# The Impact of John Nash on Economics and Game Theory

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# Introduction

- John Forbes Nash, Jr., died on May 23, 2015 in a car crash. He was returning from Norway, where he had been awarded the 2015 Abel Prize (which, with the Fields Medal, is the mathematics' Nobel prize, for “for striking and seminal contributions to the theory of nonlinear partial differential equations and its applications to geometric analysis”).
- In 1994, Nash was awarded the Nobel Memorial Prize in Economic Sciences (along with John Harsanyi and Reinhard Selten) .

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- My goal is to explain why a mathematician received the Nobel prize, and why it was shared.
- There is also a tragic and compelling story of Nash's mental illness and recover, well described in *A Beautiful Mind* by Sylvia Nassar (1998); the book was subsequently made into a movie with Russell Crowe playing Nash.

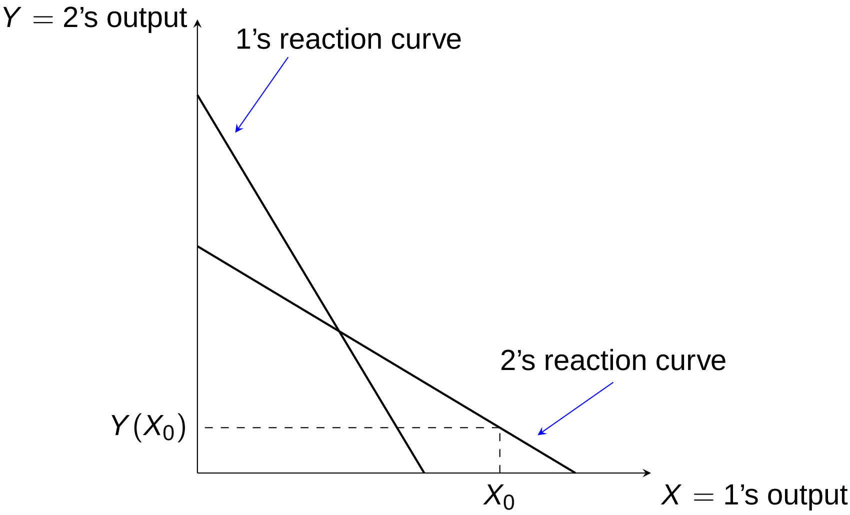
# Neoclassical economics

- Brutal summary: in the first half of the Twentieth Century, economics had made tremendous advances in our understanding of
  - preferences and ordinal utility,
  - competitive decision making, and
  - value theory (marginal value in consumption rather than labor theory of value).
- Broadly speaking, this went under the rubric of price theory.
- The partial equilibrium analysis of individual markets was also extended to general equilibrium analysis.
- Key to this is the notion of competitive equilibrium, which assumes buyers and sellers behave nonstrategically.

# Strategic Behavior

- Essentially, no formal modeling of strategic interactions.
- There were some attempts to models strategic behavior of firms, beginning with Cournot (1838).
- Cournot (1838) described two owners of water springs, each choosing a quantity to produce.

# Cournot dynamics and stable equilibrium



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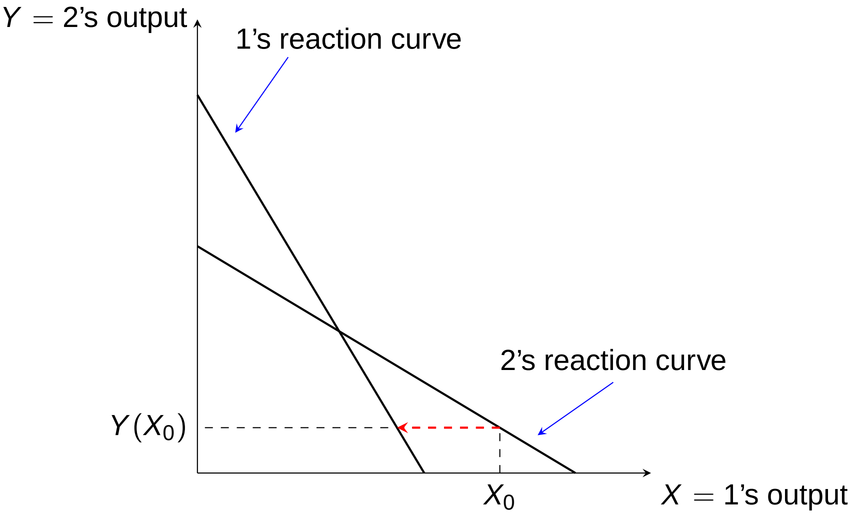

