring — Burnside Green functor — Burnside Tambara functor

fix a finite group G throughout

Burnside ring

Def.

The *Burnside ring* of G is

$$A(G) = \mathrm{Gr}(\mathrm{Fin}G)$$
$$= \mathbb{Z}[\{G/H\}_{H \leq G}]/\sim$$

with $+$ = disjoint union and $\times$ = Cartesian product

Ex.

$$A(C_p) \cong \mathbb{Z}[t_p]/(t_p^2 - pt_p)$$

for $t_p = C_p/e$

and $1 = C_p/C_p$

Thm (Dress).

The prime ideals of the Burnside ring are

$$\mathrm{Spec}(A(G)) = \{\ker \phi_q^H \mid H \leq G, q \text{ prime or } 0\}$$

where $\phi_q^H : A(G) \to \mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$

$$X \mapsto |X^H|$$

Ex.

$$\ker \phi_q^e = \langle t_p - p, q \rangle$$
$$\ker \phi_q^{C_p} = \langle t_p, q \rangle$$

Burnside Green functor

Def.

The *Burnside Green functor* of G is given by

$$\underline{A}_G = \{A(H)\}_{H \leq G}$$

with restrictions, transfers, and conjugation maps

Ex.

$\underline{A}_{C_p}$

$$A(C_p) \cong \mathbb{Z}[t_p]/(t_p^2 - pt_p)$$

T: $1 \mapsto t_p$

R: $t_p \mapsto p$

$$A(e) \cong \mathbb{Z}$$

Def.

An *ideal* is $\underline{I} = \{I(H)\}_{H \leq G}$ with $I(H) \subseteq A(H)$ an ideal so that $\underline{I}$ is closed under R, T, and c

An ideal $\underline{P}$ is *prime* if for any ideals $\underline{I}$ and $\underline{J}$, if $\underline{\mathrm{I}} \cdot \underline{\mathrm{J}} \subseteq \underline{P}$ then $\underline{\mathrm{I}} \subseteq \underline{P}$ or $\underline{\mathrm{J}} \subseteq \underline{P}$

Ex.

(nt_p)	$(t_p - p)$
(n)	(0)

Burnside Green functor

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$1 \mapsto t_p$

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$t_p \mapsto p$

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Def.

An *ideal* is $I = \{I(H)\}_{H \leq G}$ with $I(H) \subseteq A(H)$ an ideal so that I is closed under R, T, and c

An ideal P is *prime* if for any ideals I and J, if $\mathrm{I} \cdot \mathrm{J} \subseteq P$ then $\mathrm{I} \subseteq P$ or $\mathrm{J} \subseteq P$

Thm (Lewis).

$(t_p - p)$

(0)

Spec$(\underline{A}_G) = \{P(H,q)\}$ for $H \leq G$, q prime or 0

Burnside Tambara functor

Def.

The *Burnside* ~~*Green*~~ *Tambara functor* of G is given by

$$\underline{A}_G = \{A(H)\}_{H \leq G}$$

with restrictions, transfers, and conjugation maps …and norms

Ex. $\underline{A}_{C_p}$

$$A(C_p) \cong \mathbb{Z}[t_p]/(t_p^2 - pt_p)$$

T: $1 \mapsto t_p$

N

R: $t_p \mapsto p$

$$A(e) \cong \mathbb{Z}$$

Def.

An *ideal* is $I = \{I(H)\}_{H \leq G}$ with $I(H) \subseteq A(H)$ an ideal so that I is closed under R, T, and c …and N

