# AIM WORKSHOP 2024 TALK NOTES: THE SCISSORS CONGRUENCE $K$-THEORY OF POLYTOPES IS A THOM SPECTRUM

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The starting idea is that there is a category $\mathcal{P}_G^X$ whose $K$-theory encodes information about scissors congruence of polytopes in $X = \mathbb{E}^n, \mathbb{H}^n, S^n$ with respect to some subgroup $G$ of isometries of $X$. To simplify things, we will just focus on $X = \mathbb{E}^n, \mathbb{H}^n$.

**Definition 1.** Define $\mathcal{P}_G^X$ to be the category with covers[1] whose objects are polytopes in $X$ and whose morphisms are polytope inclusions (where we are allowed to act by the isometries in $G$ before including). The *covers* are those multimorphisms $\{f_i : P_i \to P\}_{i \in I}$ such that $P = \biguplus_{i \in I} f_i(P_i)$.

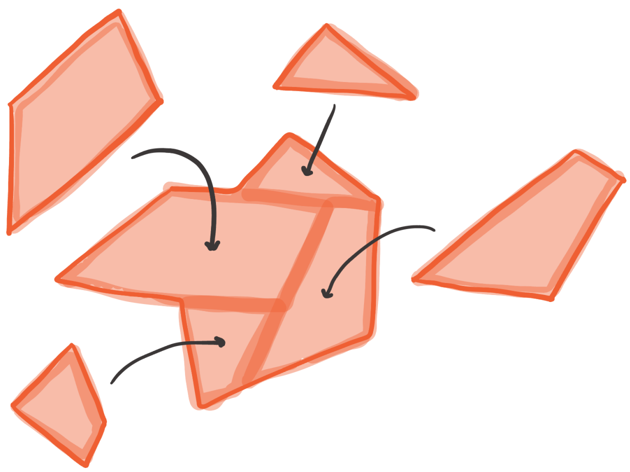

**Figure 1: An example of a cover in $\mathcal{P}_G^X$.**

The $K$-theory construction for categories with covers from [Boh+23] first produces a symmetric monoidal category called the *category of covers*. For $\mathcal{P}_G^X$, the category of covers $W(\mathcal{P}_G^X)$ has objects $\{P_i\}_{i \in I}$, finite tuples of polytopes in $X$, and morphisms $\{P_i\}_{i \in I} \to \{Q_j\}_{j \in J}$ consist of a set map $f : I \to J$ so that for each $j \in J$, there is a multimorphism

$$\{f_i : P_i \to Q_j\}_{i \in f^{-1}(j)}$$

which is a cover in $\mathcal{P}_G^X$. This is a symmetric monoidal category under disjoint union, and the $K$-theory of $\mathcal{P}_G^X$ is defined to be the Segal $K$-theory of $W(\mathcal{P}_G^X)$. The fact that the spectrum $K(\mathcal{P}_G^X)$ encodes scissors congruence problems is justified by the computations of the lowest two $K$-groups:

- There is an isomorphism $K_0(\mathcal{P}_G^X) \cong \mathcal{P}(X, G)$. Recall that $\mathcal{P}(X, G)$ is the *polytope algebra*, which is the free abelian group on polytopes in $X$ modulo the relations generated by $[P \uplus Q] = [P] + [Q]$ and $[P] = [g \cdot P]$ for $g \in G$. This identification follows from work of Zakharevich [Zak17b] (see also [Boh+23]).
- The next $K$-group, $K_1(\mathcal{P}_G^X)$, can be computed as the group of "scissors automorphisms" — all the ways that a polytope can be scissors congruent to itself. A partial presentation

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[1] The category $\mathcal{P}_G^X$ is actually an *assembler*, which is a stronger notion than a category with covers [Zak17b]. A crucial feature of assemblers is that any two covers of a given object admit a common refinement.

for $K_1(\mathcal{P}_G^X)$ follows from [Zak17a] and this presentation for $K_1$ (of an assembler) is shown to be complete in forthcoming work of Kupers–Lemann–Malkiewich–Miller–Sroka.