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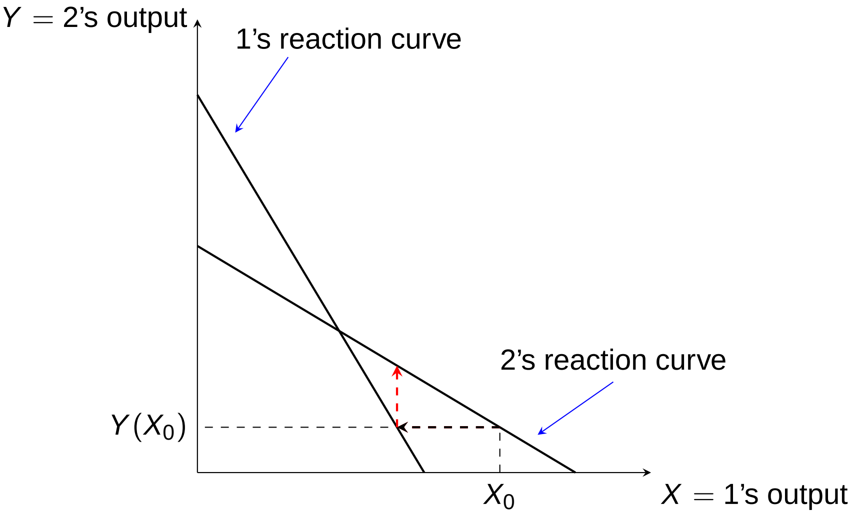

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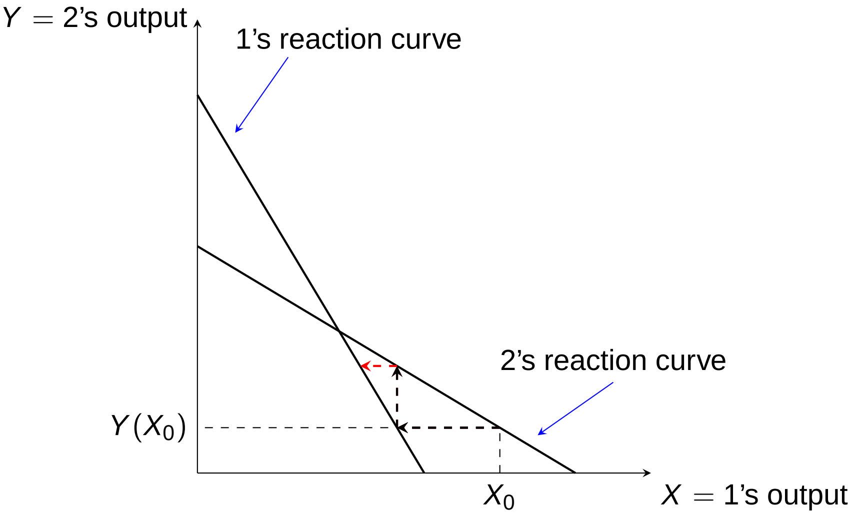

