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Who Wants a Good Reputation?

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We examine a market in which long-lived firms face a short-term incentive to exert low effort, but could earn higher profits if it were possible to commit to high effort. There are two types of firms, "inept" firms who can only exert low effort, and "competent" firms who have a choice between high and low effort. There is occasional exit, and competent and inept potential entrants compete for the right to inherit the departing firm's reputation. Consumers receive noisy signals of effort choice, and so competent firms choose high effort in an attempt to distinguish themselves from inept firms. A competent firm is most likely to enter the market by purchasing an average reputation, in the hopes of building it into a good reputation, than either a very low reputation or a very high reputation. Inept firms, in contrast, find it more profitable to either buy high reputations and deplete them or buy low reputations.

1. INTRODUCTION

Describing an eventually unsuccessful joint venture between Time Inc. and a former advertising executive who was otherwise unconnected with Time, the *Wall Street Journal* commented, "More significantly, the company had given him the right to exploit one of its most prized assets: the formidable Time Life name."[1] This was the first time that Time Inc. had licensed the Time Life name to an outsider. "Countless copycat investors ... jumped in largely on the strength of the Time Life brand name ... ."

Why doesn't access to a good reputation ensure success, as the copycat investors hoped it would? How do firms and entrepreneurs, such as the advertising executive, value the opportunity to exploit a good reputation? Which firms find good reputations most valuable?

This paper addresses these questions. We examine a market in which firms face a moral hazard problem: they have a short-term incentive to exert low effort, but could earn higher profits if it were possible to commit to high effort. We view reputation as a commitment device that allows firms to solve the moral hazard problem. In particular, there are two types of firms in our model, "inept" firms, who can only exert low effort, and "competent" firms, who have a choice between high and low effort. Consumers observe noisy signals of these effort choices. We focus on equilibria in which competent firms choose high effort in order to distinguish themselves from inept firms, and we interpret the consumers' posterior expectation that a firm is competent as the firm's *reputation*.

Since consumers receive noisy signals of effort choice, competent firms can at best be only partially successful in their attempts to separate themselves from inept firms. Indeed,

1. Both quotes are from "Tale of the Tapes: How One Media Deal Became Hazardous To Investors' Health," *Wall Street Journal* 136 (31), February 13, 1997, page 1.

this lack of precise signals would destroy the ability to build a reputation altogether were it not for an aspect of the Time Life example that plays a key role in our analysis: while customers had no way of knowing that Time Life provided only its name to the venture, leaving all operational decisions in the hands of the outsider, customers also could not exclude such a possibility. We capture this feature by assuming that firms occasionally leave the market. Competent and inept potential entrants bid for the right to a departing firm's name (and so reputation), with consumers sometimes being unable to observe such replacements. These replacements not only allow reputation building to occur, but also provide a simple model of the market for reputations. We show that competent firms find average reputations most valuable, in that an entering competent firm is more likely to purchase an average reputation, in the hopes of building it into a good reputation, than either a low reputation or a high reputation. Inept firms, in contrast, find it profitable to either buy high reputations and deplete them (reminiscent of the Time Life example) or buy low reputations, which remain low.

Reputation as an asset. We view a reputation as an asset which, like more familiar physical and financial assets, requires investment to create and maintain. Our model accordingly aims to capture three features:

First, reputations are built gradually. The consumer goodwill or trust that lies at the heart of a reputation accumulates in response to sustained, high-quality performance. Only after the firm has compiled a record of high quality does it enjoy the fruits of its reputation.

Second, reputations dissipate gradually. A firm that stops investing in its physical capital stock typically does not experience an immediate decline in productivity. Similarly, a firm that no longer maintains its reputation can initially rest on its laurels, with its reputation only gradually losing value as consumers adjust to the new performance level.

Finally, reputations can be managed. A firm's physical-capital investment profile is seldom uniform over its lifetime, with relatively high initial investment levels often giving way to lower investment levels when the firm is mature. Similarly, we expect firms to manage their reputations, with initial periods of high investment in reputation building possibly being followed by subsequent periods in which the reputation is sustained with lower investment levels. As a result, it may often be the case that "number two" tries harder.[2]

The standard approach to reputations posits that consumers believe the firm may be a "commitment" or "Stackelberg" type.[3] In our setting, a Stackelberg firm always chooses high effort.[4] Competent firms then acquire their reputations by masquerading as Stackelberg firms. It is a straightforward implication of Fudenberg and Levine (1992) that a sufficiently patient competent firm in such an environment achieves an average discounted payoff close to its commitment payoff (*i.e.* the payoff if it could credibly commit to the Stackelberg action). To obtain this payoff, the competent firm spends long periods of time choosing high effort with high probability.

2. The analysis closest in spirit to ours is Gale and Rosenthal (1994), who examine a model in which a firm gradually acquires a reputation and then depletes it upon learning that an exogenously generated shock will soon force the firm out of the market. They do not discuss the induced market for reputations. Papers which adopt a quite different approach to the need to build relationships from small beginnings include Datta (1996), Diamond (1989), and Watson (1995, 1999).

3. This approach was pioneered by Kreps, Milgrom, Roberts and Wilson (1982), Kreps and Wilson (1982), and Milgrom and Roberts (1982).

4. More generally, the Stackelberg action is the firm's optimal action if its choice is observed by the consumers (so the firm behaves as a "Stackelberg leader").

In the simplest standard reputation model with only two types, competent (or "ordinary") and Stackelberg, there is no reputation *building*. Instead, reputations spring to life, in the sense that a *de novo* firm begins with consumers immediately assigning high probability to the firm's choosing high effort, and with the probability assigned to the firm's being a Stackelberg type then steadily declining.[5] However, equilibria that exhibit reputation building can be constructed by expanding the model to include three types, a Stackelberg type, a competent type, and an inept type (who always chooses the myopically optimal action). In the initial periods, consumers may then put substantial probability on the firm being inept, and an ordinary firm builds its reputation as this probability falls (this is the mechanism underlying the reputation model of Diamond (1989)).

Reputation as separation. The standard reputation model can thus be extended to capture many of the features of a reputation that we consider important, but it relies crucially on consumers believing in the possibility of a Stackelberg type. This may be reasonable in some circumstances, but is questionable in others. In our model, for example, the Stackelberg type must choose high effort in every period. Such behaviour is typically justified by assuming that the Stackelberg action is strictly dominant.[6] But high effort is a dominant strategy in the stage game only if the Stackelberg type has quite different payoffs from the ordinary firm.

An inept type, in contrast, may require less stringent assumptions. In our model, an inept type need differ from competent types only in having higher costs of high effort. Perhaps pessimistically, we think that inept types are often more plausible than Stackelberg types. It is thus important to investigate whether reputation building can occur in the absence of Stackelberg types. Reputation building then becomes an exercise in separating oneself from inept types rather than pooling with Stackelberg types.

Competent firms distinguish themselves from inept firms in our model by choosing high effort. They do so because a firm's value is increased when consumers think it is competent, as long as consumers also believe that competent firms choose high effort. However, if the firm's type is determined once and for all in the initial period, then consumers will eventually become virtually convinced that the firm is competent, with any additional signals prompting only a minuscule revision in their beliefs. As a result, the incentive to choose high effort collapses and the equilibrium unravels, ensuring that there is no pure-strategy high effort equilibrium.[7] A key ingredient of our reputation model is thus the *perpetual* possibility that a competent firm might be replaced by an inept firm. This possibility arises naturally out of our interest in markets for reputations, which are based on the ability of a firm to buy another firm and its associated reputation. The constant possibility that the type of the firm has changed, and hence the continued desire of the competent firm to separate itself from bad firms, leads to equilibria in which

5. In any equilibrium, the Stackelberg type chooses high effort with probability one, while the competent firm mixes between high and low effort. If the competent firm were to choose high effort with probability one in some period, then consumers would not adjust posteriors in response to the signal in that period, and so the competent firm would optimally choose low effort, disrupting the equilibrium.

6. Fudenberg and Levine (1992) are concerned with providing bounds on equilibrium payoffs, and so it is important for them that the Stackelberg type always choose the Stackelberg action in every equilibrium, which requires that the action be strictly dominant in the *repeated* game. On the other hand, if one is only concerned with the existence of *an* equilibrium in which the Stackelberg type always chooses the Stackelberg action, it is enough that the action be dominant for that type in the *stage* game.

7. A related feature was first described, in the context of a signal jamming model by Holmström (1999), who also described the role of changing characteristics in removing it. Section 4 discusses Holmström (1999) in more detail. Mixed strategy equilibria exist in which high effort is chosen with positive probability, even without changing characteristics. Section 4 explains why we ignore these equilibria.

competent firms always choose high effort, gradually building and then maintaining a reputation for competency and high effort.[8]

In a complementary paper, Tadelis (1999) studies the market for firm names in an adverse selection environment. In Tadelis' model, like ours, unobservable changes in ownership are crucial for names (or reputations) to have value. In every equilibrium of his model, some "bad types" must purchase "good names," a finding consistent with our result that whenever consumers assign a very high probability to a firm being competent, the possibility of an inept replacement causes this probability to fall.

Preview. The following section presents the model. Section 3 examines equilibria under the assumption that the types of entering firms are exogenously fixed. Section 4 discusses the role of firm exit in supporting this equilibrium. Sections 5–6 allow the types of entering firms to be determined endogenously and examine which types of firms are most likely to buy which reputations. Section 7 discusses extensions of the analysis, examining cases in which the competent firm can manage its reputation by choosing from multiple effort levels that are not available to the inept firm, and considering alternative ways that a competent firm might make its type known. Section 8 concludes. Most of the proofs are collected in the Appendix.

2. PRELIMINARIES

We consider an entrepreneur who possesses a "name." This name may be a location for a business, a brand name for a product or service, the exclusive right to use a particular technology, or some similar means of identification. There may be close substitutes for this name, but the name itself is unique. Time is discrete and has an infinite horizon. The entrepreneur can sell the right to use the name to a single firm, for the duration of that firm's lifetime in the market. The lifetime of a firm is exponentially distributed, with an exogenous probability $\lambda \in (0, 1)$ that in each period the firm exits. Upon departure of a firm, the entrepreneur sells the right to use the name to a new firm, who then retains the name until exiting.[9] We think of firm exit as reflecting exogenously-generated reasons to leave the market, such as a decision to retire. A firm who owns the right to use the name maximizes the discounted sum of expected payoffs, with discount factor δ.

Consumers cannot observe firm exits or replacements. For example, the ownership of a restaurant might change without changing the restaurant's name and without consumers being aware of the change, or Time Life may commission an outsider to undertake a publishing venture that consumers cannot differentiate from previous publishing ventures. At the same time, consumers know that such replacements are possible, and take this into account when forming their expectations.

Consumers repeatedly purchase an experience good or service from the firm. The experience good generates two possible utility levels, normalized to 0 and 1. We describe a utility of 1 as a good outcome, denoted g, and a utility of 0 as a bad outcome, denoted

8. For a model without replacements that has an equilibrium in which competent firms always choose high effort, see Hörner (1999). In Hörner, there is a continuum of firms competing for customers. A firm may thus lose customers to other firms after a bad signal, providing incentives for high effort.

9. Allowing the current firm to own the name (and so sell it upon exit) complicates the game, since the value to a firm of changing consumer beliefs now depends on the resale value of the name, as well as expected revenue considerations. Our results continue to hold in such a model, except that we have only been able to prove Proposition 4 under additional sufficient conditions that may conflict with the sufficient conditions for existence.

b. In each period, the firm exerts effort, which determines the probability of a good outcome. There are two possible effort levels, high (H) and low (L). There are also two possible types of firm, "inept" (I) and "competent" (C). An *inept* firm can only exert low effort, while a *competent* firm can exert either high or low effort.

High effort yields a good outcome for consumers with probability $1-\rho > 1/2$. With probability $\rho > 0$, high effort results in a bad outcome. Conversely, low effort yields a bad outcome with probability $1-\rho$, and a good outcome with probability ρ.[10] Low effort is costless, while high effort entails a cost of c, where $0 < c < 1-2\rho$. The latter inequality ensures that if consumers knew that the firm was competent, and could verify its effort before purchasing the good, then they would be willing to pay a premium for high effort sufficient to make high effort optimal for the competent firm. All consumers receive the same outcome, which is observed by the firm and market (we return to this assumption below).

The prior probability that the firm in the market at time zero is competent is given by ϕ_0. The probability that a competent firm replaces an exiting firm is $\theta \in (0,1)$. We initially take θ to be exogenously fixed. Sections 5 and 6 focus on endogenous θ. When taking θ to be fixed, we assume $\phi_0 \in [\lambda\theta, 1-\lambda(1-\theta)]$. While it is natural to further assume $\theta = \phi_0$, nothing depends upon this equality.