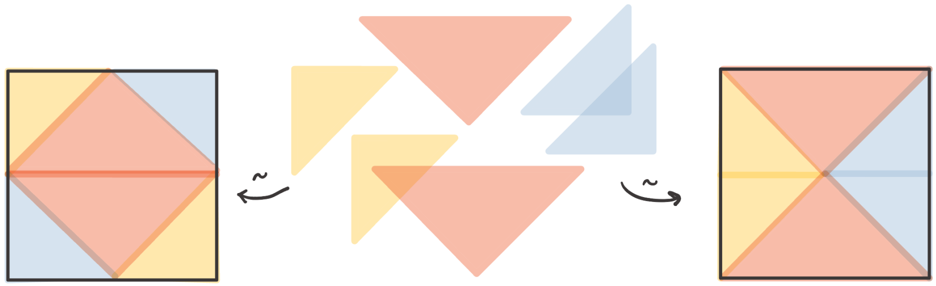


**Figure 2: An example of a scissors automorphism.**

The higher $K$-groups should thus encode "higher scissors congruence" problems, but $K$-theory is famously difficult to compute. However, in this case, $K(\mathcal{P}_G^X)$ turns out to be something more computable.[2] The first part of the argument reduces the problem from using arbitrary $G \leq \mathrm{Isom}(X)$ to the trivial subgroup.

**Theorem 2** ([Boh+23])**.** *There is an equivalence of spectra*

$$K(\mathcal{P}_G^X) \simeq K(\mathcal{P}_1^X)_{hG},$$

*where the subscript $hG$ denotes homotopy orbits.*

It thus suffices to understand $K(\mathcal{P}_1^X)$, the $K$-theory of "no moving" scissors congruence. The next theorem identifies this spectrum using ideas like the Barratt–Priddy–Quillen theorem, apartment maps, and abstract manipulations of spectra.

**Theorem 3** ([Mal23])**.** *There is an equivalence of spectra*

$$\begin{aligned} K(\mathcal{P}_1^X) &\simeq \bigvee \mathbb{S} \\ &\simeq PT(X)^{-TX} \\ &\to ST(X)^{-TX} \end{aligned}$$

*and the last map is an equivalence if $X = \mathbb{E}^n, \mathbb{H}^n$.*

The first equivalence that appears is joint work of Malkiewich–Zakharevich. Here, $ST(X)$ is the suspension of the Tits complex which (by the Solomon–Tits theorem) is also equivalent to a wedge of spheres. The space $PT(X)$ is constructed as a total homotopy cofiber

$$PT(X) = (\mathrm{hocolim}_{\varnothing \subsetneq U \subseteq X} U) \,/\, (\mathrm{hocolim}_{\varnothing \subsetneq U \subsetneq X} U)$$

where the indexing diagram is over geodesic subspaces (which are contractible for $X = \mathbb{E}^n, \mathbb{H}^n$) and inclusions. After making sure these identifications are suitably equivariant, this theorem combines with the previous one to produce an identification

$$K(\mathcal{P}_G^X) \simeq \left(ST(X)^{-TX}\right)_{hG}.$$

This result gives us access to computations, such as the following examples.

**Corollary 3.1** ([Boh+23])**.** *There is a trace*[3] *map $K_i(\mathcal{P}_G^X) \to H_i(G^\delta, \mathbb{R})$ where $G^\delta$ is $G$ regarded as a discrete group.*

**Theorem 4** ([Mal23])**.** *The spectrum homology is $H_*(K(\mathcal{P}_G^X)) \cong H_*(G^\delta, Pt(X)^t)$ and consequently*

$$K_i(\mathcal{P}_G^X) \otimes \mathbb{Q} \cong H_i(G^\delta, Pt(X)^t) \otimes \mathbb{Q}.$$

[2]Although what "more computable" means for homotopy theorists might make other mathematicians unhappy.