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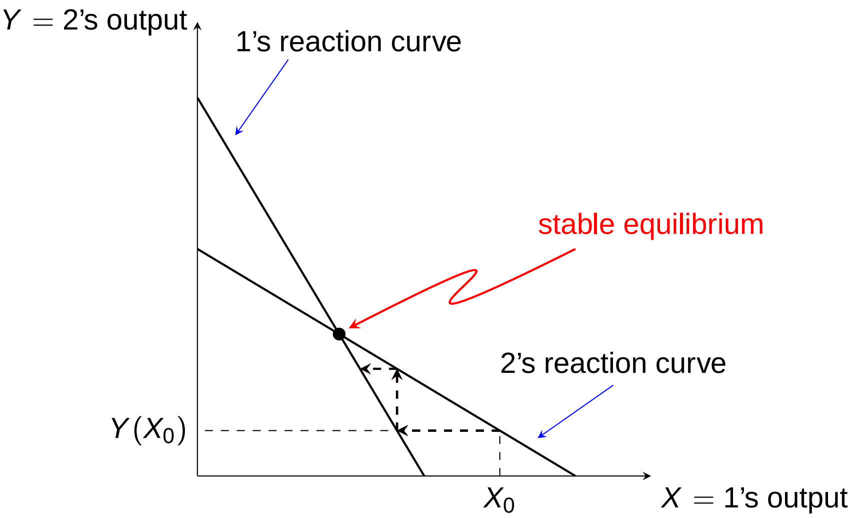

# Criticisms

- Fellner (1949): It is contradictory for each firm to act as if its competitor's quantity will not respond to changes in the firm's output.
- Myopic optimization.
- How to model firm behavior?

# Game Theory

The formal analysis of strategic behavior.

- Borel (1921) introduced the notion of a method of play (méthode de jeu): “a code that determines for every possible circumstance...exactly what the person should do.”
- von Neumann (1928) provided a complete explication of this idea (Spielmethode), leading to the modern notion of strategy and normal form: a strategy gives a complete contingent plan of behavior. Since a strategy can be chosen before game begins, can think of all players as simultaneously choosing their strategies.
- von Neumann (1928) also proves the celebrated minimax theorem for zero-sum games.

# Zero-Sum Games

- Games with strictly opposing interests. Games with a winner and a loser (battles, chess, poker, ...)
- Two player game: $((S_1, u_1), (S_2, u_2))$, where
  - $S_i$ is player $i$, $i = 1, 2$, is player $i$ strategy space, and
  - $u_i : S_1 \times S_2 \rightarrow \Re$ is player $i$'s payoff function ($u_i(s_1, s_2)$ is player $i$'s reward when 1 plays $s_1$ and 2 plays $s_2$).
- The game is zero sum if

$$u_1(s_1, s_2) = -u_2(s_1, s_2).$$

# Example

| | $L$ | $C$ | $R$ |
|---|---|---|---|
| $T$ | $1, -1$ | $3, -3$ | $2, -2$ |
| $M$ | $-1, 1$ | $5, -5$ | $-2, 2$ |
| $B$ | $0, 0$ | $4, -4$ | $3, -3$ |

- How should the row player play? How should the column player play?
- The row player wishes to maximize his payoff ($T$ is a good choice against $L$, while $M$ and $B$ are good choices against $C$).
- The column player wishes to minimize row player's payoff ($L$ is a good choice against $T$ and $B$, while $R$ is a good choice against $M$).

# Min Max I

- The row player can guarantee himself

$$\max_{s_1} \min_{s_2} u_1(s_1, s_2).$$

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- *TL* is a sensible prediction, since

$$\max_{s_1} \min_{s_2} u_1(s_1, s_2) = \min_{s_2} \max_{s_1} u_1(s_1, s_2).$$

# Min Max III

- In general, all we have is

$$\max_{s_1}\min_{s_2} u_1(s_1, s_2) \leq \min_{s_2}\max_{s_1} u_1(s_1, s_2).$$

- In matching pennies,

$$\max_{s_1}\min_{s_2} u_1(s_1, s_2) = -1 < 1 = \min_{s_2}\max_{s_1} u_1(s_1, s_2).$$

| | $H$ | $T$ |
|---|---|---|
| $H$ | $1, -1$ | $-1, 1$ |
| $T$ | $-1, 1$ | $1, -1$ |

# The Min Max Theorem

- But suppose players can randomize (play mixed strategies).
- Player $i$'s mixed strategy is $\sigma_i \in \Delta(S_i)$, where $\sigma_i(s_i)$ is the prob $i$ chooses $s_i$.
- Payoffs from $(\sigma_1, \sigma_2)$ are expected payoffs.