There is a continuum of identical consumers, of unit mass. Since the product produced by the firm is an experience good, consumers observe neither the effort expended in its production nor the utility it will yield before purchase. Moreover, it is not possible to write contracts conditional on these properties. Since there is a continuum of consumers, no single consumer can affect the future play of the game. We accordingly treat consumers as myopic, in the sense that the only issue for a consumer in period t is the probability she assigns to the product inducing a good outcome in that period. For specificity, we assume that each consumer pays her expected utility given that probability. The important feature is that the firm's revenue in a period is increasing in consumers' beliefs over the firm's effort choice in that period,[11] and independent of the true effort choice.

The sequence of events is as follows. At the beginning of period t, consumers assign a probability ϕ_t to the firm being competent, and have an expected utility p_t from consuming the good (the normalization on utility levels means that p_t is also the probability consumers assign to receiving the good outcome). If the firm is competent, it makes its unobserved effort choice. Output is then produced, and the firm receives revenues of p_t, regardless of its type and regardless of the realized utility in that period. Consumers, the firm, and market next observe the realized valuation of the good and update beliefs about the type of firm and hence their expected utility. Finally, with probability λ, the firm is replaced.

We assume that all consumers receive the *same* realization of utility outcomes, and that this realization is *public* (so that ϕ_t and p_t are the same for all consumers). In particular, the firm and the market both observe the outcome at the end of the period. In a more realistic model, each consumer would receive an idiosyncratic, private outcome, so that some consumers receive a good utility outcome while others receive a bad utility outcome in each period. Such a model is tractable when entrants' types are exogenously determined,

10. We could dispense with the symmetry assumption, that high effort produces a good outcome with the same probability that low effort produces a bad outcome, without affecting the results (with an exception identified in Footnote 20), but at the cost of additional notation.

11. More specifically, if F and G are two distributions describing consumer beliefs over high effort choice by the firm in period t, and if F first-order stochastically dominates G, then the firm's revenue in period t under F is higher than under G.

leading to equilibria in which reputations have the asset-like features in which we are interested.[12] However, the combination of idiosyncratic consumer realizations and endogenously-determined entrant types makes the model intractable.

Unfortunately, a model with common, public realizations has many equilibria, even when there are no replacements and the firm is known to be competent, including equilibria in which high effort is chosen but reputation has none of the asset-like features that we consider essential to the study of the market for reputations. For example, for δ close to 1 and ρ close to 0, there exists an equilibrium in which the firm initially exerts high effort, and continues with high effort as long as consumers receive good utility realizations, switching to low effort forever as soon as consumers receive a bad realization. In our view, such equilibria not only fail to capture the asset-like features of reputations, but depend upon an implausible degree of coordination between firm behaviour and consumer beliefs about firm behaviour. This type of behaviour is precluded in a model in which consumers receive private, idiosyncratic utility realizations.[13] In this paper, we eliminate these equilibria by requiring behaviour to be Markov. Trivially, in the absence of uncertainty over firm types, there is a unique Markov equilibrium, and in this equilibrium, the firm always chooses low effort.

3. REPUTATIONS WITH EXOGENOUS REPLACEMENTS

In the presence of uncertainty concerning the firm's type and replacements, the state variable is the consumers' posterior probability that the current firm is competent, denoted ϕ, with prior probability ϕ_0. A Markov strategy for the competent firm is a mapping

$$\tau: [0, 1] \to [0, 1],$$

where $\tau(\phi)$ is the probability of high effort when the consumers' posterior probability that the current firm is competent is ϕ. The inept firm makes no choices, and hence has a trivial strategy.

With probability one, there will be an infinite number of replacement events, infinitely many of which will introduce new competent firms into the game. By restricting attention to strategies that only depend on consumers' posteriors, we are requiring that a new competent firm, entering when consumers have belief ϕ, behave in the same way as an existing competent firm when consumers have the same belief ϕ.[14] While restricting firms to such strategies may rule out some equilibria, any equilibrium under this assumption will again be an equilibrium without it. We sometimes refer to a firm strategy as the competent firm's (or competent type's) strategy, although it describes the behaviour of all new competent firms as well.

Consumer behaviour is described by the Markov belief function

$$p: [0, 1] \to [0, 1],$$

where $p(\phi)$ is the probability consumers assign to receiving a good outcome, given posterior ϕ. Revenues for the firm when consumers have posterior ϕ are then $p(\phi)$ (since the utility of a good outcome is 1 and the utility of a bad outcome is 0).

We denote by $\varphi(\phi|x)$ or ϕ_x the posterior probability that the current firm is competent, given a realized outcome $x \in \{g, b\}$ and a prior probability ϕ. If a competent firm

12. We analyse such a model in Mailath and Samuelson (1998).

13. It is easy to verify that the only pure strategy equilibrium of the model with idiosyncratic consumers has the firm choosing low effort in every period. Mixed strategies introduce significant complications and little is known of the structure of mixed equilibria in models of this type.

14. In equilibrium we are thus requiring different firms to behave identically in identical situations, yielding a *symmetric* Markov perfect equilibrium.

always exerts high effort, then posterior beliefs are

$$\varphi(\phi|g) \equiv \phi_g = (1-\lambda)\frac{(1-\rho)\phi}{(1-\rho)\phi + \rho(1-\phi)} + \lambda\theta, \tag{1}$$

and

$$\varphi(\phi|b) \equiv \phi_b = (1-\lambda)\frac{\rho\phi}{\rho\phi + (1-\rho)(1-\phi)} + \lambda\theta. \tag{2}$$

In a *Markov perfect equilibrium*, firms maximize profits, consumers' expectations are correct, and consumers use Bayes rule to update their posterior probabilities:

Definition 1. A *Markov perfect equilibrium* is the triple (τ, p, φ) such that

(a) $\tau(\phi)$ is maximizing for all ϕ.

(b) $p(\phi) = \{(1-\rho)\tau(\phi) + \rho(1-\tau(\phi))\}\phi + \rho(1-\phi)$,

(c) $\varphi(\phi|g) = (1-\lambda)\dfrac{[(1-\rho)\tau(\phi) + \rho(1-\tau(\phi))]\phi}{\{[(1-\rho)\tau(\phi) + \rho(1-\tau(\phi))]\phi + \rho(1-\phi)\}} + \lambda\theta$,

(d) $\varphi(\phi|b) = (1-\lambda)\dfrac{[\rho\tau(\phi) + (1-\rho)(1-\tau(\phi))]\phi}{\{[\rho\tau(\phi) + (1-\rho)(1-\tau(\phi))]\phi + (1-\rho)(1-\phi)\}} + \lambda\theta$.

A strategy for the firm uniquely determines the equilibrium updating rule that consumers must use if their beliefs are to be correct, as well as the equilibrium pricing rule.

Proposition 1. *Suppose* $\lambda \in (0, 1)$, $\phi_0 \in [\lambda\theta, 1-\lambda(1-\theta)]$, *and* $\delta \in (0, 1)$.

(1.1) *If* $\theta \in (0, 1)$, *then there exists* $\bar{c} > 0$ *such that for all* $0 \leqq c < \bar{c}$, *the pure strategy profile in which the competent firm always chooses high effort is a Markov perfect equilibrium.*

(1.2) *If* $\theta = 0$, *then for any* $\phi' \in (0, 1-\lambda)$, *there exists* $\bar{c} > 0$ *such that for all* $0 \leqq c < \bar{c}$, *the pure strategy profile in which the competent firm chooses high effort in period t if and only if* $\phi_t \geqq \phi'$ *is a Markov perfect equilibrium.*

Proof. (1.1) Suppose the competent firm always chooses high effort and fix ϕ. Then $\varphi(\varphi(\phi|g)|g) \equiv \phi_{gg} > \phi_g > \phi > \phi_b > \phi_{bb}$ and $\phi_{gx} > \phi_{bx}$ for $x \in \{g, b\}$. The value function of the competent firm is given by

$$V_C(\phi) = p(\phi) - c + \delta(1-\lambda)[(1-\rho)V_C(\phi_g) + \rho V_C(\phi_b)].$$

The payoff from exerting low effort and thereafter adhering to the equilibrium strategy is

$$V_C(\phi; L) \equiv p(\phi) + \delta(1-\lambda)[\rho V_C(\phi_g) + (1-\rho)V_C(\phi_b)].$$

Thus

$$\begin{aligned} V_C(\phi) - V_C(\phi; L) &= \delta(1-\lambda)(1-2\rho)(p(\phi_g) - p(\phi_b)) - c \\ &\quad + \delta^2(1-\lambda)^2(1-2\rho)\{(1-\rho)[V_C(\phi_{gg}) - V_C(\phi_{bg})] + \rho[V_C(\phi_{gb}) - V_C(\phi_{bb})]\} \\ &\geqq \delta(1-\lambda)(1-2\rho)[p(\phi_g) - p(\phi_b)] - c, \end{aligned} \tag{3}$$

since V_C is monotonic in ϕ.[15]

15. Let $f_t(\phi, \phi_0, t_0)$ be the distribution of consumer posteriors ϕ at time $t > t_0$ induced by strategy τ of (5) given period-t_0 posterior ϕ_0. Then $f_t(\phi, \phi_0, t_0)$ first-order stochastically dominates $f_t(\phi, \phi_0', t_0)$ for all $t > t_0$ and $\phi_0 > \phi_0'$. The same is then true for the distribution of revenues, which suffices for the monotonicity of V_C.

An equilibrium in which the competent firm always exerts high effort requires $V_C(\phi) - V_C(\phi; L) \geqq 0$ for all feasible ϕ. From (3), a sufficient condition for this inequality is

$$p(\phi_g) - p(\phi_b) \geqq \frac{c}{\delta(1-\lambda)(1-2\rho)}. \tag{4}$$

Now choose a $\phi_0 \in [\lambda\theta, 1-\lambda(1-\theta)]$. Posterior probabilities are then contained in the interval $[\lambda\theta, 1-\lambda(1-\theta)]$. In addition, the minimum of $p(\phi_g) - p(\phi_b) = p(\varphi(\phi|g)) - p(\varphi(\phi|b))$ over $\phi \in [\lambda\theta, 1-\lambda(1-\theta)]$ is strictly positive, because p and φ are continuous. We can thus find a value of c sufficiently small that (4) holds for all $\phi \in [\lambda\theta, 1-\lambda(1-\theta)]$. Moreover, an argument analogous to that for the one-stage deviation principle for infinite horizon games shows that, because $V_C(\phi) - V_C(\phi; L) \geqq 0$ for all ϕ, no deviation from always choosing high effort is profitable for the competent firm, ensuring that we have an equilibrium.

(1.2) Suppose $\theta = 0$. Because $\lim_{\phi \to 0} p(\phi_g) - p(\phi_b) = 0$, there is no value of c small enough to ensure that (4) holds for all $\phi \in [\lambda\theta, 1-\lambda(1-\theta)] = [0, 1-\lambda]$. However, consider the strategy

$$\tau(\phi) = \begin{cases} 1, & \text{if } \phi \geqq \phi', \\ 0, & \text{if } \phi < \phi', \end{cases} \tag{5}$$

so that the competent firm exerts high effort if and only if the posterior exceeds a cutoff level ϕ', where $\phi' \in (0, 1-\lambda)$. If the posterior falls short of this cutoff level, so $\phi < \phi'$, then the firm chooses low effort and hence no further updating of beliefs occurs, leading to the continued choice of low effort, and a value for the competent firm of $V_C(\phi) = \rho/(1-\delta(1-\lambda))$.

The strategy given by (5) will be an equilibrium as long as $V_C(\phi) - V_C(\phi; L) \geqq 0$, and hence as long as (4) holds, for all $\phi \geqq \phi'$. But since the minimum of $p(\phi_g) - p(\phi_b) = p(\varphi(\phi|g)) - p(\varphi(\phi|b))$ over $\phi \in [\phi', 1-\lambda]$ is strictly positive for any $\phi' \in (0, 1-\lambda)$, we can again find a value of c for any such ϕ' for which (4) is satisfied, and hence the strategy given in (5) is an equilibrium. ||

Observe that the efficiency condition $c < 1-2\rho$ alone does not suffice for existence of a high-effort equilibrium. The costs of high effort are borne immediately. The benefits are only partially recaptured, in the future, as a result of the favourable consumer belief revision induced by high effort. The existence of a high-effort equilibrium thus requires values of c smaller than $1–2\rho$.