[3]Although Cary may want to call this a regulator.

If $G$ contains the subgroup $T(n)$ of translations, then the $K$-groups of $\mathcal{P}_G^X$ are rational, and in 1-dimensional geometries the $K$-groups for $G = T(1), \mathrm{Isom}(X)$ can be entirely computed as wedge sums of $\mathbb{R}$. The theorem above further motivates the study of the homology of infinite-groups-made-discrete.

The goal of this talk/note is not to explain these computations, but instead unpack Theorem 2 and Theorem 3, for the purposes of understanding the techniques involved so that they might be adapted to other situations.

**Getting rid of the $G$.** Theorem 2 is actually an instance of a more general theorem for any $G$-category with covers $\mathcal{C}$ (meaning the objects and morphisms have $G$-actions and these actions are compatible with the various structure maps).

**Definition 5.** Let $\mathcal{C}$ be a $G$-category with covers. The *Grothendieck construction* is the category with covers $\mathcal{C}_{hG} := G \int \mathcal{C}$ whose objects are the same as those of $\mathcal{C}$, but a morphism $(g, f)\colon c \to d$ is a pair of $g \in G$ and $f\colon g \cdot c \to d$. A multimorphism $\{(g_i, f_i)\colon c_i \to c\}_{i \in I}$ is a cover precisely when $\{f_i\colon g_i \cdot c_i \to c\}_{i \in I}$ is a cover in $\mathcal{C}$.

If $\mathcal{C}$ is a $G$-category with covers, its $K$-theory spectrum $K(\mathcal{C})$ is a spectrum with $G$-action. The following theorem says that the homotopy orbits[4] of this Borel $G$-spectrum are modeled by $K(\mathcal{C}_{hG})$.

**Theorem 6** ([Boh+23])**.** *For a $G$-category with covers $\mathcal{C}$, there is an equivalence of spectra*

$$K(\mathcal{C}_{hG}) \xrightarrow{\sim} K(\mathcal{C})_{hG}.$$

In other words, this $K$-theory construction commutes with homotopy colimits; this can be interpreted as an instance of Thomason's theorem which says that the Grothendieck construction models homotopy colimits in categories.

**Example 7.** Consider the category $\mathcal{P}_1^X$ of "no moving" scissors congruence, where the only morphisms allowed are literal subset inclusions of polytopes. This category with covers has a $G$-action for $G \leq \mathrm{Isom}(X)$, and so we may do the Grothendieck construction $(\mathcal{P}_1^X)_{hG}$. The objects of this category are again polytopes, but now we may first precompose by an isometry in $G$ before including; these are precisely the morphisms in $\mathcal{P}_G^X$. Similarly investigating the covers implies that $(\mathcal{P}_1^X)_{hG} \cong \mathcal{P}_G^X$. Theorem 2 is therefore a special case of the theorem above.

