Efficient Non-Contractible Investments in Finite Economies*

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Abstract

Investors making complementary investments typically do not have incentives to invest efficiently when they cannot contract prior to their decisions. When they bargain over the surplus generated by their investments, they will usually not obtain the full fruits of the investment. Intuitively, this hold-up problem should be ameliorated if, in the bargaining stage, each agent has alternatives to the partner he is bargaining with. We characterize the matching and division of surplus in finite economies for any initial investment decisions. We provide conditions on those decisions that guarantee that each agent will capture the change in the aggregate social surplus that results from any investment change he makes. We further show that for any given problem, there exists a bargaining rule by which pairs split their surplus that will support efficient investment choices in equilibrium. We also show, however, that overinvestment or underinvestment can occur for natural bargaining rules. *Journal of Economic Literature* classification numbers: C78, D41, D51.

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1. Introduction

Investors making complementary investments typically do not have incentives to invest efficiently when they cannot contract prior to their decisions. When investors bargain over the surplus generated by their investments, they will usually not obtain the full fruits of the investment. Intuitively, this *hold-up problem* should be ameliorated for a given agent if, in the bargaining stage, there are alternative partners for the agent he is bargaining with. When there are close substitutes for any given agent, competition among those potential partners can indeed ameliorate the hold-up problem, resulting in more efficient investments. Cole, Mailath, and Postlewaite [5] analyzed a two-sided matching model in which a continuum of buyers and sellers make investment decisions prior to a matching stage. Subsequent to those investments, agents match, produce, and split the surplus that results from that production. We showed there that if bargaining respected outside options in the sense that the resulting allocation was in the core of the assignment game, efficient investment decisions could always be supported in equilibrium.

With a continuum of agents, nearly all agents have essentially perfect substitutes. Further, no agent can affect other agents' payoffs through his own investment. For some problems, a continuum plausibly captures intense competition among agents for partners, but for other problems, there is a natural pairing of partners, with each agent facing inferior alternatives should he leave the match which is most efficient. In this paper we present a finite agent model that allows us to analyze the case of imperfect competition among potential partners, the effect that reduced competition has on investments, and the possibility that an individual agent's investment can affect other agents' payoffs. We show that the finiteness of the set of agents is not necessarily a barrier to efficient investment decisions. Agents' investment decisions will depend on the bargaining process that determines the split of the surplus to any pair, and for any efficient investment decisions, there is a bargaining rule that respects the outside options represented by rematching which will support those efficient investment decisions.

Although the sets of competitive and core allocations of the economies for fixed investments are the same, the existence of a bargaining rule that provides incentives for efficient investment is not assured by either the first or second welfare theorems. It is true that the economy with complete markets for investments has a Walrasian equilibrium and that the equilibrium is efficient. The prices in this equilibrium cannot, however, be used to construct a bargaining rule for the agents. The difficulty is that for an agent to determine whether a particular investment is optimal, he must compare the consequences of that investment to alternative investments. But competitive prices only determine surplus divisions for *equilibrium* choices. In a world with a finite number of agents, competitive prices are typically not market clearing when nonequilibrium choices have been made. In contrast, we require that the bargaining be consistent with both feasibility and outside options for *all* investments, not just equilibrium ones.[1]

The existence of a bargaining rule consistent with efficiency does not necessarily mean that efficient investments are guaranteed, however. Bargaining rules are not a choice variable of the participants; it is reasonable to think of bargaining rules that have historically evolved within a given society. Any given bargaining rule governing a society will generally *not* be consistent with efficient equilibrium investments for most configurations of agent characteristics.

Even if the "right" bargaining rule is in force, equilibrium investment decisions need not be efficient. While there is always *some* bargaining rule that will support efficient investment, there will often be multiple equilibria, some of which exhibit inefficient equilibrium choices stemming from coordination failure: each side is investing inefficiently, but neither side finds it beneficial to unilaterally alter investment.