As θ approaches one, the cost of high effort must shrink in order for an equilibrium to exist in which the firm exerts high effort. In equilibrium, the difference between the value of choosing high effort and the value of choosing low effort must exceed the cost of high effort. However, the value functions corresponding to high and low effort approach each other as $\phi \to 1$, because the values diverge only through the effect of current outcomes on future posteriors, and current outcomes have very little affect on future posteriors when consumers are currently quite sure of the firm's type. The smaller the probability of an inept replacement, the closer the posterior expectation of a competent firm can approach unity, and hence the smaller must be the cost in order to support high effort. Replacements ensure that ϕ can never reach unity, and hence there is always a wedge between the high-effort and low-effort value functions (see Figure 1). As long as the cost of the former is sufficiently small, high effort will be an equilibrium.

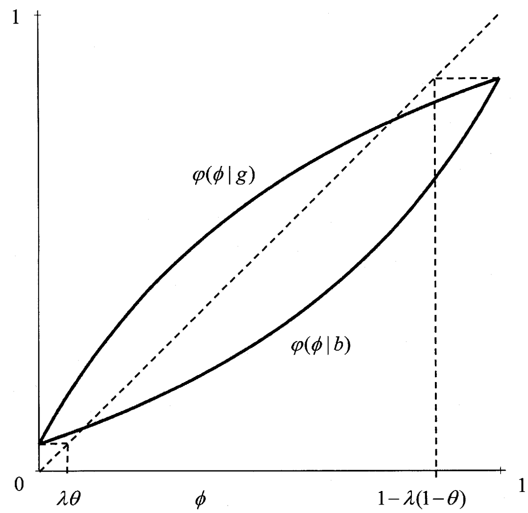

FIGURE 1
The updating rules with exogenous replacements

If replacements always introduce inept firms ($\theta = 0$), then there is no lower bound on the posterior probability of an inept firm. This again destroys the proposed equilibrium, as the high-effort and low-effort value functions also approach one another as $\phi \to 0$. For very low posteriors, the actions of the firm have so little effect on consumer posteriors as to render high effort suboptimal.

In this case, an alternative equilibrium exists in which competent firms sometimes (but not always) choose high effort. We need only choose ϕ' in (5) so that (4) holds for all feasible posteriors $\phi \geqq \phi'$. The competent firm then exerts high effort as long as the posterior probability that they are competent remains sufficiently high ($\phi \geqq \phi'$). The firm reverts to low effort whenever a string of bad luck reduces the posterior below the cutoff level ϕ', causing the competent firm to abandon the attempt to convince consumers of its type and resign itself to a life of low effort. As a result, competent firms exert high effort over the course of an initial period until, as will eventually happen (with probability one), their reputation falls below ϕ' and they are consigned to a life of low effort.

Notice that this latter class of equilibria poses a coordination problem. For any fixed cost level c, there will be a variety of critical posteriors ϕ' for which the strategies given by (5) are an equilibrium. Equilibria with lower values of ϕ' are more efficient, in the sense that they support high effort over a larger collection of posteriors, and hence a competent firm can expect to exert high effort over a longer period of time. Finally, notice that the strategies given by (5) constitute an equilibrium, for sufficiently small values of c, even if $\theta > 0$. Hence, the equilibrium in which competent firms *always* exert high effort (when $\theta > 0$) is joined by a host of less efficient equilibria, characterized by (5) (with $\phi' > \lambda\theta$), in which competent firms initially exert high effort and then are eventually absorbed in a low-effort trap.

4. THE ROLE OF REPLACEMENTS

The possibility that a competent firm may be replaced by an inept one ($\lambda > 0$ and $\theta < 1$) is necessary for the existence of the reputation equilibrium described in Proposition 1. We thus have the seemingly paradoxical result that it can be good news for the firm to have consumers constantly fearing that the firm might "go bad." However, the purpose of a reputation is to convince consumers that the firm is competent and hence will choose high effort. The problem with maintaining a reputation in the absence of inept replacements ($\lambda = 0$ or $\theta = 1$) is that the firm does too good a job of convincing consumers it is competent. Consumers eventually become so convinced the firm is competent (*i.e.* the posterior ϕ becomes so high), that subsequent evidence can only shake this belief very slowly. Once this happens, the incentive to choose high effort disappears, as the current cost savings of low effort overwhelm the small adverse belief revision. But then the incentive to convince consumers the firm is competent also disappears, and the equilibrium collapses. If replacements continually introduce the possibility that the firm has become inept, then the firm cannot be "too successful" at convincing consumers it is competent. Instead, there is an upper bound, short of unity, on the posterior ϕ. The incentive to choose high effort in order to reassure consumers that the firm is still competent always remains, and so there is an equilibrium in which the competent firm always exerts high effort.

In the remainder of this section, we clarify the role of replacements by characterizing equilibrium behaviour in the model *without* replacements:

Proposition 2. *Suppose there are no replacements, i.e.* $\lambda = 0$.

(2.1) *There is a unique Markov perfect equilibrium in pure strategies, and in this equilibrium, the competent firm chooses low effort in every state.*

(2.2) *A mixed-strategy Markov perfect equilibrium with* $\tau(\phi) > 0$ *for some* ϕ *exists if*

$$\rho + c(1 - \delta\rho)/\{\delta(1 - 2\rho)\} < 1. \tag{6}$$

(2.3) *In any Markov perfect equilibrium with* $\tau(\phi) > 0$ *for some* ϕ,

$$\limsup_{\phi \to 1} \tau(\phi) > \liminf_{\phi \to 1} \tau(\phi). \tag{7}$$

Without replacements, the only pure-strategy equilibrium calls for competent firms to always exert low effort. The previous paragraph explained why there are no equilibria in which competent firms always choose high effort, and the remaining task is to show that there are no pure-strategy equilibria in which competent firms *sometimes* choose high effort. There are mixed equilibria in which high effort is chosen with positive probability, but, as shown in (7), these equilibria require discontinuities in the firm's behaviour and hence consumer expectations of that behaviour. Such discontinuous behaviour entails the same implausible degree of coordination on the part of consumers that prompted us to impose a Markov restriction on behaviour, leading us to restrict attention to pure strategy equilibria.

Proof. The last two claims of Proposition 2 are proved in the Appendix. We prove the first claim here. Notice first that if the competent firm chooses low effort at any state, then there is no updating at that state and, by the Markov assumption, the state is absorbing. In every subsequent period, the competent firm then chooses low effort, yielding a continuation value of $\rho/(1 - \delta)$. Hence, the competent firm must choose high effort in state ϕ_0 if it is ever to do so.

Define a sequence of states $\{\phi_k\}_{k=0}^{\infty}$ as follows: ϕ_0 is the prior, and $\phi_{k+1} \equiv \varphi(\phi_k | g)$, where the updating assumes the competent firm is choosing high effort at ϕ_k. Note that $\phi_k \to 1$ as $k \to \infty$. The symmetry specification on outcome probabilities (*i.e.* $\Pr\{g|H\} = \Pr\{b|L\}$) implies $\varphi(\phi_k | b) = \phi_{k-1}$, where again φ assumes the competent firm is choosing high effort at ϕ_k.[16] Hence, if a competent firm chooses high effort in a Markov perfect equilibrium at ϕ_{k-1} and ϕ_k, then it must choose high effort after observing a bad outcome at the state ϕ_k. As a result, the discussion at the beginning of this subsection, arguing that there is no equilibrium in which the competent firm always chooses high effort, also allows us to conclude that it cannot be an equilibrium to choose high effort at every element of the sequence $\{\phi_k\}_{k=0}^{\infty}$. Hence, there is some k' such that high effort is chosen at every $k \leqq k'$ and low effort is chosen at $k'+1$. The incentive constraint for choosing high effort at $\phi_{k'}$ then implies that the continuation value at $\phi_{k'-1}$ (reached via a bad utility outcome) is less than the continuation value at $\phi_{k'+1}$ (reached via a good utility outcome). But this is impossible, as the latter continuation value, $\rho/(1-\delta)$, can be at least equalled at posterior $\phi_{k'-1}$ simply by always choosing low effort. ||

Replacements allow us to obtain a high-effort equilibrium by bounding the posterior of the consumers away from one. Alternatively, we could assume directly that the type of the firm (but not the firm itself) changes randomly over time (for example, a competent firm becomes inept next period with probability $\lambda(1-\theta)$, while an inept firm becomes competent with probability $\lambda\theta$). In an influential paper, Holmström (1999) discussed the role of market uncertainty in providing incentives in the face of moral hazard, including the role of fluctuating "types" in obtaining this market uncertainty. Holmström (1999) examined a signal-jamming model in which uncertainty about a worker's productivity is shared by the worker and the market. The worker exerts high effort in order to increase the market's estimate of his productivity. In the absence of fluctuating productivity, the market becomes sufficiently convinced of the worker's high productivity that the worker eventually reverts to low effort. In addition to its signal-jamming aspect, Holmström's model differs significantly from ours in assuming that a higher-productivity worker is intrinsically more valuable, even if he exerts low effort. In contrast, there is no value to knowing that one of our firms is competent, but exerts low effort. As a result, a "nicely-behaved" pure-strategy equilibrium exists in Holmström's model, in which high effort is temporarily exerted, even in the absence of fluctuating productivity. Fluctuating productivity gives rise to equilibria in which high effort is *always* exerted. Because a higher reputation is valuable in our model only to the extent that it increases the likelihood of high effort, no pure-strategy equilibrium exists in which high effort is ever exerted.

Proposition 2.1 crucially depends on the symmetry assumption $\Pr\{g|H\} = (1-\rho) = \Pr\{b|L\}$. Suppose instead that high effort gives a good outcome with probability $1-\rho_H$, low effort a good outcome with probability $1-\rho_L$, and $(1-\rho_H)^m\rho_H \neq (1-\rho_L)^m\rho_L$ for all $m \in \mathcal{N}$. Then there is a pure strategy equilibrium (for low c and high δ) in which high effort is sometimes taken: $\tau(\phi)=1$ for all $\phi \in \{\phi_k\}_{k=0}^{\infty}$ and $\tau(\phi)=0$ for $\phi \notin \{\phi_k\}_{k=0}^{\infty}$, where $\{\phi_k\}_{k=0}^{\infty}$, the sequence of posteriors reached via good utility realizations, was defined in the proof of Proposition 2.1. The key to this equilibrium is the observation that now $\varphi(\phi_{k+m}|b) \neq \phi_k$ for any m and k. Hence, consumer updating after a bad utility realization generates a belief that cannot be reached by any sequence of only good outcomes. The profile then maintains the incentive for high effort by attaching the punishment of expectations of future low effort to any posterior $\varphi(\phi_k|b)$. This is a Markov implementation of the non-Markovian strategy which starts with high effort, continues with high effort while good outcomes are observed, and switches to low effort forever at the first bad outcome. Since $\phi_k \to 1$ and $\varphi(\phi_k|b) \to 1$ as $k \to \infty$, in such an equilibrium, there are states arbitrarily close to 1 at which the competent firm chooses high effort *and* states

16. We discuss below the implications of an asymmetric specification of outcome probabilities.

arbitrarily close to 1 at which it chooses low effort (*i.e.* $\limsup_{\phi\to 1}\tau(\phi) = 1 > 0 = \liminf_{\phi\to 1}\tau(\phi)$).

Proposition 2.3, which requires no symmetry assumption on ρ_H and ρ_L, indicates that such a discontinuity is necessarily the case in *any* Markov equilibrium with high effort.[17] Imposing the requirement $\limsup_{\phi\to 1}\tau(\phi) = \liminf_{\phi\to 1}\tau(\phi)$ eliminates all the "non-Markov" Markov equilibria, so that the only "nicely-behaved" Markov equilibrium has the competent firm always choosing low effort.

5. ENDOGENOUS REPLACEMENTS

In this section, the probability that a replacement firm is competent is determined by market forces. In each period, there is again a probability λ that the current firm leaves the market. When this occurs, the entrepreneur sells the name to a new firm. There are always a large number of potential new firms who are inept. Formally, we only require two, but we think of this as an endeavour where it is easy to be inept, and hence where there is always an ample supply of inept firms. We normalize the opportunity cost of potential inept firms to 0.