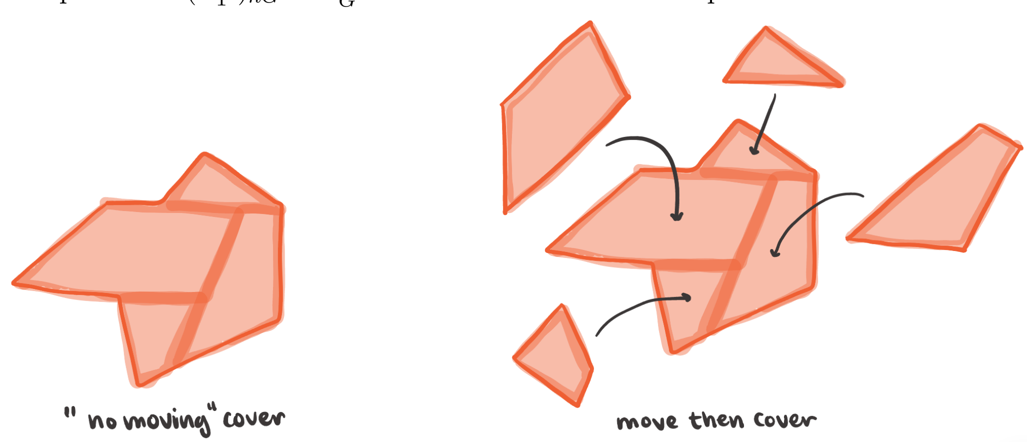



**Figure 3: A cover in $\mathcal{P}_1^X$ versus a cover in $(\mathcal{P}_1^X)_{hG}$.**

The trace map from Corollary 3.1 is induced by volume in the following way. View $\mathbb{R}$ as a category with covers $\mathcal{E}\mathbb{R}$ whose objects are real numbers and with a unique morphism between any two objects. There is a cover $\{x_i \to x\}_{i \in I}$ if $x = \sum_i x_i$, and the $K$-theory of this

[4]Recall that homotopy orbits are a homotopy coherent way of taking orbits. The important fact to know for our purposes is that if an equivariant map $f\colon X \to Y$ between $G$-spaces is an underlying equivalence (not necessarily on all fixed points), then $f$ induces an equivalence $X_{hG} \simeq Y_{hG}$.

category is the Eilenberg MacLane spectrum $H\mathbb{R}$. Then volume is a functor of $G$-categories with covers (where $\mathcal{E}\mathbb{R}$ has trivial $G$-action)

$$vol\colon \mathcal{P}_G^X \to \mathcal{E}\mathbb{R}$$

since $P = \biguplus_{i\in I} P_i$ implies $vol(P) = \sum_i vol(P_i)$ and $vol(g\cdot P) = vol(P)$ (i.e. covers go to covers). After taking $K$-theory, we obtain a map

$$K(\mathcal{P}_1^X)_{hG} \to (H\mathbb{R})_{hG} \simeq \Sigma_+^\infty BG \wedge H\mathbb{R}$$

yielding the trace map from Corollary 3.1 which can be used, e.g., to produce non-trivial elements in higher $K$-groups.

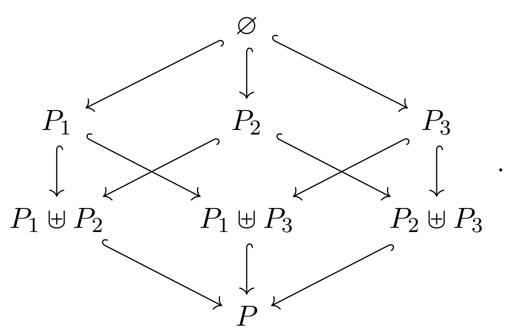
**Computing "no moving" $K$-theory.** The upshot of the previous subsection is that it suffices to study the $K$-theory of $\mathcal{P}_1^X$, as long as we do so in a suitably equivariant way. We will split up this computation (which is Theorem 3) into two pieces:

(1) $K(\mathcal{P}_1^X) \simeq \bigvee \mathbb{S}$,
(2) $K(\mathcal{P}_1^X) \simeq PT(X)^{-TX}$.

The first idea for (1) is to express the category $\mathcal{P}_1^X$ as a colimit over categories whose $K$-theory is easier to compute.

**Definition 8.** Let $\{P_i\}_{i\in I}$ be a finite collection of polytopes with disjoint interiors. Define $\mathcal{P}_{\{P_i\}}^X$ to be the full subcategory of $\mathcal{P}_1^X$ whose objects are polytopes of the form $Q = \biguplus_{j\in J\subseteq I} P_i$.

**Example 9.** Consider $P = P_1 \uplus P_2 \uplus P_3$. Then $\mathcal{P}_{\{P_i\}}^X$ is the following poset:




The observant reader may note that this just the poset for the power set of $I = \{1,2,3\}$.