Intuitively, the "right" bargaining rule requires agents to fully appropriate the value of their decisions (to use the language of Makowski and Ostroy [14]). The simplest version of the sufficient conditions for this to be true involves binding outside options, so that all agents' payoffs are completely determined by the payoffs that any single agent receives. With a finite population, this requires that multiple agents are choosing the attributes that are also chosen by other agents. If each agent is idiosyncratic (for example, has different costs of acquiring attributes), then efficient attribute choices will not imply binding outside options. Efficiency then results only if the bargaining between agents results in a particular outcome (see, in particular, the discussion after Proposition 4). On the other hand, outside options do limit the agreements that agents can come to, and the richer the set of chosen attributes, the closer to binding the outside options become. A plausible (but incorrect) conjecture is that as the number of agents becomes large, outside options become binding and so in large economies, we have full appropriation. The conjecture fails even when the set of agents is

[1] This is discussed in more detail in Section 6.

rich (in the sense that each agent has a close competitor in exogenous characteristics), precisely because attributes are endogenous: agents may not have a close competitor in attributes.

The outline of the paper is as follows. In the next section, we present the formal model (Section 2) and characterize the bargaining outcomes (Section 3). Section 4 provides sufficient conditions for agents to receive the social value of their investments, and Section 5 compares the cases in which agents can and cannot contract prior to investing. Finally, Section 6 discusses the relationship between our results and the welfare theorems, while Section 7 describes related literature.

2. The investment problem

An *investment problem* Γ is the collection $\{I, J, B, S, \psi, c, v\}$, where

- I and J are disjoint finite sets of buyers and sellers;
- B and S are, respectively, the set of possible attributes (income, wealth, or willingness to pay) buyers can choose from and the set of possible attributes (quality of good) for sellers;
- $\psi : B \times I \to \Re_+$, where $\psi(b, i)$ is the cost to buyer i of attribute b;
- $c : S \times J \to \Re_+$, where $c(s, j)$ is the cost to seller j of attribute s; and
- $v : B \times S \to \Re_+$, where $v(b, s)$ is the surplus generated by a buyer with attribute b matching with a seller with attribute s.

We assume B and S are compact subsets of $\Re_+$. We assume (without loss of generality) that there are equal populations of buyers and sellers.[2] We assume that v displays complementarities in attributes (v is supermodular): for $b < b'$ and $s < s'$, $v(b', s) + v(b, s') \leq v(b, s) + v(b', s')$. Equivalently, if v is $\mathcal{C}^2$, $\partial^2 v / \partial b \partial s \geq 0$. We will sometimes assume that the surplus function is *strictly* supermodular, i.e., $v(b', s) + v(b, s') < v(b, s) + v(b', s')$ for all $b < b'$ and $s < s'$. We also assume v is continuous and strictly increasing in b and in s, and that $\psi(\cdot, i)$ and $c(\cdot, j)$ are continuous and strictly increasing in b and in s, respectively.

We model the bargaining and matching process that follows the attribute choices as a cooperative game. Given a fixed distribution of attributes of buyers and sellers, the resulting cooperative game is an *assignment game*: there are two

[2] The case of more buyers than sellers, for example, is handled by adding additional sellers with attribute 0 and setting $v(b, 0) = 0$ for all b.

buyer's share (x_i)	2	$4\frac{1}{2}$
buyer's attribute (b_i)	2	3
buyer (i)	1	2
seller (j)	1	2
seller's attribute (s_j)	2	3
seller's share (p_j)	2	$4\frac{1}{2}$

Figure 2.1: An example with two buyers and sellers.

populations of agents (here, buyers and sellers), with each pair of agents (one from each population) generating some value. To distinguish this assignment game from the assignment game we describe in Section 5, we call this assignment game the *ex post assignment game* (indicating that attribute choices are taken as fixed). An outcome in the assignment game is a *matching* (each buyer matching with no more than one seller and each seller matching with no more than one buyer) and a *bargaining outcome* or *payoff* (a division of the value generated by each matched pair between members of that pair). We denote the buyer's share of the surplus by $x \geq 0$ and the seller's share by $p \geq 0$, with $x + p \leq v(b, s)$.[3]

Definition 1. *A* ***matching*** *m is a function $m : I \to J \cup \{\emptyset\}$, where m is one-to-one on $m^{-1}(J)$, and $\emptyset$ is interpreted as no match.*