Competent firms are scarce. Whether there is a potential competent firm, and the opportunity cost of that firm, is randomly determined in each period, independently across periods. In each period, with probability $\nu + \kappa D(d_C)$, there is a potential competent firm whose opportunity cost of participating in the market is less than or equal to $d_C \geqq 0$. We assume $\nu \in (0, 1)$, $\kappa \geqq 0$, $\nu + \kappa \leqq 1$, and D is a strictly increasing, differentiable cumulative distribution function on $[0, \infty)$ with $D(0) = 0$. Hence, ν is the probability that there is a competent firm with opportunity cost zero. With probability $\kappa D(d_C)$, there is a competent firm whose opportunity cost exceeds zero but not d_C. With probability $1 - \nu - \kappa$, there is no potential competent firm. We have thus assumed that competent firms have higher opportunity costs than inept firms. While we find this assumption natural, it is unnecessary.[18]

We assume there is at most one competent firm. When the current firm exits, the right to the name is sold by a sealed-bid, second-price auction. The second-price auction is convenient because it ensures that the right to the name is sold to the firm with the highest net valuation. Coupled with our assumption that there is at most one competent firm and at least two inept firms among the potential entrants, this allows us to easily identify equilibrium prices and the circumstances under which entrants are likely to be either competent or inept.

If we set $\kappa = 0$, the model is formally identical to that of exogenous replacements, with ν taking the place of θ, the probability of an entrant's being competent. In particular, with probability $1 - \nu$, there is no competent firm and the entrant is inept. With probability ν, there is a competent firm who will win the second-price auction, giving a competent replacement.

Let V_C (V_I) denote the value function for the competent (inept) firm. The *price* of a name currently characterized by the posterior ϕ is the net (of opportunity cost) value of

17. The mixed strategy constructed in Proposition 2.2 uses mixed behaviour by the competent firm at states reached by good outcomes so that (as in the asymmetric case) updating after a bad outcome yields a posterior that could not be reached after only good outcomes.

18. Our model effectively covers the case in which there is a positive probability that the opportunity cost of potential competent entrants is lower than that of inept entrants: Since $V_C > V_I$, that event is equivalent to the one in which the opportunity cost of potential competent firms is zero. We do take opportunity costs to be exogenous. Tadelis (1999) generates the opportunity costs of competent and inept firms endogenously, resulting in higher opportunity costs for competent firms.

the name to the second highest bidder, who will be an inept firm with zero opportunity cost and value $V_I(\phi)$, giving a price of $V_I(\phi)$. A competent firm with opportunity cost d_C thus buys the name if

$$V_C(\phi) \geqq d_C + V_I(\phi). \tag{8}$$

One of the inept firms buys the name if the inequality is reversed and strict. The tie breaking rule in (8) is irrelevant, since Lemma D in the Appendix implies that, for any ϕ, there is a zero probability that, under the distribution D, an opportunity cost for the competent firm arises that yields equality in (8). The probability that the replacement is competent is then $D(V_C(\phi) - V_I(\phi))$, which depends upon the consumers' posterior ϕ.

We seek an equilibrium in which competent firms always exert high effort. In such an equilibrium, posterior beliefs of the consumers are given by

$$\varphi(\phi|g) = (1-\lambda)\frac{(1-\rho)\phi}{(1-\rho)\phi + \rho(1-\phi)} + \lambda\nu + \lambda\kappa D(V_C(\varphi(\phi|g)) - V_I(\varphi(\phi|g))), \tag{9}$$

and

$$\varphi(\phi|b) = (1-\lambda)\frac{\rho\phi}{\rho\phi + (1-\rho)(1-\phi)} + \lambda\nu + \lambda\kappa D(V_C(\varphi(\phi|b)) - V_I(\varphi(\phi|b))). \tag{10}$$

The functions $\varphi(\phi|g)$ and $\varphi(\phi|b)$ enter both sides of (9) and (10). This reflects the fact that beliefs depend upon the likelihood that entrants are competent or inept firms, which in turn depends upon beliefs. The beliefs of the consumers are then a fixed point of (9) and (10).

Given that competent firms exert high effort, the equilibrium belief function for consumers is given by

$$p(\phi) = (1-2\rho)\phi + \rho. \tag{11}$$

The value function of the inept firm is then

$$V_I(\phi) = (1-2\rho)\phi + \rho + \delta(1-\lambda)\{\rho V_I(\varphi(\phi|g)) + (1-\rho)V_I(\varphi(\phi|b))\}, \tag{12}$$

and the value function of the competent firm is

$$V_C(\phi) = (1-2\rho)\phi + \rho - c + \delta(1-\lambda)\{(1-\rho)V_C(\varphi(\phi|g)) + \rho V_C(\varphi(\phi|b))\}. \tag{13}$$

From (13), a necessary condition for always exerting high effort to be consistent with equilibrium, is that for all possible posteriors ϕ,[19] a one period deviation to low effort not be profitable, *i.e.*

$$\delta(1-\lambda)(1-2\rho)\{V_C(\varphi(\phi|g)) - V_C(\varphi(\phi|b))\} \geqq c. \tag{14}$$

Moreover, as in the proof of Proposition 1, an argument analogous to that of the one-stage deviation principle for infinite horizon games shows that (14) is also sufficient.

Definition 2. The triple (τ, p, φ) is a *reputation equilibrium* if the competent firm chooses high effort in every state ($\tau(\phi) = 1$ for all ϕ), the expectation updating rules and the value functions of the firms satisfy (9)–(13), and the competent firm is maximizing at every ϕ.

19. Strictly speaking, we should only be requiring (14) for those posteriors that can be reached from the initial prior after some finite history $h^t \in \{g, b\}^t$. However, nothing is lost by requiring (14) for all posteriors and doing so avoids awkward statements.

Proposition 3. *Suppose* $\nu > 0, \lambda > 0, \delta(1-\lambda) < \rho(1-\rho)/(1-3\rho+3\rho^2)$, *and* D' *is bounded. Then there exists* $\kappa^* > 0$ *and* $c^* > 0$ *such that a reputation equilibrium exists for all* $\kappa \in [0, \kappa^*]$ *and* $c \in [0, c^*]$.

Proof. See the Appendix. ||

The difficulty in establishing the existence of an equilibrium in this case arises from the linkage between the posterior updating rules and the firms' value functions. In the case of exogenous replacements, the updating rules are defined independently of the value functions. We could accordingly first calculate posterior beliefs, use these calculations to obtain value functions, and then confirm that the proposed strategies are optimal given the value functions. With endogenous replacements, the value functions enter the updating rules given by (9) and (10). As a result, we must now use a fixed-point argument to establish the existence of mutually consistent updating rules and value functions, and much of the proof is concerned with this fixed point argument. After concluding that consistent value functions and updating rules exist, we show that, as long as c and κ are not too large, the proposed strategies are optimal.

We again require that c be sufficiently small that the potential future gains of maintaining a reputation can exceed the current cost. The requirements that $\nu > 0$ and $\lambda > 0$ ensure that there exist $\underline{\phi}$ and $\bar{\phi}$, with $0 < \underline{\phi} < \bar{\phi} < 1$, for any allowable values of κ, such that for any φ satisfying (9) and (10), $\varphi(\phi|x) \in [\underline{\phi}, \bar{\phi}]$ for all $\phi \in [0, 1]$ and $x \in \{g, b\}$. As in the case of Proposition 1, this bounding of posterior probabilities away from the ends of the unit interval is necessary in order to preserve the incentive for competent firms to exert high effort.

The inequality restriction on δ, λ and ρ in Proposition 3 ensures that the one-period discounted "average" derivative of the no-replacement updating rule is strictly less than one. Coupled with the requirements that D' is bounded and that κ is not too large, this ensures that the value functions have uniformly bounded derivatives. This in turn allows us to construct a compact set of potential value functions to which a fixed point argument can be applied to yield consistent belief updating rules and value functions. Taking κ to be small also ensures that the type of an entering firm is not too sensitive to the difference $V_C(\phi) - V_I(\phi)$. Otherwise, the possibility arises that for some values of ϕ, consumers and potential entrants might coordinate on an equilibrium in which entrants are likely to be competent, because the value of a competent firm is high, because consumers expect entrants to be competent. For other values of ϕ, entrants may be unlikely to be competent, because the value is low, because consumers expect inept entrants. This allows us to introduce sharp variations in the value function $V_C(\phi)$, potentially destroying the convention that higher reputations are good, which lies at the heart of a reputation equilibrium.

6. WHO BUYS GOOD REPUTATIONS?

We now turn our attention to the market for reputations. In particular, which posteriors are most likely to attract competent firms as replacements, and which are most likely to attract inept firms? A competent firm is more likely to enter as the difference $V_C(\phi) - V_I(\phi)$ increases. The Appendix contains a proof of the following:

Proposition 4. *Suppose a reputation equilibrium exists for all* $\kappa < \kappa^*$. *For any* $\xi > 0$, *there is a* $\kappa^\dagger \leqq \kappa^*$ *such that for any* $\kappa < \kappa^\dagger$, $V_C(\phi) - V_I(\phi)$ *is strictly increasing for* $\phi < \frac{1}{2} - \xi$ *and strictly decreasing for* $\phi > \frac{1}{2} + \xi$.

Replacements are more likely to be competent firms for intermediate values of ϕ and less likely to be competent firms for extreme values of ϕ.[20] Hence, firms with low reputations are relatively likely to be replaced by inept firms. Good firms find it too expensive to build up the reputation of such a name. On the other hand, firms with very good reputations are also relatively likely to be replaced by inept firms. These names are attractive to competent firms, who would prefer to inherit a good reputation to having to build up a reputation, and who would maintain the existing, good reputation. However, these names are even more attractive to inept entrants, who will enjoy the fruits of running down the existing high reputation (recall that if consumers believe that the firm is almost certainly competent, then bad outcomes do not change consumer beliefs by a large amount).[21]

Replacements are more likely to be competent firms for intermediate reputations. These are attractive to competent firms because less expenditure is required to build a reputation than is the case when the existing firm has a low reputation. At the same time, these reputations are less attractive than higher reputations to inept entrants, because the intermediate reputation offers a smaller stock that can be profitably depleted.

As a result, we can expect reputations to exhibit two features. First, there will be churning: high reputations will be depleted while intermediate reputations will be enhanced. Secondly, low reputations are likely to remain low.

7. EXTENSIONS

This section seeks additional insight into the key aspects of our model. We study the case of exogenously-determined entrant types throughout, with θ being the probability that an entrant is competent, in order to focus attention on the points of interest.[22]

7.1. *Multiple effort levels*

Coordinating expectations and actions. The subtleties in establishing the existence of an equilibrium with endogenous replacements arise because there may be multiple ways to coordinate consumer expectations and firm entry decisions. We investigate this coordination feature further by allowing the firm three possible actions.

Suppose the competent firm has three possible actions, denoted low, medium and high, or $\{L, M, H\}$. Denote the cost to the competent firm of medium effort by c_M and the probability of a bad outcome under medium effort by ρ_M (recall that ρ is the probability of a bad outcome under high effort, as well as the probability of a good outcome under low effort). Assume

$$\rho < \rho_M < 1 - \rho, \qquad 0 < c_M < c, \tag{15}$$

and

$$\rho < 1 - \rho_M - c_M < 1 - \rho - c. \tag{16}$$

20. The function $V_C(\phi) - V_I(\phi)$ has its maximum near 1/2 because of the symmetry assumption $\Pr\{g|H\} = \Pr\{b|L\}$. This result holds without the symmetry assumption, but with the maximum possibly no longer being near 1/2.

21. In a pure adverse selection environment, Tadelis (1999) similarly identifies two effects: a "reputation maintenance effect" (reflecting the higher value that competent firms assign to owning a name for any posterior) and a "reputation start-up effect" (which roughly reflects the change in the difference $V_C - V_I$ as ϕ becomes large).

22. Allowing entrant types to be endogenously determined should retain the basic properties of the analysis, but would require an involved existence argument in each case.

The competent firm thus finds both high effort and medium effort better than low effort, if it could commit to such effort levels and consumer expectations were formed accordingly. However, medium effort is inferior to high effort.

We now ask whether there are equilibria in which the competent firm builds a reputation for choosing medium effort. Why would the firm do so, when high effort is more efficient? Given a candidate equilibrium in which medium effort is always exerted, there are two obstacles to the profitability of high effort. The first involves timing: high effort requires an immediate expenditure in return for favourable adjustments in beliefs that are not realized until the future. The second involves consumer expectations: if the candidate equilibrium involves medium effort, then high effort increases the likelihood that consumers attach to the event that the firm is a competent firm *exerting medium effort*. There are cases in which high effort would be undertaken if it could quickly convince the consumers that the firm would exert future high effort, but the combination of the delay in consumer reactions and the continued equilibrium expectation of medium effort can make high effort unprofitable, leading to an equilibrium in which the firm always exerts medium effort.