**Theorem (von Neumann 1928)**

*Suppose $S_1$ and $S_2$ are finite. Then,*

$$\max_{\sigma_1 \in \Delta(S_1)} \min_{\sigma_2 \in \Delta(S_2)} u_1(\sigma_1, \sigma_2) = \min_{\sigma_2 \in \Delta(S_2)} \max_{\sigma_1 \in \Delta(S_1)} u_1(\sigma_1, \sigma_2).$$

*Equivalently, there exists a mixed strategy profile $(\sigma_1^*, \sigma_2^*)$ such that*

$$u_1(s_1, \sigma_2^*) \leq u_1(\sigma_1^*, \sigma_2^*) \leq u_1(\sigma_1^*, s_2) \qquad \forall s_1 \in S_1, s_2 \in S_2.$$

# *Theory of Games and Economic Behavior*

von Neumann and Morgenstern (1944, first edition)

- Introduced the formal analysis of coalitions (and expected utility in 1947 edition, to make sense of expected payoffs).
- The book switches focus from individual optimizing behavior to coalitions, and the maximum payoff that each coalition can guarantee itself.
- von N-M proposed a solution set (the von N-M stable set), a collection of specifications of payoffs to each player that captured notions of coalitional stability.
- But difficult to calculate, and can be empty (though this was an open question till 1969).

# Game Theory in 1948

The state of game theory when Nash went to the Princeton Mathematics Department to do his Ph.D. in 1948:

- The notion of strategy (and the associated notion of the normal form).
- *Theory of Games and Economic Behavior* was viewed by many as a transformative book, introducing the formal analysis of strategic behavior, conflict and cooperation.
    - Popularized by a page one article on a Sunday edition of *The New York Times* (10 March 1946).
- But there was no analysis of individual optimizing behavior in games with either more than two players or nonzero-sum payoffs.

# Nash's contributions

- Noncooperative Games, Ph.D. Dissertation Princeton, 1950, and *Annals of Mathematics* 1951 (announced in PNAS 1950).
  - Introduced the distinction between noncooperative and cooperative game theory.
  - Defined equilibrium point (now called "Nash equilibrium"), the sine qua non for analysis of individual optimizing behavior in general games, and proved existence for finite normal form games.
- The Bargaining Problem, *Econometrica* 1950.
  - Introduced an axiomatic approach to bargaining, and proved it uniquely characterized a solution, now called the "Nash bargaining solution."
- Two Person Cooperative Games, *Econometrica* 1953.
  - Provided a link between cooperative and non-cooperative (bargaining) theory (leading to the "Nash program").

# Cooperative and Noncooperative Games

*"This book [von Neumann and Morgenstern] also contains a theory of n-person games of a type which we would call cooperative. This theory is based on an analysis of the inter-relationships of the various coalitions which can be formed by the players of the game.*

*Our theory, in contradistinction, is based on the absence of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others."*

Nash 1951

# Equilibrium Points

## Definition

Given an $n$ player game in normal form, $((S_1, u_1), \ldots, (S_n, u_n))$, a strategy profile $(s_1^*, \ldots, s_n^*)$ is an equilibrium point if, for all players $i$ and all strategies $s_i \in S_i$,

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## Theorem

*Every finite normal form game has an equilibrium point (in mixed strategies).*

# The Proof

- The first proof applied Kakutani’s fixed point theorem.
- The second proof applied Brouwer’s fixed point theorem, and was the basis of the first satisfactory proofs of the existence of Walrasian equilibrium).