There are two main kinds of "overlap" we can have between these categories. First, if $\{Q_j\} = \{P_i\} \amalg \{Q\}$, then there is an inclusion $\mathcal{P}_{\{P_i\}}^X \subseteq \mathcal{P}_{\{Q_j\}}^X$. Second, if $\{P_k'\} \to \{P_i\}$ is a cover, then $\mathcal{P}_{\{P_i\}}^X \subseteq \mathcal{P}_{\{P_k'\}}^X$. These generate all the possible overlaps, so if we glue along these identifications we obtain

$$P_1^X = \operatorname{colim} P_{\{P_i\}}^X.$$

*Remark* 10. This colimit is actually a filtered colimit, essentially because covers have common refinements. It follows that $K(\mathcal{P}_1^X) \simeq \operatorname{hocolim} K(\mathcal{P}_{\{P_i\}}^X)$ (see [Mal23, Lemma 4.2 and 4.3]).

Using the Barratt–Priddy–Quillen theorem, we can identify

$$K(\mathcal{P}_{\{P_i\}}^X) \simeq \prod_i K(\mathrm{FinSet}) \simeq \prod_i \mathbb{S}.$$

Hence $K(\mathcal{P}_1^X) \simeq \operatorname{hocolim} \prod_i \mathbb{S}$ and the gluing maps become inclusion of factors (corresponding to the first type of overlap) and diagonal maps on $\mathbb{S}$ (corresponding to the second). The final step to show is that $\operatorname{hocolim} \prod_i \mathbb{S}$ is equivalent to a wedge of spheres (see [Mal23, Theorem 4.8]). The proof of this result makes use of the fact that $\pi_0(K(\mathcal{P}_1^X)) \cong P(X,1)$ is a free Abelian group.

For (2), we want to show that $K(\mathcal{P}_1^X) \simeq PT(X)^{-TX}$. The first reduction is that (for $X = \mathbb{E}^n, \mathbb{H}^n$) it instead suffices to show

$$\Sigma^X K(\mathcal{P}_1^X) \simeq \Sigma^\infty PT(X)$$

in a suitably equivariant way. By $\Sigma^X$, we mean smashing with $S^X$, the one-point compactification of $X$. From the proof of (1), we know

$$\begin{aligned}\Sigma^X K(\mathcal{P}_1^X) &\simeq \Sigma^X \operatorname{hocolim} \prod_i \mathbb{S} \\ &= S^X \wedge \operatorname{hocolim} \prod_i \Sigma^\infty(S^0) \\ &\simeq \operatorname{hocolim} \prod_i \Sigma^\infty(S^X).\end{aligned}$$

Now, we observe that $S^X \simeq P/\partial P$ for every polytope $P \subseteq X$ via a collapse map. The maps that were inclusions of factors remain inclusions of factors, and the maps that were diagonals become pinch maps (this can be seen by considering what happens when $P$ is covered by smaller pieces).

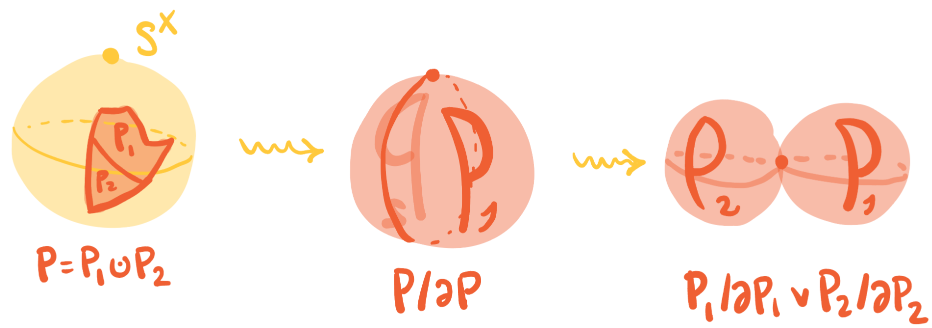



**Figure 4: The collapse map $S^X \xrightarrow{\sim} P/\partial P$ turns covers into wedges.**

Remarkably, there are no coherence issues! We thus continue with the chain of equivalences:

$$\begin{aligned}&\simeq \operatorname{hocolim} \prod_i \Sigma^\infty(P_i/\partial P_i) \\ &\simeq \Sigma^\infty \operatorname{hocolim} \prod_i P_i/\partial P_i \\ &\simeq \Sigma^\infty \operatorname{hocolim} \bigvee_i P_i/\partial P_i.\end{aligned}$$

The upshot is that it suffices to study the space $\operatorname{hocolim} \bigvee_i P_i/\partial P_i$ and show it is stably equivalent to $PT(X)$. This is where apartment maps come in. We will describe the map in the case that $P$ is a simplex — in fact, a 2-simplex (aka a triangle) since that's not too hard to visualize.

Recall that the apartment map on $P = \operatorname{span}\{x_0, x_1, x_2\}$ sends a vertex $x_i$ to the 0-cell in $T(X)$ corresponding to the 0-dimensional subspace $\{\varnothing \subsetneq \operatorname{span}\{x_i\} \subseteq X\}$. In $ST(X)$, there isn't a single 1-cell connecting $\{\varnothing \subsetneq \operatorname{span}\{x_0\} \subseteq X\}$ with $\{\varnothing \subsetneq \operatorname{span}\{x_1\} \subseteq X\}$, but there is a span

$$\{\varnothing \subsetneq \operatorname{span}\{x_0\} \subseteq X\} \xrightarrow{\subsetneq} \{\varnothing \subsetneq \operatorname{span}\{x_0, x_1\} \subseteq X\} \xleftarrow{\supsetneq} \{\varnothing \subsetneq \operatorname{span}\{x_1\} \subseteq X\}$$

and so we may send the edge connecting $x_0$ and $x_1$ in $P$ to the path in $ST(X)$ determined by this span. Note that there is a choice of parametrization, but the natural choice sends the middle of the edge connecting $x_0$ and $x_1$ to the 0-cell $\{\varnothing \subsetneq \operatorname{span}\{x_0, x_1\} \subseteq X\}$. Continuing in this way, we may view $P$ as a subspace of the cone $CT(X) = \operatorname{hocolim}_{\varnothing \subsetneq U \subseteq X} U$; in particular, if we first barycentrically subdivide $P$, then the apartment map is the inclusion of $sd(P)$ as a sub-complex. Note that the barycenter of $P$ is sent to the 0-cell $\{\varnothing \subsetneq X \subseteq X\}$ and the boundary of $P$ is sent to a subcomplex of $\{\varnothing \subsetneq U \subsetneq X\}$. Hence the apartment map for $P$ defines $P/\partial P \to ST(X)$.