We assume throughout that c is small enough that, in the absence of medium effort, it would be a "uniformly strict" equilibrium for the competent firm to always exert high effort (in the sense that there is an amount $\eta > 0$, for all feasible ϕ, by which the left side of (14) exceeds the right side). The critical parameters are then ρ_M and c_M, which we assume satisfy (15)–(16) throughout the following discussion.

There are three cases of interest. First, if c_M is sufficiently large, for fixed ρ_M, then medium effort affords little cost savings over high effort. In this case, the "high-effort" equilibrium persists, and there is no "medium-effort" equilibrium (*i.e.* an equilibrium in which the competent firm always exerts medium effort). To verify this, let $V_M(\phi)$ ($V_H(\phi)$) be the value function for a competent firm when the competent firm always exerts (and is expected to always exert) medium (high) effort. If medium effort is to be an equilibrium, the following counterpart of (14), ensuring that deviating to high effort is unprofitable, must hold

$$\delta(1-\lambda)(\rho_M-\rho)[V_M(\varphi(\phi|g))-V_M(\varphi(\phi|b))] \geqq c-c_M. \tag{17}$$

For fixed ρ_M, if c_M approaches c, then eventually (17) fails, ensuring there is no medium-effort equilibrium. Ensuring that the high-effort equilibrium persists requires showing that medium effort is not a profitable deviation, or, for all feasible ϕ,

$$\delta(1-\lambda)(\rho_M-\rho)[V_H(\varphi(\phi|g))-V_H(\varphi(\phi|b))] \geqq c-c_M, \tag{18}$$

which will hold for c_M sufficiently close to c.

Second, if c_M is close to zero and ρ_M is close to ρ, then medium effort is quite inexpensive and almost as efficient as high effort. As a result, there is no high-effort equilibrium, while there is a medium-effort equilibrium. There is no high-effort equilibrium because (18) does not hold when ρ_M is close to ρ and c_M close to zero. A medium-effort equilibrium exists if (17) holds and no deviation to low effort is profitable. The latter condition is equivalent to

$$\delta(1-\lambda)(1-\rho_M-\rho)[V_M(\varphi(\phi|g))-V_M(\varphi(\phi|b))] \geqq c_M. \tag{19}$$

But (17) and (19) are satisfied for c_M is close to zero and ρ_M close to ρ.

Finally, if c_M is close to zero, ρ_M is close to $1-\rho$, and $(1-\rho_M-\rho)/c_M$ is large, then medium effort is relatively efficient, compared to low effort, but does not provide nearly the probability of a good outcome that high effort does. Now, both high-effort and medium effort equilibria exist. A high-effort equilibrium exists if (18) holds. Since the high-effort profile is a uniformly strict equilibrium when medium effort is not available, if c_M is sufficiently close to zero and ρ_M is sufficiently close to $1-\rho$, (18) is satisfied. A medium-effort equilibrium exists if (17) and (19) hold. Since $V_M(\varphi(\phi|g)) - V_M(\varphi(\phi|b))$ converges to zero (in the sup norm) as ρ_M approaches $1-\rho$, (17) and (19) hold with strict inequality if $c_M = 0$ and ρ_M is sufficiently close to $1-\rho$, and hence also hold for slightly larger values of c_M.

The final possibility highlights the role of consumer expectations. In this case, the firm would find it advantageous to take high effort if doing so allowed it to establish a reputation for choosing high effort, but will not do so if the payoff is a reputation for taking medium effort.

Spending a reputation. A competent firm faced with a choice from three effort levels may work very hard to build up its reputation when the latter is low, and then relax to enjoy the fruits of its labours once it has amassed a higher reputation. To illustrate this possibility, suppose again that the firm has a third action available, which we now call extraordinary effort, denoted E. Calculating the value functions and best replies becomes extremely complicated when the firm's actions are not constant. We accordingly simplify the illustration by assuming that excessive effort yields a good outcome with probability one, and that low effort yields a bad outcome with probability one (with high effort still yielding a good outcome with probability $1-\rho$). We further assume

$$1-\rho-c > 1-c_E,$$

which ensures that high effort is still efficient. Then there exists an equilibrium in which competent firms choose high effort if $\phi > \phi^*$ and excessive effort if $\phi \leqq \phi^*$, for some $\phi^* \in (\underline{\phi}, \bar{\phi})$. Hence, as long as the posterior exceeds ϕ^*, the firm enjoys its reputation by exerting high effort. Good utility realizations raise the consumers' posterior to $\bar{\phi}$, while bad reputations reduce the posterior. Excessive effort then ensures that the posterior jumps to $\bar{\phi}$, but is too expensive to warrant its use at relatively high posteriors. Once the posterior falls as low as ϕ^*, however, the firm engages in inefficiently costly excessive effort in order to restore its reputation. A bad outcome here reduces the consumer posterior to $\underline{\phi}$, where it remains until the next good outcome induces the posterior $\bar{\phi}$. These strategies are an equilibrium if ϕ^* is chosen so that the counterpart of equation (14),

$$\delta(1-\lambda)\rho[V(\bar{\phi}) - V(\varphi(\phi|b))] \leqq c_E - c, \tag{20}$$

holds for posteriors $\phi > \phi^*$, with the reverse (weak) inequality holding for posteriors $\phi \leqq \phi^*$ (in this case $\varphi(\phi|b) = \underline{\phi}$), where V is the value function induced by the posited strategies.

We thus have a market in which firms alternate between periods in which they work inefficiently hard to build up reputations and efficiently enjoy their reputations. More realistic but complicated versions of the equilibrium would relax the assumptions that excessive effort always, and low effort never, yields a good utility realization.

7.2. *Announcements*

The strongest assumption in our model is the restriction that consumers observe neither the type of a newly entering firm, nor even the fact that the existing firm has been replaced.

It clearly stretches credibility that consumers will never be able to observe the replacements of firms. However, our results require only that firm changes be *sometimes* unobserved. We obtain equivalent results if entries were observed with some probability $v \in (0, 1)$ and unobserved with probability $1 - v$.

Left to their own devices, consumers may well remain unaware of some changes of ownership. But a firm that has newly purchased the right to use a name may want consumers to know of the ownership change. This is especially likely to be the case if the current reputation attached to the name is lower than the prior expectation attached to entrants. New firms concerned that consumers may not have observed their entry may then seek ways to announce their presence. Signs proclaiming "under new management" and "grand opening" are commonplace. This section explores some of the strategies that entrants might adopt to announce their presence to consumers.

In each of the following models, there remains an equilibrium in which all announcements are ignored and our previous results hold. However, we concentrate on the possible existence and properties of equilibria in which announcements are effective.

Under new management. We first suppose firms can make public announcements that they have newly purchased the right to use the name. However, these announcements are cheap talk in two respects: they are costless, and consumers cannot verify whether claims to have newly purchased the name are correct. It is immediately apparent that such announcements cannot convey information. In particular, such an announcement will be valuable only if it increases the posterior belief ϕ that the firm is competent. But both competent and inept firms would like ϕ to be higher, as would both existing firms and entrants. Any announcement that had an effect on ϕ would then be made regardless of the firm's identity, ensuring that the announcement is uninformative.

Meet our new chef. Announcements of changes in the characteristics of a firm are often accompanied by invitations to verify these changes. The chefs in good restaurants sometimes mingle with the customers, especially early in their tenure, as do mechanics and other service technicians. Firms call press conferences to display new members of their management staff, as do professional sports teams with new athletes. Firms sometimes invite customers to tour their new facilities. To the extent that the personnel or facilities in question are responsible for the quality of the firm's product, these activities make ownership changes verifiable.[23]

Let us suppose that whenever the right to use the name has changed, the new firm can make a costless, verifiable announcement of this change before the beginning of the next period, though no information concerning the quality of the new firm can be verified. Upon hearing such an announcement, consumers will expect the new firm to be competent with probability θ. As a result, no firm purchasing a name whose current reputation satisfies $\phi > \theta$ will announce the change of ownership, while a firm for whom $\phi < \theta$ earns an instant increase in reputation from such an entry announcement. A candidate for an equilibrium then calls for announcements if and only if $\phi < \theta$. As a result, the reputation updating for posteriors $\phi > \theta$ will proceed according to (1)–(2), as is the case without

23. Throughout, we have viewed a change in the identity of the firm as a change in the holder of the right to use the name. More generally, the relevant change involves a change in the input that is responsible for the ability of the firm to produce high quality. Hence, a change of chef in a restaurant, for a given proprietor, is the equivalent of a change of ownership in our model if the chef is the primary determinant of the restaurant's quality, but not if quality depends mostly upon a manger who chooses what dishes to offer, what ingredients to order, and what level of service to offer.

announcements. For posteriors $\phi < \theta$, the lack of an announcement indicates that no change of ownership has occurred. The reputation updating then proceeds according to (1)–(2) with $\lambda = 0$.

Because the lack of an announcement when $\phi < \theta$ ensures the absence of an ownership change, the smallest possible posterior is $\underline{\phi} = 0$. The equilibrium then takes the form constructed in the proof of (1.2) of Proposition 1. There will then be a posterior ϕ' with the property that a competent firm chooses low effort when $\phi < \phi'$ and high effort if $\phi > \phi'$. New firms announce their presence if and only if they acquire a reputation less than θ.[24] Consumers thus observe changes in the ownership of names whose reputation falls short of that of a randomly drawn entrant, and observe no ownership changes of higher reputations. The inability to observe the latter creates an incentive for the competent firm of reputation above ϕ' to build its reputation by choosing high effort.

Newly opened after remodelling. Changes of ownership are often accompanied by the remodelling of a firm's facilities, especially in the service industry. Walls are moved and repainted. Artwork, carpeting and furniture is replaced. The result is often a facility that is different, but not obviously superior to the previous one. We view such remodelling as a costly signal. If signals are not verifiable, and hence costless signals are uninformative, firms may still convey information through the use of costly signals.

We are interested in the ability to use costly but nonverifiable signals to convey information about the user of a name. We assume the firm can choose how much to spend on sending a costly signal after the realization of the consumers' utility, but before the replacement event is realized.[25] The signal and its cost is observed by consumers before they purchase.

It is again clear that a nonverifiable signal, however costly, cannot usefully convey information about ownership changes. Any reputation revision prompted by such a signal is equally valuable to a new or existing firm. However, a competent firm may be able to reveal his *type* by sending a costly signal.

We accordingly describe an equilibrium in which costly signals are sent only by competent firms. At the beginning of the period following such a signal, the posterior probability consumers assign to the firm being competent is the maximum possible, $\bar{\phi} \equiv 1 - \lambda(1 - \theta)$. We focus on equilibria of the following form: there is a critical posterior ϕ^* such that if (and only if) $\phi < \phi^*$ any competent firm spends k to send a signal, yielding a posterior of $\bar{\phi}$. An inept firm for whom $\phi < \phi^*$ sends no signal, yielding a posterior in the beginning of the next period of $\lambda\theta$. A firm who is believed to be inept retains this reputation until sending a signal at cost k, which will happen when a competent firm replaces the existing inept firm. No signals are sent for posteriors $\phi \geqq \phi^*$. The mechanism by which the firm contrives to spend the cost k is immaterial, as long as consumers can verify that it has been spent. Remodelling facilities is one possibility, but publicly burning the money would serve as well.

24. If $\theta > \phi'$, then it is a strict best response for new firms acquiring reputations less than θ to announce their arrival. If $\theta < \phi'$, then there is never a strict incentive for such firms to announce a replacement, though it is a weak best response to do so.

25. An alternative assumption would be to have the costly signal sent after the replacement event, so that a new competent firm could attempt to signal its arrival. The difficulty is that in a separating equilibrium, such a signal leads to a posterior of 1, at which point the competent firm must choose low effort, disrupting the putative equilibrium. In order to support an equilibrium in which the competent firm chooses high effort immediately after the signal, a bad realization must have some information content. This will be the case if the competent firm may have been replaced by an inept firm after the signal.