Definition 2. *A bargaining outcome $(\mathbf{x}, \mathbf{p}) \in \Re_+^I \times \Re_+^J$ is* ***feasible for the matching*** *m if $x_i + p_{m(i)} \leq v(b_i, s_{m(i)})$ whenever $m(i) \neq \emptyset$, $x_i = 0$ whenever $m(i) = \emptyset$, and $p_j = 0$ whenever $j \notin m(I)$. A bargaining outcome is* ***feasible*** *if it is feasible for some matching.*

To illustrate the matching-bargaining process, suppose there are two buyers, $\{1, 2\}$, and two sellers, $\{1, 2\}$. For now, we fix the attributes of the buyers and sellers as in Figure 2.1. The surplus generated by a pair (b, s) is given by the product of their attributes, $v(b, s) = b \cdot s$. Figure 2.1 displays one particular bargaining outcome for this environment with each of the two columns representing a matched pair and the split of the surplus for that pair. Total surplus is maximized by the indicated matching, and the split of the surplus for the pairs is unique if the sharing rule is symmetric with respect to buyers and sellers.

Suppose now that attributes are not fixed, but are chosen from the set $\{2, 3\}$. We focus on the behavior of seller 1, with the attributes of the other agents

[3]Note that shares are amounts, not fractions.

x_i	3	$4\frac{1}{2}$
b_i	2	3
i	1	2
j	1	2
s_j	3	3
p_j	3	$4\frac{1}{2}$

Figure 2.2: Seller 1 with attribute $s = 3$.

unchanging.[4] If the surplus is always divided equally and seller 1 chose instead $s = 3$, then the matching and surplus division are as in Figure 2.2.

In this example, equal division violates equal treatment: The two sellers have the same attribute but receive different payoffs. But then seller 1 could make buyer 2 a marginally better offer than he gets when matched with seller 2. In other words, there is a pair of agents who by matching and appropriately dividing the resulting surplus can make themselves better off. We take into account each agents' outside options by requiring that no pair of agents can, by matching and sharing the resulting surplus, make themselves strictly better off:

Definition 3. *A bargaining outcome* $(\mathbf{x}, \mathbf{p})$ *is* ***stable*** *if it is feasible and for all* $i \in I$ *and* $j \in J$,

$$x_i + p_j \geq v(b_i, s_j). \tag{2.1}$$

A matching associated with a stable bargaining outcome is a ***stable matching****.*

In a feasible and stable bargaining outcome, $x_i + p_{m(i)} = v(b_i, s_{m(i)})$, and so there are no transfers across matched pairs. As usual in assignment games, stable bargaining outcomes are core allocations of the assignment game and conversely, where the characteristic function of the assignment game has value $V(A)$ at a coalition $A \subset I \cup J$ given by the maximum of the sum of surpluses of matched pairs (the maximum is taken over all matchings of buyers and sellers in A).[5] Since

[4] The following cost functions for the two buyers and for seller 2 ensure (assuming the bargaining is monotonic) that their optimal choice of attributes are given in Figure 2.1: $\psi(2,1) = \psi(b,2) = c(s,2) = 0, \psi(3,1) = 10$.

[5] Assignment games have received considerable attention in the literature. The core of any assignment game is nonempty and coincides with the set of Walrasian allocations (see Kaneko [11] and Quinzii [16]). Our case is particularly simple, since v is supermodular.

While the set of stable bargaining outcomes coincides with the core, the notion itself is not inherently cooperative. Equilibrium outcomes of almost any noncooperative game with frictionless matching will be stable. See, for example, Felli and Roberts [7].

buyer attributes are described by the vector $\mathbf{b}$ and seller attributes are described by the vector $\mathbf{s}$, we sometimes write $V(\mathbf{b},\mathbf{s})$ for $V(I \cup J)$.

We are thus modelling the game facing buyers and sellers as one of simultaneously choosing attributes and, subsequent to the choice of attributes, matching and sharing the surplus generated by the matches. We restrict attention to matches and payoffs that are stable, given the choice of attributes. Since v is supermodular, it is straightforward to show directly that there always exists a stable payoff for any vector of attribute choices.