# Examples

- The prisoners' dilemma as a partnership game:

| | $E$ | $S$ |
|---|---|---|
| $E$ | $2, 2$ | $-1, 3$ |
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  - $S$ strictly dominates $E$ and so $SS$ is a Nash equilibrium.
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- The stag hunt (Rousseau)

| | Stag | Hare |
|---|---|---|
| Stag | $9, 9$ | $0, 5$ |
| Hare | $5, 0$ | $5, 5$ |

  - Jointly hunting the stag is socially efficient, but risky.

# Cournot?

- Viewed as a one period interaction, Cournot's stable equilibrium is a Nash equilibrium.
- But that is not how Cournot viewed it.
- And Nash, following in the footsteps of von Neumann and Morgenstern, is clear that the solution has full generality, appropriate for analyzing any strategic interaction, not just duopoly.

# The Problem with Noncooperative Games

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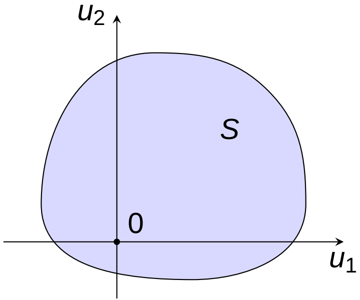
- Since everything needs to be specified, how to model bargaining?
- Two agents, 1 and 2, must agree on an outcome $x \in X$. If no agreement, an inefficient outcome $d \in X$ results.
- Suppose each agent has a payoff function $u_i$ defined on $X$.
- Then, the two agents must choose a point in the set of feasible utilities $S = \{(u_1(x), u_2(x)) : x \in X\}$. Suppose $S$ is convex ("nice") and normalize $u_i(d) = 0$.



# Nash's (bargaining) solution

## Nash 1950

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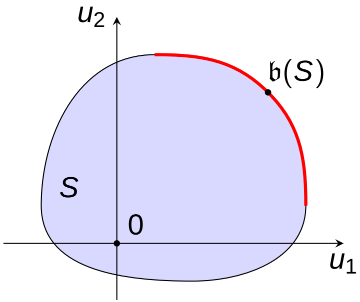

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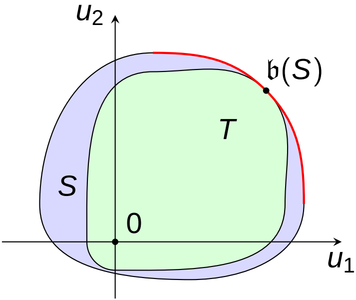
3. if $S$ is symmetric, then $\mathfrak{b}_1(S) = \mathfrak{b}_2(S)$, and
4. If $T \subset S$, and $\mathfrak{b}(S) \in T$, then $\mathfrak{b}(T) = \mathfrak{b}(S)$.



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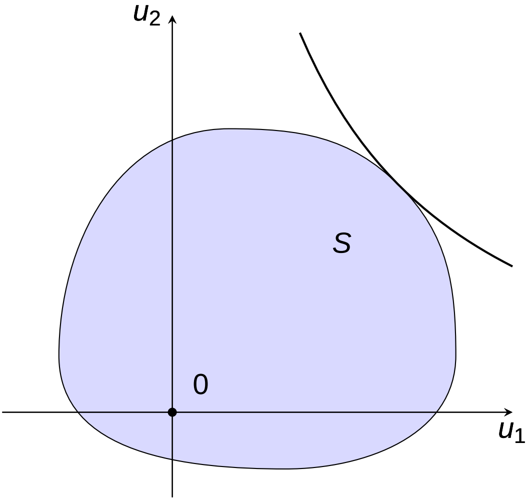
4. If $T \subset S$, and $\mathfrak{b}(S) \in T$, then $\mathfrak{b}(T) = \mathfrak{b}(S)$.