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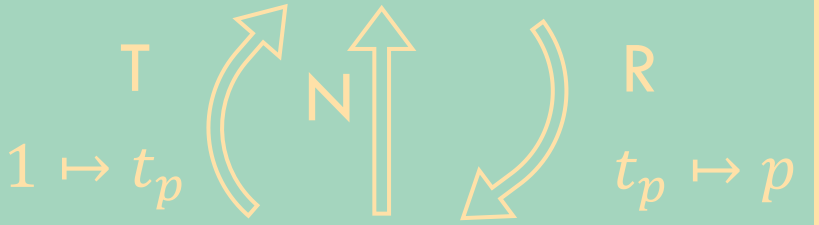
Ex. $\underline{A}_{C_p}$

$$A(C_p) \cong \mathbb{Z}[t_p]/(t_p^2 - pt_p)$$



$$A(e) \cong \mathbb{Z}$$

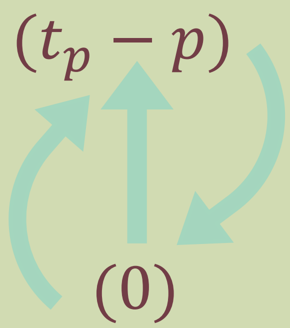
Def.

An *ideal* is $I = \{I(H)\}_{H \leq G}$ with $I(H) \subseteq A(H)$ an ideal so that I is closed under R, T, and c

…and N

An ideal P is *prime* if for any ideals I and J, if $\mathrm{I} \cdot \mathrm{J} \subseteq P$ then $\mathrm{I} \subseteq P$ or $\mathrm{J} \subseteq P$

Thm (C.-Ginnett).



$\mathrm{Spec}(\underline{A}_G) = \{P(H, q)\}$ for $G = C_N, H \leq G$, q prime or 0

Main Results

- For any finite group G,

$$\{P(H,p) \mid H \leq G, p \text{ prime or } 0\} \subseteq \operatorname{Spec}(\underline{A}_G)$$

built from $\ker \phi_p^K$ s

Main Results

- For any finite Abelian group G,

$$\{P(H,p) \mid H \leq G, p \text{ prime or } 0\} \subseteq \operatorname{Spec}(\underline{A}_G)$$

with certain containment relations.

- $P(H,0) \subseteq P(H,p)$
- $P(K,0) \subseteq P(H,0)$ if $H \leq K$
- $P(K,p) \subseteq P(H,p)$ if $H \leq_{\hat{p}} K$

look at "non-p part"

Main Results

- For $G = C_N$,

$$\{P(i,p) \mid i \mid n, p \text{ prime or } 0\} = \mathrm{Spec}(\underline{A}_G)$$

with certain containment relations.

- $P(i,0) \subseteq P(i,p)$
- $P(j,0) \subseteq P(i,0)$ if $i \mid j$
- $P(j,p) \subseteq P(i,p)$ if $i \mid_{\hat{p}} j$

look at "non-p part"

Main Results

- For $G = C_N$,

$$\{P(i,p) \mid i \mid n, p \text{ prime or } 0\} = \operatorname{Spec}(\underline{A}_G)$$