In this equilibrium, competent firms choose high effort and inept firms choose low effort. Both competent and inept firms refrain from costly signals as long as their reputations are sufficiently high. Eventually, however, the randomness in realized outcomes will cause a firm's reputation to slip below the critical reputation ϕ^* Good firms then send a costly signal, boosting their reputations to $\bar{\phi}$, while inept firms resign themselves to a future reputation of $\lambda\theta$. The latter persists until a new, competent firm appears, who enhances his reputation by the only means possible, sending the costly signal. In order to support these strategies as an equilibrium, it must be profitable for the competent, but not the inept, firm to send the costly signals, for any posterior $\hat{\phi} < \phi^*$. A necessary condition is that the cost k be such that

$$\delta V_I(\bar{\phi}, \phi^*, k) - \delta \frac{\rho}{1-(1-\lambda)\delta} \leqq k \leqq \delta V_C(\bar{\phi}, \phi^*, k) - \delta \frac{\rho}{1-(1-\lambda)\delta}, \tag{21}$$

where $\delta\rho/(1-(1-\lambda)\delta)$ is the expected value, to both a competent and inept firm, of failing to send the signal and hence inducing a reputation of $\lambda\theta$, and where we write the value functions as $V_I(\bar{\phi}, \phi^*, k)$ and $V_C(\bar{\phi}, \phi^*, k)$ to emphasize that continuation values depend upon the posterior at which signals are sent and the cost of the signals.

For any value of ϕ^* and k satisfying (21), we can ensure that we have an equilibrium by choosing out-of-equilibrium beliefs so that a firm who sends a signal when its posterior satisfies $\phi > \phi^*$ is believed to be competent with probability ϕ. There is then no return from signalling when $\phi > \phi^*$, while (21) ensures that competent firms (and only competent firms) will signal when $\phi < \phi^*$, ensuring that sufficient conditions for an equilibrium hold.

This equilibrium is reminiscent of the "excessive effort" equilibrium of the previous subsection. In the latter, a signal took the form of exerting excessive effort, and only competent firms could send signals. In the current context, both types of firms have the ability to send signals. The equilibrium requirement that inept firms choose not to send signals then introduces an additional incentive constraint in (21) that does not appear in (20).

In general, there will be many combinations of the critical posterior ϕ^* and signalling cost k that satisfy (21), and hence multiple equilibria in which firms use costly signals to identify their types. For a given choice of ϕ^*, refinements such as the intuitive criterion can be applied to the out-of-equilibrium beliefs supporting these equilibria to select the equilibrium in which the competent firm sends the least costly signal consistent with separating from the inept firm. However, there will still be multiple values of ϕ^* consistent with (21), and hence multiple equilibria. Higher values of ϕ^* correspond to cases in which firms use frequent signals to constantly keep their reputations high. Lower values of ϕ^* correspond to cases in which signals are infrequently used but accomplish large increases in reputations. Different equilibria thus arise because consumers and firms can coordinate on different values of the critical posterior that triggers a signal. A similar coordination problem between consumers and firms allowed us to construct multiple equilibria, by choosing different values of ϕ' in (5), when proving Proposition 1.

Limited-time introductory offer. We have assumed that the firm prices at the consumer's reservation price in each period, given the firm's current reputation. However, a common way for firms to send signals to consumers is by appropriately choosing their prices, with low prices potentially serving as a costly signal of a competent firm. In our model, setting a price less than the consumers' reservation price is equivalent to burning money, and hence equivalent to the costly signalling possibilities we have just considered.

Expanding our model to allow the firm more discretion in setting prices then leads to equilibria equivalent to those in which the firm can send costly signals.

We have assumed that consumers are homogeneous, and each consumer purchases the good in each period. The scope for introductory pricing may be expanded if consumers are heterogeneous. In particular, consumers who have formed relatively pessimistic posterior expectations may then cease patronizing the firm altogether, and hence cease collecting information about the firm. If the right to use the name passes to a competent firm, then the latter may find it profitable to incur costs to bring these consumers back to the firm, and may find introductory pricing specials more effective than burning money in doing so.

I'm your new doctor. Our model of reputations depends upon the assumption that customers cannot always tell who currently owns the right to use the name. This is likely to be the case in many markets, including service industries such as restaurants, auto repair, and the Time Life example with which we opened. In other cases in which reputations are commonly said to be traded, any such uncertainty is highly unlikely. For example, private medical and dental practices often command high prices. Because very little in the way of physical assets typically changes hands in such a transaction, much of the price is ascribed to reputation. But patients cannot help but notice that their doctor or dentist has changed, making both our model and the extension to voluntary, verifiable announcements of ownership changes inapplicable. Our suspicion is that the bulk of the price for such practices represents compensation for in-place physical assets, which may dwarf the replacement cost of these assets. The key is to notice that patients have no reason to believe that the purchasing doctor is better than the expected outcome they could obtain by returning to the market, despite the selling doctor's glowing recommendation, nor do they have reason to believe that the purchasing doctor is worse. The presence of even extremely small costs of returning to the market to seek a new doctor will thus suffice to retain them at the current practice, making the mere collection of patient records a valuable asset.[26]

8. CONCLUSION

Does a good reputation ensure a firm's success? Obviously not, as any firm can be sufficiently unlucky as to lose a superb reputation. Just as obviously, a good reputation is better than a bad one. But our analysis shows that these considerations can interact in unexpected ways. In a market with potentially unobserved firm turnover, for example, the current reputation of a firm may not be a good predictor of its future success. Instead, a selection bias arises in the process by which firms acquire reputations, with relatively capable firms tending to buy medium reputations, leaving high reputations to be acquired and spent by less capable firms. This acquisition pattern arises because the advantage enjoyed by a competent firm is the ability to boost consumers' posterior expectations of the firm's quality, by exerting high effort. But the consumer posteriors that are most easily influenced are intermediate posteriors, giving competent firms a comparative advantage in medium reputations.

26. An interesting prediction of this suspicion is that medical practices which include physical facilities and support staff should exceed the value of those sold without staff and facilities, by an amount exceeding the physical value of the facilities and costs of obtaining new staff, since patient familiarity with existing facilities will increase the costs of switching.

The more general implication of our analysis is that embedding reputation considerations in a market can produce new insights into the economics of building and maintaining a reputation. We have taken only the first step in modelling the market, with many factors still being exogenously fixed that future work might usefully bring within the purview of the model.

APPENDIX

A.1. *Proof of Proposition* 2.

(2.2). We first construct a mixed strategy equilibrium. This mixed strategy will have the property $\limsup_{\phi\to 1}\tau(\phi)>0=\liminf_{\phi\to 1}\tau(\phi)$.

The profile specifies that if ϕ' is a state that results from a realization of b, then $\tau(\phi')=0$. If $\tau(\phi')=0$, then ϕ' is an absorbing state ($\varphi(\phi'|x)=\phi'$ for $x=b,g$), and $p(\phi')=\rho$, so that $V(\phi')=\rho/(1-\delta)$. In order for the seller to be willing to randomize at a state ϕ, it must be the case that

$$V(\varphi(\phi|g))=V(\varphi(\phi|b))+\frac{c}{\delta(1-2\rho)}=\frac{\rho}{(1-\delta)}+\frac{c}{\delta(1-2\rho)}.$$

Thus,

$$V(\phi)=p(\phi)-\tau(\phi)c+\delta\frac{\rho}{(1-\delta)}+(\tau(\phi)(1-\rho)+(1-\tau(\phi))\rho)\frac{c}{(1-2\rho)}$$

$$=p(\phi)+\delta\frac{\rho}{(1-\delta)}+\frac{\rho c}{(1-2\rho)}.$$

But if $\phi=\varphi(\phi^*|g)$, then $V(\phi)=V(\varphi(\phi^*|g))=\rho/(1-\delta)+c/\delta(1-2\rho)$. This then yields an equation in $\tau(\phi)$

$$p(\phi)+\delta\frac{\rho}{(1-\delta)}+\frac{\rho/c}{(1-2\rho)}=\frac{\rho}{(1-\delta)}+\frac{c}{\delta(1-2\rho)},$$

or

$$\phi\tau(\phi)=\rho+\frac{c(1-\delta\rho)}{\delta(1-2\rho)}.$$

Denote the solution of this equation $\hat{\tau}$.

$$\hat{\tau}(\phi)=\frac{\rho}{\phi}+\frac{c(1-\delta\rho)}{\delta(1-2\rho)\phi},\tag{A.1}$$

which is a well defined probability for $\phi>\rho+c(1-\delta\rho)/\{\delta(1-2\rho)\}$ (by assumption, $\rho+c(1-\delta\rho)/\{\delta(1-2\rho)\}<1$). Fix a probability $\phi_1>\rho+c(1-\delta\rho)/\{\delta(1-2\rho)\}$. Let $\hat{\varphi}$ denote the updating rule based on $\hat{\tau}$. Finally, let $\{\phi_k\}_{k=1}^{\infty}$ be the sequence of states given by ϕ_1 and $\phi_{k+1}=\hat{\varphi}(\phi_k|g)$ for $k\geqq 2$. Note that $\hat{\tau}$ is decreasing in ϕ, so that $\phi_k<\hat{\varphi}(\phi_{k+1}|b)$ (and so it is consistent to specify $\tau(\phi^k)>0$ and $\tau(\hat{\varphi}(\phi^{k+1}|b))=0$). The mixed strategy equilibrium is (τ^*,p^*,φ^*), where

$$\tau^*(\phi)=\begin{cases}\hat{\tau}(\phi), & \text{if } \phi=\phi_k,\ \text{some } k,\\ 0, & \text{otherwise,}\end{cases}$$

and p^* and φ^* are given by (b), (c) and (d) from Definition 1.

(2.3). We now prove that any equilibrium with high effort must have the property

$$\limsup_{\phi\to 1}\tau(\phi)>\liminf_{\phi\to 1}\tau(\phi).$$

Define $\phi'\equiv\inf\{\phi'':\tau(\phi)=0\ \forall\phi>\phi''\}$ and $\tau'\equiv\limsup_{\phi\uparrow\phi'}\tau(\phi)$. Suppose $\phi'<1$ and $\tau'>0$. Then, for all $\varepsilon>0$, there exists $\phi\in(\phi'-\varepsilon,\phi')$ satisfying $\tau(\phi)>\tau'/2$, and so for ε sufficiently small, $\varphi(\phi|g)>\phi'$. But then $V(\varphi(\phi|g))=\rho/(1-\delta)\leqq V(\varphi(\phi|b))$, and so $\tau(\phi)=0$, a contradiction. Thus either $\phi'=1$ or $\tau'=0$. Note this implies that if $\phi'<1$, then $\tau(\phi')=0$.

Suppose, then, that $\phi'=1$. Also suppose, en route to a contradiction, that $\limsup_{\phi\to 1}\tau(\phi)=\liminf_{\phi\to 1}\tau(\phi)=\bar{\tau}$. For all $\nu>0$ and T there exists $\eta>0$ such that $\phi\in(1-\eta,1]$ implies $\varphi(\phi|h^t)\in(1-\nu,1]$ for all

$h^t \in \{b, g\}^t$ and $t = 0, 1, \ldots, T$. For all $\varepsilon > 0$ there exists $\underline{\nu} > 0$ such that for all $\nu \in (0, \underline{\nu})$, $\tau(\phi) \in (\bar{\tau} - \varepsilon, \bar{\tau} + \varepsilon)$ for all $\phi \in (1 - \nu, 1]$. Letting $\underline{p} \equiv \{(1-\rho)(\bar{\tau} - \varepsilon) + \rho(1 - \bar{\tau} + \varepsilon)\}(1 - \nu) + \rho\nu$ and $\bar{p} \equiv (1 - \rho)(\bar{\tau} + \varepsilon) + \rho(1 - \bar{\tau} - \varepsilon)$, $\underline{p} \leqq p(\phi) \leqq \bar{p}$ for all $\phi \in (1 - \eta, 1]$. Then,

$$V(\varphi(\phi|g)) \leqq \textstyle\sum_{t=0}^{T} \delta^t \{\bar{p} - (\bar{\tau} - \varepsilon)c\} + \delta^T/(1 - \delta),$$

while

$$V(\varphi(\phi|b)) \geqq \textstyle\sum_{t=0}^{T} \delta^t \{\underline{p} - (\bar{\tau} + \varepsilon)c\},$$

so that

$$\begin{aligned} V(\varphi(\phi|g)) - V(\varphi(\phi|b)) &\leqq \textstyle\sum_{t=0}^{T} \delta^t \{\bar{p} - \underline{p} + 2\varepsilon c\} + \delta^T/(1 - \delta) \\ &= \textstyle\sum_{t=0}^{T} \delta^t \{2\varepsilon(1 - 2\rho - \nu\rho + c) - \nu\bar{\tau}(1 - 2\rho)\} + \delta^T/(1 - \delta). \end{aligned}$$

For T large, ε and ν small, this last expression is smaller than $c/(\delta(1 - 2\rho))$. So, for $\phi \in (1 - \eta, 1)$, if $\tau(\phi) > 0$, setting $\tau = 0$ is a profitable deviation.