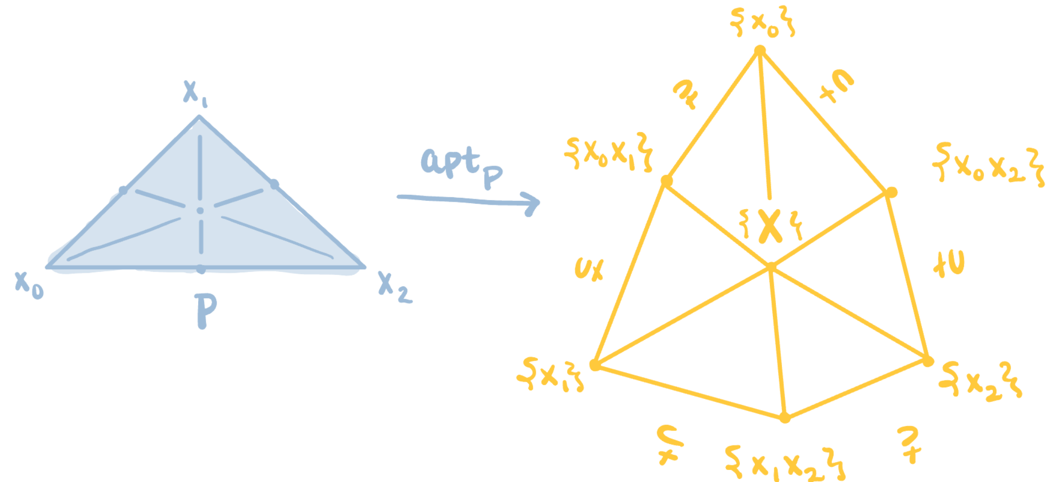



**Figure 5: The apartment map on $P$.**

This defines one map from $P/\partial P \to ST(X)$, but as noted above there are choices of parametrization. And subdividing $P$ (covering it by smaller triangles) induces a different map $P/\partial P \to ST(X)$ than the apartment map on $P$; in particular, the map induced by a cover $P = \biguplus_{i\in I} P_i$ is the apartment maps of each $P_i$ "glued together."

For instance, if we cover $P$ by two triangles $P_1, P_2$ as in the figure below, the apartment map on $P_1$ glued to the apartment map on $P_2$ is not the apartment map on $P$, but it is homotopic to it.

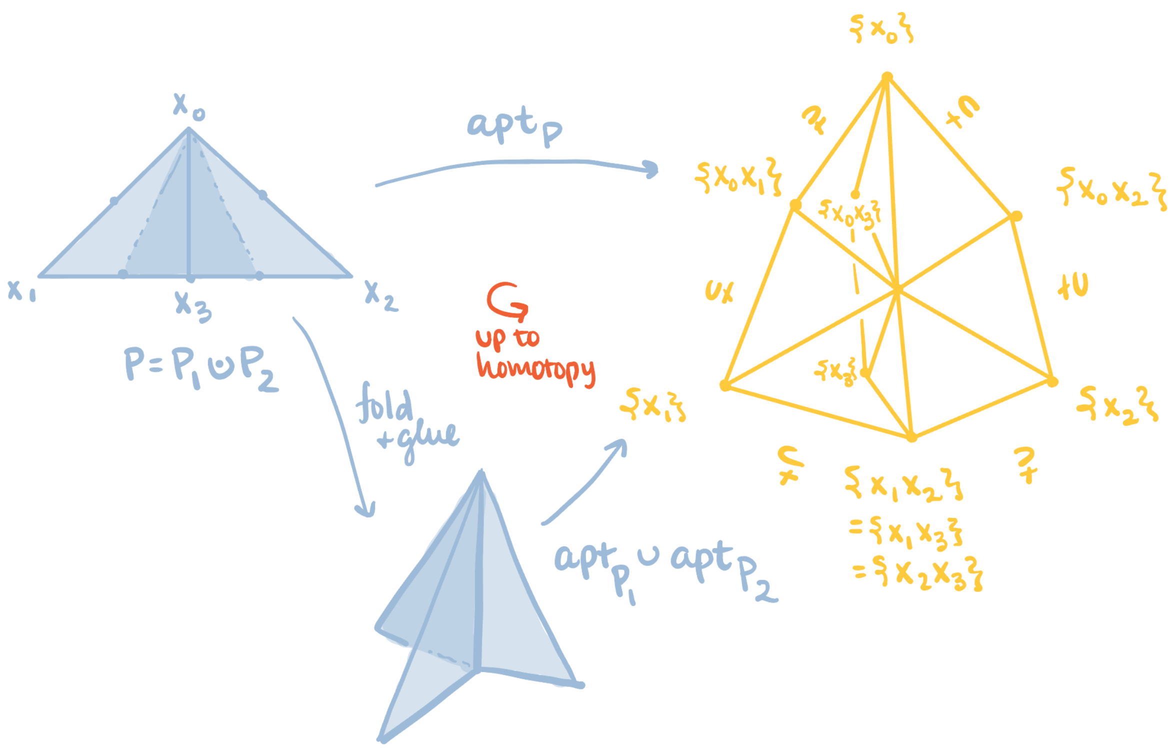
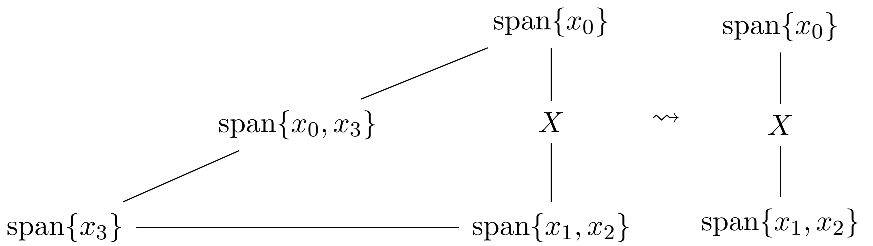