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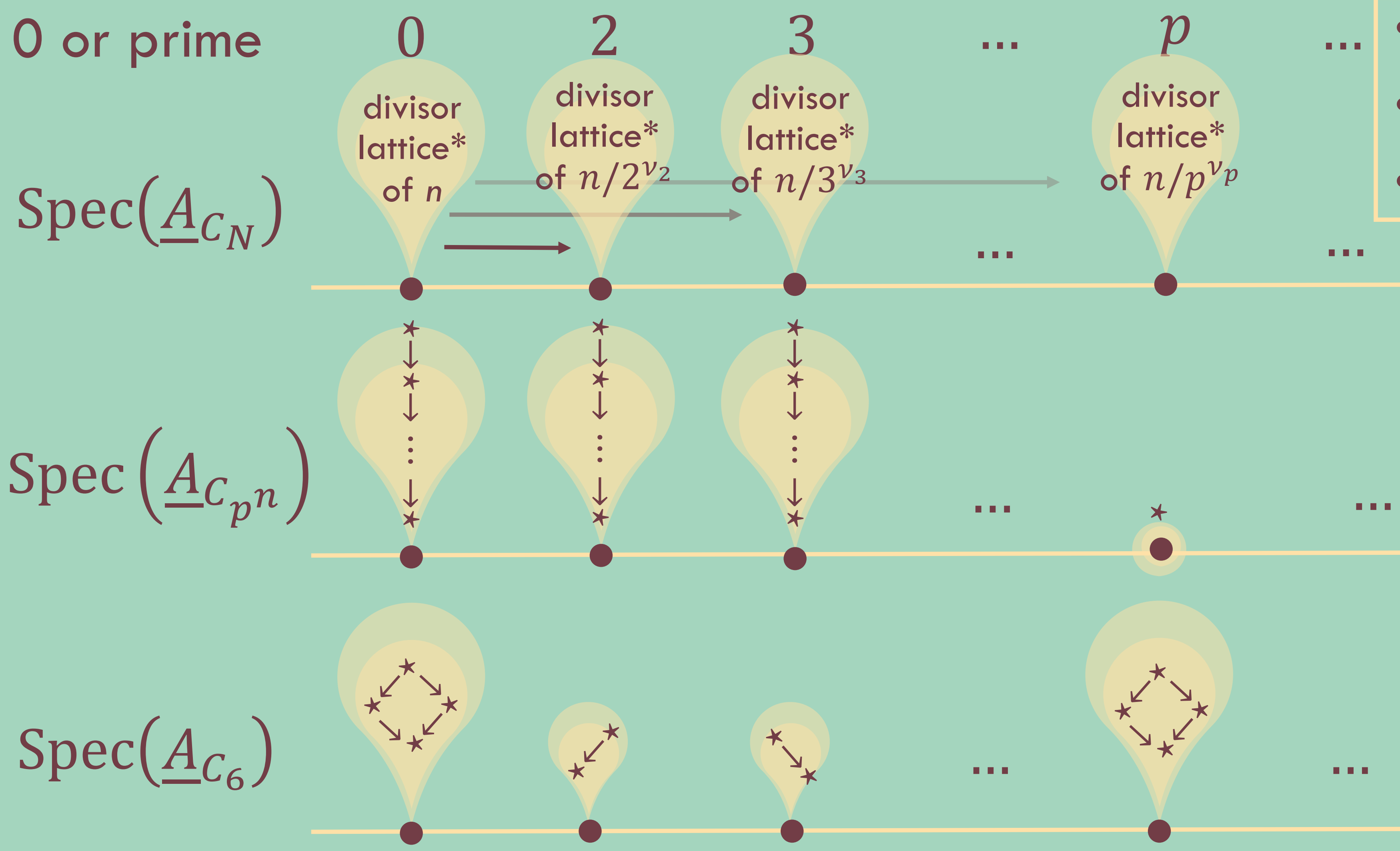
- $P(j,0) \subseteq P(i,0)$ if $i \mid j$
- $P(j,p) \subseteq P(i,p)$ if $i \mid_{\hat{p}} j$

if $i = p^k j$ then $P(j,p) = P(i,p)$

(for $p \neq 0$)

$$\frac{i}{p^{\nu_i}} \,\Big|\, \frac{j}{p^{\nu_j}}$$

Main Results



- $P(i,0) \subseteq P(i,p)$
- $P(j,0) \subseteq P(i,0)$ if $i \mid j$
- $P(j,p) \subseteq P(i,p)$ if $i \mid_{\hat{p}} j$

if $i = p^k j$ then $P(j,p) = P(i,p)$

the case $N = p^n$ is a theorem of Nakaoka and our proof relies on his result

Conjecture: this result holds for all (Abelian?) groups G

References

- M. Calle and S. Ginnett

  The spectrum of the Burnside Tambara functor of a cyclic group

  Journal of Pure and Applied Algebra: Vol. 27, Iss. 8 (2023)

- A. Dress

  Notes on the theory of finite groups

  Universität Bielefeld notes (1971)

- L. G. Lewis

  The theory of Green functors

  Unpublished notes (1980)

- H. Nakaoka

  The spectrum of the Burnside Tambara functor on a finite cyclic *p*-group

  Journal of Algebra: Vol. 398, Iss. 1 (2014)

Thanks for listening!