The remaining possibility that must be ruled out is $\phi' < 1$. But we argued above that this implies that $\tau' = 0$, which means that $\phi' < 1$ is similar to the case $\phi' = 1$, since $\tau' = \limsup_{\phi \uparrow \phi'} \tau(\phi) = 0$ implies that for all $\varepsilon > 0$ and T there exists $\eta > 0$ such that $\phi \in (\phi' - \eta, \phi']$ implies $\varphi(\phi|h^t) \in (\phi' - \varepsilon, \phi']$ for all $h^t \in \{b, g\}^t$ and $t = 0, 1, \ldots, T$. The last argument can now be applied to show that setting $\tau = 0$ is a profitable deviation for any type $\phi < \phi'$ but sufficiently close to ϕ' with $\tau(\phi) > 0$. ||

A.2. *Proof of Proposition* 3.

The proof is involved and requires some lemmas. For the reader's convenience, we have collected the lemmas at the end of this subsection.

Step 1. We first show that any admissible posterior updating rule implies a unique pair of value functions. Fix $\eta \in (\delta(1 - \lambda)(1 - 3\rho + 3\rho^2)/(\rho - \rho^2), 1)$, and let $\mathfrak{X} \equiv \{f \in \mathscr{C}^1([0, 1], [0, 1]^2): \delta|\rho f_1'(\phi) + (1 - \rho)f_2'(\phi)| \leqq \eta,$ $\delta|(1 - \rho)f_1'(\phi) + \rho f_2'(\phi)| \leqq \eta, |f_1'(\phi)| \leqq 2/\rho, |f_2'(\phi)| \leqq 2/\rho$ for all $\phi \in [0, 1]\}$. Any function $\varphi(\phi) \equiv (\varphi(\phi|g), \varphi(\phi|b)) \in \mathfrak{X}$ is a potential posterior updating rule, giving, for any prior probability ϕ, the posteriors $(\varphi(\phi|g), \varphi(\phi|b))$ that follow a good and bad utility realization. The set $\mathfrak{X}$ is a convex compact subset of the normed space $\mathscr{C}^1([0, 1], [0, 1]^2)$, with the $\mathscr{C}^1$-norm,

$$\|f\|_1 = \max\left\{\sup_\phi |f_1(\phi)|, \sup_\phi |f_1'(\phi)|, \sup_\phi |f_2(\phi)|, \sup_\phi |f_2'(\phi)|\right\}.$$

The set $\mathfrak{X}$ is nonempty. In particular, let φ_0 denote the exogenous updating rule, *i.e.* κ is set equal to zero in (9) and (10). Now it is straightforward to verify that, for $x \neq y \in \{g, b\}$, $0 \leqq \rho\varphi_0'(\phi|x) + (1 - \rho)\varphi_0'(\phi|y) \leqq (1 - \lambda)(1 - 3\rho + 3\rho^2)/(\rho - \rho^2) < \eta/\delta$, and so $\varphi_0 \equiv (\varphi_0(\cdot|g), \varphi_0(\cdot|b)) \in \mathfrak{X}$.

Let $Y \equiv (1 - \rho)/(1 - \delta)$ and $\mathfrak{Y} \equiv \{f \in \mathscr{C}^1([0, 1], [-Y, Y]^2): \sup_x |f_1'(x)| \leqq (1 - 2\rho)/(1 - \eta), \sup_x |f_2'(x)| \leqq (1 - 2\rho)/(1 - \eta)\}$. Interpret an element of $\mathfrak{Y}$ as a pair of possible value functions, one for the inept firm and one for the competent firm. Fix an updating rule $\varphi \equiv (\varphi(\cdot|g), \varphi(\cdot|b)) \in \mathfrak{X}$, and let $\Psi^\varphi: \mathfrak{Y} \to \mathscr{C}^1([0, 1], \Re^2)$ denote the mapping whose coordinates are the functions

$$\Psi_1^\varphi(V_I, V_C)(\phi) = (1 - 2\rho)\phi + \rho + \delta(1 - \lambda)\{\rho V_I(\varphi(\phi|g)) + (1 - \rho)V_I(\varphi(\phi|b))\},$$

and

$$\Psi_2^\varphi(V_I, V_C)(\phi) = (1 - 2\rho)\phi + \rho - c + \delta(1 - \lambda)\{(1 - \rho)V_C(\varphi(\phi|g)) + \rho V_C(\varphi(\phi|b))\}.$$

The mapping Ψ^φ is a contraction on $\mathfrak{Y}$ (Lemmas A and B), and so has a unique fixed point. For any updating rule φ, this fixed point identifies the unique value functions that are consistent with φ, in the sense of satisfying (12) and (13). Let $\Phi: \mathfrak{X} \to \mathfrak{Y}$ denote the mapping that associates, for any updating rule φ in $\mathfrak{X}$, the fixed point of Ψ^φ. The mapping Φ is continuous (Lemma C).

Step 2. We now show that there exist updating rules and value functions that are consistent, in the sense that using Φ to obtain value functions from an updating rule and then applying (9)–(10) returns the original updating rules. Let $\hat{\varphi}$ denote the updating rule obtained from φ and $\Phi(\varphi) = (V_I, V_C)$ by using (9)–(10):

$$\hat{\varphi}(\phi|g) = \varphi_0(\phi|g) + \lambda\kappa D[V_C(\varphi(\phi|g)) - V_I(\varphi(\phi|g))],$$

and

$$\hat{\varphi}(\phi|b) = \varphi_0(\phi|b) + \lambda\kappa D[V_C(\varphi(\phi|b)) - V_I(\varphi(\phi|b))].$$

Since

$$\hat{\varphi}'(\phi|x) = \varphi_0'(\phi|x) + \lambda\kappa D'[V_C(\varphi(\phi|x)) - V_I(\varphi(\phi|x))] \cdot \{V_C'(\varphi(\phi|x)) - V_I'(\varphi(\phi|x))\} \cdot \varphi'(\phi|x),$$

we have (for $x \neq y \in \{g, b\}$)

$$\begin{aligned}|\rho\hat{\varphi}'(\phi|x) + (1-\rho)\hat{\varphi}'(\phi|y)| \leqq & |\rho\varphi_0'(\phi|x) + (1-\rho)\varphi_0'(\phi|y)| \\ & + \lambda\kappa \sup_{d_C} D'(d_C) \cdot \sup_{\phi} |V_C'(\phi) - V_I'(\phi)| \\ & \times \sup_{\phi} |\rho\varphi'(\phi|x) + (1-\rho)\varphi'(\phi|y)|,\end{aligned}$$

which is less than or equal to η/δ for κ small (but nonzero).

Hence, for sufficiently small values of κ (say $\kappa \leqq \kappa^{**}$), the mapping $\Upsilon^{\kappa}(\varphi) = \hat{\varphi}$ is a mapping from $\mathfrak{X}$ into $\mathfrak{X}$. The space $\mathscr{C}^1([0,1],[0,1]^2)$ with the $\mathscr{C}^1$-norm is a locally convex, linear topological space. The set $\mathfrak{X}$ is convex and compact. Moreover, the mapping $\Upsilon^{\kappa}(\varphi) = \hat{\varphi}$ is clearly continuous (because Φ is continuous), and hence, by the Schauder–Tychonoff theorem (Dunford and Schwartz (1988, p. 456)), has a fixed point. For each value of κ, we denote a posterior updating function which is a fixed point of the mapping Υ^{κ} by φ_{κ}, and let $(V_I^{\kappa}, V_C^{\kappa}) = \Phi(\varphi_{\kappa})$ denote the corresponding value functions. Together, φ_{κ} and $(V_I^{\kappa}, V_C^{\kappa})$ satisfy (9)–(13).

Step 3. We now verify (14). There is a unique triple $(\varphi_0, V_I^0, V_C^0)$ satisfying (9), (10), (12) and (13) when $\kappa = 0$. Since $\nu > 0$ and $\lambda > 0$, there exist $\underline{\phi}, \underline{\phi}_0, \bar{\phi}_0$, and $\bar{\phi}, 0 < \underline{\phi} < \underline{\phi}_0 < \bar{\phi}_0 < \bar{\phi} < 1$, such that $\varphi_0(\phi|x) \in [\underline{\phi}_0, \bar{\phi}_0]$ for all $\phi \in [0,1]$ and $x \in \{g, b\}$. Moreover, there exists c_0^* such that $\delta(1-\lambda)(1-2\rho)\{V_C^0(\varphi_0(\phi|g)) - V_C^0(\varphi_0(\phi|b))\} > c$ for all $\phi \in [\underline{\phi}, \bar{\phi}]$ and $c < c_0^*$ (Lemma D).

Fix $c^* < c_0^*$. The sequential compactness of $\mathfrak{X}$ and $\mathfrak{Y}$ then implies the existence of κ^* ($\leqq \kappa^{**}$) such that for all $\kappa \leqq \kappa^*$, $\varphi_{\kappa}(\phi|x) \in [\underline{\phi}, \bar{\phi}]$ for all $\phi \in [0,1]$ and $x \in \{g, b\}$, and $\delta(1-\lambda)(1-2\rho)\{V_C^{\kappa}(\varphi_{\kappa}(\phi|g)) - V_C^{\kappa}(\varphi_{\kappa}(\phi|b))\} > c^*$ for all $\phi \in [\underline{\phi}, \bar{\phi}]$. ||

Lemma A. $\Psi^{\varphi}(\mathfrak{Y}) \subset \mathfrak{Y}$.

Proof. Denote the image of (V_I, V_C) under Ψ^{φ} by $(\hat{V}_I, \hat{V}_C)$. We verify that $(\hat{V}_I, \hat{V}_C) \in \mathfrak{Y}$ for all $(V_I, V_C) \in \mathfrak{Y}$. Clearly, both $\hat{V}_I$ and $\hat{V}_C$ are $\mathscr{C}^1$, and it is straightforward that $|\hat{V}_I(\phi)|, |\hat{V}_C(\phi)| \leqq Y$. Now,

$$\begin{aligned}|\hat{V}_I'(\phi)| & \leqq (1-2\rho) + \delta(1-\lambda)|\rho\varphi'(\phi|g) + (1-\rho)\varphi'(\phi|b)|(1-2\rho)/(1-\eta) \\ & \leqq (1-2\rho) + \eta(1-2\rho)/(1-\eta) \\ & = (1-2\rho)/(1-\eta).\end{aligned}$$

A similar calculation holds for $|\hat{V}_C'(\phi)|$, and so Ψ^{φ} maps $\mathfrak{Y}$ into $\mathfrak{Y}$. ||

Lemma B. Ψ^{φ} *is a contraction.*

Proof. First note that

$$\begin{aligned}& \sup_{\phi} |\Psi_1^{\varphi}(V_I, V_C)(\phi) - \Psi_1^{\varphi}(\hat{V}_I, \hat{V}_C)(\phi)| \\ & \quad \leqq \delta(1-\lambda)\left\{\sup_{\phi} \rho|V_I(\varphi(\phi|g)) - \hat{V}_I(\varphi(\phi|g))| + \sup_{\phi}(1-\rho)|V_I(\varphi(\phi|b)) - \hat{V}_I(\varphi(\phi|b))\right\} \\ & \quad \leqq \delta(1-\lambda)\sup_{\phi}|V_I(\phi) - \hat{V}_I(\phi)|,\end{aligned}$$

and similarly that $\sup_{\phi}|\Psi_2^{\varphi}(V_I, V_C)(\phi) - \Psi_2^{\varphi}(\hat{V}_I, \hat{V}_C)(\phi)| \leqq \delta(1-\lambda)\sup_{\phi}|V_C(\phi) - \hat{V}_C(\phi)|$.