Then, $\mathfrak{b}(S, d)$ is unique and solves

$$\max_{(u_1, u_2) \in S} u_1 u_2.$$

# Maximizing the Nash product



# Nash Program

The Nash demand game:

- Each player simultaneously announces a demand $\hat{u}_i$.
- If $(\hat{u}_1, \hat{u}_2) \in S$, then each player receives his demand.
- If $(\hat{u}_1, \hat{u}_2) \notin S$, then each player receives zero.

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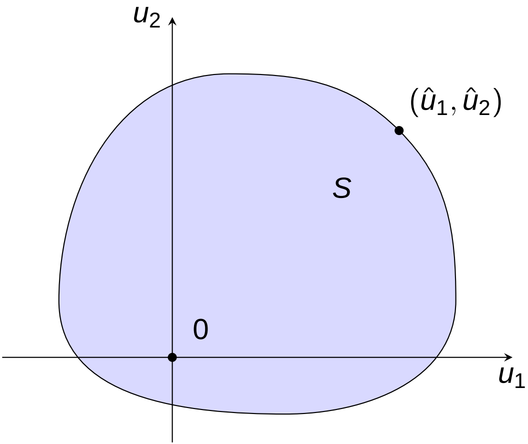
- Game has a lot of equilibria.

# Equilibrium selection




The demand $(\hat{u}_1, \hat{u}_2)$ is feasible under $S$.

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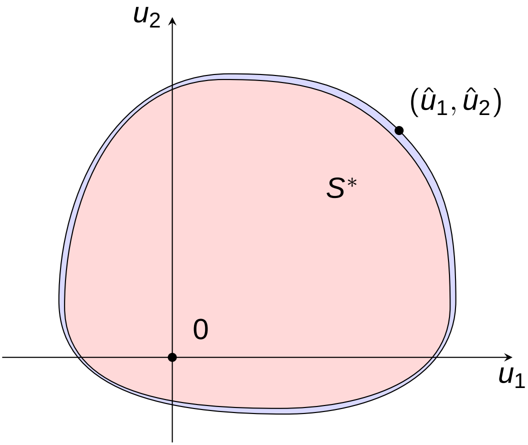


The demand $(\hat{u}_1, \hat{u}_2)$ is not feasible under $S^*$.
All equilibria converge to Nash bargaining solution as probability of $S$ converges to 1.

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The theory of noncooperative games after Nash in the late fifties still had important shortcomings:

- Some Nash equilibria are implausible.
- The simultaneous choice representation in the normal form appeared to ignore dynamic issues related to the credibility of promises of future rewards and punishments.
- The normal form representation appears to require that all players (when simultaneously choosing their strategies at the beginning) have identical ex ante information.

# Implausible equilibria and sequential irrationality

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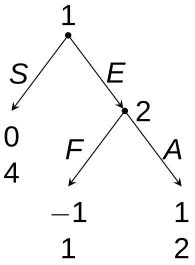


| | $F$ | $A$ |
|---|---|---|
| $S$ | $0, 4$ | $0, 4$ |
| $E$ | $-1, 1$ | $1, 2$ |

- Selten (1965, 1975) showed how to appropriately refine Nash equilibrium to eliminate implausible eq and capture sequential rationality.

# Ex ante symmetric information

- Harsanyi (1965-68) showed how to model settings in which players are ex ante asymmetrically informed, and how this modelling (by constructing a so-called Bayesian game), the existing tools and insights of noncooperative game theory can be used to analyze games of incomplete information.
- This also provided a deeper interpretation of mixed strategies, in which players do not actually use roulette wheels (Harsanyi, 1973).

# And now?

- Tremendous success in the application of Bayesian games to the study and implementation of auctions.
- Provides the basis for modern studies of institutions, organizations, and the internal organization of firms.
- Besides economics and related fields (such as finance, management, operations research, and political science), Nash equilibrium plays an important role in fields as diverse as philosophy, linguisitics, computer science, and biology.