**Figure 6: The apartmentlike map on $P$ specified by the cover $P = P_1 \uplus P_2$.**

In particular, the new vertex $x_3$ in the middle of the edge $x_1x_2$ is sent to the 0-cell $\{\varnothing \subsetneq \text{span}\{x_3\} \subseteq X\}$ whereas the apartment map for $P$ sends that point to the 0-cell $\{\varnothing \subsetneq \text{span}\{x_1, x_2\} \subseteq X\}$. However, since $\text{span}\{x_1, x_2\} = \text{span}\{x_1, x_3\} = \text{span}\{x_2, x_3\}$, there is a homotopy between the apartment map defined by this cover and the apartment map defined by $P$ which crushes the triangle to its edge:




Every cover defines a similarly "glued together apartment map" which is not the apartment map on $P$, but it is *apartment-like*[5] in that it sends every face of $P$ to the subcomplex of $\operatorname{hocolim}_{\varnothing \subsetneq U \subseteq X} U$ that contains the span of that face. It turns out that the space $\mathcal{A}$ of apartment-like maps on $\operatorname{hocolim} \bigvee_i P_i/\partial P_i$ is contractible, so up to homotopy there is a unique choice, and moreover any particular apartment-like map $\operatorname{hocolim} \bigvee_i P_i/\partial P_i \to ST(X)$ is an equivalence.

**Bringing the $G$ back.** In order to combine Theorem 2 and Theorem 3 to compute $K(\mathcal{P}_G^X)$, we need to ensure all the equivalences in Theorem 3 are suitably equivariant. Here, "suitably" just means Borel, i.e. it is enough to construct an equivariant map which is an underlying homotopy equivalence. Almost everything goes through just fine except for the final step

$$\operatorname{hocolim} \bigvee_i P_i/\partial P_i \to PT(X)$$

which is usually not equivariant. However, the space $\mathcal{A}$ of apartment-like maps has a $G$-action by congugation, and there is a coarse equivalence $EG \to \mathcal{A}$. The adjoint of this coarse equivalence induces an equivariant map

$$EG_+ \wedge \operatorname{hocolim} \bigvee_i P_i/\partial P_i \to PT(X)$$

which is an equivalence since it is an equivalence pointwise and $EG \simeq *$. Consequently, there is a zig-zag of Borel $G$-spectra between $\Sigma^X K(\mathcal{P}_G^X)$ and $\Sigma^\infty PT(X)$ which induces

$$K(\mathcal{P}_G^X) \simeq K(\mathcal{P}_1^X)_{hG} \simeq (PT(X))_{hG}^{-TX}.$$

## References

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[Zak17b] Inna Zakharevich. "The $K$-theory of assemblers". In: *Advances in Mathematics* 304 (2017), pp. 1176–1218. arXiv: 1401.3712.

[5]In early versions of [Mal23], such maps are instead called *introspective*.