Turning to the derivatives,

$$\begin{aligned}& \sup_{\phi}|(\Psi_1^{\varphi}(V_I, V_C))'(\phi) - (\Psi_1^{\varphi}(\hat{V}_I, \hat{V}_C))'(\phi)| \\ & \quad \leqq \delta(1-\lambda)\sup_{\phi}\{\rho\varphi'(\phi|g)|V_I'(\varphi(\phi|g)) - \hat{V}_I'(\varphi(\phi|g))| + (1-\rho)\varphi'(\phi|b)|V_I'(\varphi(\phi|b)) - \hat{V}_I'(\varphi(\phi|b))|\} \\ & \quad \leqq \delta(1-\lambda)\sup_{\phi}\{\rho\varphi'(\phi|g) + (1-\rho)\varphi'(\phi|b)\}\sup_{\phi}|V_I'(\phi) - \hat{V}_I'(\phi)| \\ & \quad \leqq \eta(1-\lambda)\sup_{\phi}|V_I'(\phi) - \hat{V}_I'(\phi)|,\end{aligned}$$

while a similar calculation shows that

$$\sup_{\phi} |(\Psi_2^{\varphi}(V_I, V_C))'(\phi) - (\Psi_2^{\varphi}(\hat{V}_I, \hat{V}_C))'(\phi)| \leqq \eta(1-\lambda) \sup_{\phi} |V'_C(\phi) - \hat{V}'_C(\phi)|.$$

Thus,

$$\|\Psi^{\varphi}(V_I, V_C) - \Psi^{\varphi}(\hat{V}_I, \hat{V}_C)\|_1 \leqq \max\{\delta(1-\lambda), \eta(1-\lambda)\} \|(V_I, V_C) - (\hat{V}_I, \hat{V}_C)\|_1$$

and, as claimed, Ψ^{φ} is a contraction. ||

Lemma C. *Φ is continuous.*

Proof. Suppose $\varphi_n \to \varphi_\infty$. Since $\mathfrak{Y}$ is sequentially compact (it is an equicontinuous collection of uniformly bounded functions on a compact space), there is a subsequence, denoted $\{\varphi_m\}$, with $(V_I^m, V_C^m) \equiv \Phi(\varphi_m)$ uniformly converging to some $(V_I, V_C) \in \mathfrak{Y}$. To see that V_I satisfies (12), note that

$$\begin{aligned}
&|V_I(\phi) - (1-2\rho)\phi - \rho - \delta(1-\lambda)\{\rho V_I(\varphi_\infty(\phi|g)) + (1-\rho)V_I(\varphi_\infty(\phi|b))\}| \\
&\quad \leqq |V_I(\phi) - V_I^m(\phi)| + |V_I^m(\phi) - (1-2\rho)\phi - \rho - \delta(1-\lambda)\{\rho V_I^m(\varphi_m(\phi|g)) + (1-\rho)V_I^m(\varphi_m(\phi|b))\}| \\
&\qquad + \delta(1-\lambda)\rho|V_I(\varphi_\infty(\phi|g)) - V_I^m(\varphi_m(\phi|g))| + \delta(1-\lambda)(1-\rho)|V_I(\varphi_\infty(\phi|b)) - V_I^m(\varphi_m(\phi|b))| \\
&\quad = |V_I(\phi) - V_I^m(\phi)| + \delta(1-\lambda)\rho|V_I(\varphi_\infty(\phi|g)) - V_I^m(\varphi_m(\phi|g))| \\
&\qquad + \delta(1-\lambda)(1-\rho)|V_I(\varphi_\infty(\phi|b)) - V_I^m(\varphi_m(\phi|b))|,
\end{aligned} \tag{A.2}$$

where the equality holds because $(V_I^m, V_C^m) \equiv \Phi(\varphi_m)$.

Now, fix $\varepsilon > 0$. There exists m_ε such that for all $m \geqq m_\varepsilon$ and all ϕ, $|V_I(\phi) - V_I^m(\phi)| < \varepsilon/3$. Moreover, since V_I is uniformly continuous and $\Phi(\varphi_m)$ converges uniformly to φ_∞, m_ε can be chosen such that $|V_I(\varphi_\infty(\phi|x)) - V_I^m(\varphi_m(\phi|x))| \leqq |V_I(\varphi_\infty(\phi|x)) - V_I^m(\varphi_\infty(\phi|x))| + |V_I^m(\varphi_\infty(\phi|x)) - V_I^m(\varphi_m(\phi|x))| \leqq \varepsilon/3$ for $x \in \{g, b\}$. Thus, (A.2) is less than or equal to ε, for all $\varepsilon > 0$, and so V_I satisfies (12) for the updating rule φ_∞. Because there is a unique solution to (12) given φ_∞, it must then be that V_I^m converges to V_I. A similar argument shows that $V_C^m(\varphi_m)$ converges to $V_C(\varphi_\infty)$, giving the result. ||

Lemma D. *There exists a cost c_0^* such that $V_C^0(\varphi_0(\phi|g)) - V_C^0(\varphi_0(\phi|b)) > c/[\delta(1-\lambda)(1-2\rho)]$ for all $\phi \in [\underline{\phi}, \bar{\phi}]$ and all $c < c_0^*$.*

Proof. There exists $\zeta > 0$ such that for all $\phi \in [\underline{\phi}, \bar{\phi}]$,

$$\varphi_0(\phi|g) - \varphi_0(\phi|b) > \zeta. \tag{A.3}$$

Given $h^t \in \{g, b\}^t$, denote the consumers' posterior using φ_0 after observing the sequence $h^t = (x_1, \ldots, x_t)$ by $\varphi_0(\phi|h^t) \equiv \varphi_0(\cdots(\varphi_0(\varphi_0(\phi|x_1)|x_2)\cdots|x_t)$. The value function V_C^0 can be written as, by recursively substituting,

$$V_C^0(\phi) = \frac{\rho - c}{1-(1-\lambda)\delta} + (1-2\rho)\phi + (1-2\rho)\textstyle\sum_{t=1}^{\infty} \delta^t(1-\lambda)^t \sum_{h^t \in \{g,b\}^t} \varphi_0(\phi|h^t) \Pr(h^t|H), \tag{A4}$$

where $\Pr(h^t|H)$ is the probability of realizing the sequence of outcomes h^t given that the firm chooses high effort in every period.

Then,

$$\begin{aligned}
V_C^0(\varphi_0(\phi|g)) - V_C^0(\varphi_0(\phi|b)) &= (1-2\rho)(\varphi_0(\phi|g) - \varphi_0(\phi|b)) \\
&\quad + (1-2\rho)\textstyle\sum_{t=1}^{\infty} \delta^t(1-\lambda)^t \sum_{h^t \in \{g,b\}^t} \{\varphi_0(\phi|gh^t) - \varphi_0(\phi|bh^t)\} \Pr(h^t|H) \\
&\geqq (1-2\rho)(\varphi_0(\phi|g) - \varphi_0(\phi|b)),
\end{aligned}$$

since $\varphi_0(\phi|gh^t) - \varphi_0(\phi|bh^t) \geqq 0$ for all ϕ and all h^t. Thus, using (A.3),

$$V_C^0(\varphi_0(\phi|g)) - V_C^0(\varphi_0(\phi|b)) > (1-2\rho)\zeta$$

and so an appropriate upper bound on c is

$$c_0^* \equiv \delta(1-\lambda)(1-2\rho)^2\zeta.$$

Note that this is not a tight bound, since we used only the inequalities that pertain to the first period of the value-function calculations. ||

A.3. *Proof of Proposition* 4.

Consider first the case of exogenous entry, $\kappa = 0$. The value function $V_I^0(\phi)$ can be written as

$$V_I^0(\phi) = \frac{\rho}{1-(1-\lambda)\delta} + (1-2\rho)\phi + (1-2\rho)\sum_{t=1}^{\infty} \delta^t (1-\lambda)^t \sum_{h^t \in \{g,b\}^t} \varphi_0(\phi|h^t) \Pr(h^t|L), \tag{A.5}$$

where $\Pr(h^t|L)$ is the probability of realizing the sequence of outcomes h^t given that the firm chooses low effort in every period. From (A.4) and (A.5),

$$V_C^0(\phi) - V_I^0(\phi) = (1-2\rho)\sum_{t=1}^{\infty}\{\sum_{h^t} \delta^t(1-\lambda)^t \Pr(h^t|H)\varphi_0(\phi|h^t) - \sum_{h^t}\delta^t(1-\lambda)^t \Pr(h^t|L)\varphi_0(\phi|h^t)\} + k, \tag{A.6}$$

where k is independent of ϕ. The set of histories $\{g,b\}^t$ can be partitioned into sets of "mirror images," $\{h^t, \hat{h}^t\}$, where h^t specifies g in period $\tau \leqq t$ if and only if $\hat{h}^t$ specifies b in period $\tau \leqq t$. It suffices to show that

$$\beta(\phi) \equiv \varphi_0(\phi|h^t)\Pr(h^t|H) + \varphi_0(\phi|\hat{h}^t)\Pr(\hat{h}^t|H) - \varphi_0(\phi|h^t)\Pr(h^t|L) - \varphi_0(\phi|\hat{h}^t)\Pr(\hat{h}^t|L)$$

is convex and maximized at $\phi = \frac{1}{2}$, since (A.6) is a weighted sum of such terms. Now notice that

$$\Pr(h^t|H) = \Pr(\hat{h}^t|L) \equiv x,$$

and

$$\Pr(\hat{h}^t|H) = \Pr(h^t|L) \equiv y,$$

which implies

$$\varphi_0(\phi|h^t) = (1-\lambda)^t \frac{x\phi}{x\phi + y(1-\phi)} + (1-(1-\lambda)^t)\gamma,$$

and

$$\varphi_0(\phi|\hat{h}^t) = (1-\lambda)^t \frac{y\phi}{y\phi + x(1-\phi)} + (1-(1-\lambda)^t)\gamma,$$

where γ does not depend on ϕ. Letting $x\phi + y(1-\phi) \equiv Z_x$ and $y\phi + x(1-\phi) \equiv Z_y$, we can then calculate (where β' and β'' denote first and second derivatives)

$$\beta' = (1-\lambda)^t \left[\frac{x^2y}{Z_x^2} + \frac{xy^2}{Z_y^2} - \frac{xy^2}{Z_x^2} - \frac{x^2y}{Z_y^2}\right],$$

which equals zero when $\phi = \frac{1}{2}$. We can then calculate

$$\beta'' = -2xy(1-\lambda)^t \left[\frac{(x-y)^2}{Z_x^3} + \frac{(y-x)^2}{Z_y^3}\right] \leqq 0,$$

with the inequalities strict whenever h^t specifies an unequal number of good and bad outcomes, so that $V_C^0 - V_I^0$ is strictly concave and maximized at $\phi = \frac{1}{2}$.

Moreover, since $d\{V_C^0(\phi) - V_I^0(\phi)\}/d\phi$ is strictly decreasing, it is bounded away from zero from below for $\phi \leqq \frac{1}{2} - \xi$ and it is bounded away from zero from above for $\phi \geqq \frac{1}{2} + \xi$.

The extension to κ small but nonzero is now an immediate implication of the sequential compactness of $\mathfrak{X}$ and $\mathfrak{Y}$ in the $\mathscr{C}^1$ norm (since convergence in this norm implies uniform convergence of the first derivative). ||

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REFERENCES

DATTA, S. (1996), "Building trust" (STICERD Discussion Paper 96/305, London School of Economics),

DIAMOND, D. W. (1989), "Reputation Acquisition in Debt Markets", *Journal of Political Economy*, **97**, 828–862.

DUNFORD, N. and SCHWARTZ, J. T. (1988) *Linear Operators. Part I. General Theory* (New York: Wiley–Interscience).

FUDENBERG, D. and LEVINE, D. K. (1992), "Maintaining a Reputation When Strategies Are Imperfectly Observed", *Review of Economic Studies*, **59**, 561–579.

GALE, D. and ROSENTHAL, R. W. (1994), "Price and Quality Cycles For Experience Goods". *Rand Journal of Economics*, **25**, 590–607.

HOLMSTRÖM, B. (1999), "Managerial Incentive Problems—A Dynamic Perspective", *Review of Economic Studies*, **66**, 169–182.

HÖRNER, J. (1999), "Reputation and Competition" (University of Pennsylvania).

KREPS, D., MILGROM, P. R., ROBERTS, D. J. and WILSON, R. (1982), "Rational Cooperation In The Finitely Repeated Prisoner's Dilemma", *Journal of Economic Theory*, **27**, 245–252.

KREPS, D. and WILSON, R. (1982), "Reputation and Imperfect Information", *Journal of Economic Theory*, **27**, 253–279.

MAILATH, G. J. and SAMUELSON, L. (1998), "Your Reputation Is Who You're Not, Not Who You Would Like To Be" (CARESS Working Paper No. 98–11, University of Pennsylvania).

MILGROM, P. R. and ROBERTS, D. J. (1982), "Predation, Reputation and Entry Deterrence", *Journal of Economic Theory*, **27**, 280–312.

TADELIS, S. (1999), "What's in a Name? Reputation as a Tradeable Asset", *American Economic Review*, **89**, 548–563.

WATSON, J. (1995), "Building a Relationship" (Mimeo, University of California, San Diego).

WATSON, J. (1999), "Starting Small and Renegotiation", *Journal of Economic Theory*, **85**, 52–90.