There is, however, one important issue in considering the attribute investment decisions as a noncooperative game. In order to treat attribute choices as a noncooperative game, each agent must be able to compare the payoffs from two different attribute choices, given other agents' choices. Typically there is not a unique stable outcome associated with a vector of attributes; in fact, as we will see, there is usually a continuum of stable outcomes. Thus, there must be a selection that specifies *which* stable outcome is associated with a given attribute choice vector. We describe this selection by a *bargaining outcome function* (or, more simply, a *bargaining function*) $g : B^I \times S^J \to \Re^I_+ \times \Re^J_+$, with $g(\mathbf{b},\mathbf{s}) = (\mathbf{x},\mathbf{p})$ a stable outcome for each vector of attribute choices $(\mathbf{b},\mathbf{s})$. We denote by $x_i(\mathbf{b},\mathbf{s})$ buyer i's share when the vector of attributes is $(\mathbf{b},\mathbf{s})$ and by $p_j(\mathbf{b},\mathbf{s})$ the j-th seller's share. Observe that given g, buyers and sellers are simultaneously choosing attributes, with payoffs $x_i(\mathbf{b},\mathbf{s}) - \psi(b_i, i)$ to buyer i and $p_j(\mathbf{b},\mathbf{s}) - c(s_j, j)$ to seller j. This is a standard strategic form game. We next define a notion, *weak ex post contracting equilibrium*, that combines the requirement that every vector of attribute choices lead to a stable payoff of the induced ex post assignment game with the requirement that attribute choices are a Nash equilibrium of the strategic form game.

Definition 4. *Given an investment problem* $\Gamma = \{I, J, B, S, \psi, c, v\}$, *a* ***weak ex post contracting equilibrium*** *is a pair* $\{g^*, (\mathbf{b}^*, \mathbf{s}^*)\}$ *such that*

1. $g^* : B^I \times S^J \to \Re^I_+ \times \Re^J_+$, *where for any choice of characteristics* $(\mathbf{b},\mathbf{s})$, $g^*(\mathbf{b},\mathbf{s}) = (\mathbf{x}^*(\mathbf{b},\mathbf{s}), \mathbf{p}^*(\mathbf{b},\mathbf{s}))$ *is a stable payoff for* $(\mathbf{b},\mathbf{s})$, *and*

2. *for each* $i \in I$ *and* $b'_i \in B$, $x^*_i(\mathbf{b}^*_{-i}, b^*_i, \mathbf{s}^*) - \psi(b^*_i, i) \geq x^*_i(\mathbf{b}^*_{-i}, b'_i, \mathbf{s}^*) - \psi(b'_i, i)$, *and for each* $j \in J$ *and* $s'_j \in S$, $p^*_j(\mathbf{b}^*, \mathbf{s}^*_{-j}, s^*_j) - c(s^*_j, j) \geq p^*_j(\mathbf{b}^*, \mathbf{s}^*_{-j}, s'_j) - c(s'_j, j)$.

This equilibrium notion combines a noncooperative notion (Nash) and a cooperative notion (stability), along with the requirement that the cooperative notion apply after all histories. Each individual is best replying to the actions of everyone else, the future consequences of any attribute choice are correctly foreseen, and any attribute choice must lead to a stable payoff.

We think of the bargaining function, g, as capturing the way bargaining transpires in an investment problem. Restricting the sharing of the surpluses arising from a given vector of attribute choices $(\mathbf{b}, \mathbf{s})$ to stable payoffs constrains the allowable bargaining process. However, it still leaves considerable indeterminacy, since there is typically a multitude of stable allocations for a given vector of attributes choices. For some investment problems, that indeterminacy might be resolved through bargaining that favors the buyers to the greatest extent possible, given the constraints imposed by stability. For other problems, bargaining might resolve the indeterminacy in favor of the sellers, while in still others, bargaining might result in as equal a division as is consistent with stability.

An alternative to including the bargaining function in the definition of the equilibrium is to include it in the description of the investment problem. For example, if bargaining favors buyers, the bargaining function capturing this could be included in the specification of the investment problem, leading to a "buyer-friendly" bargaining problem. There are two difficulties with this approach. First, the bargaining function is endogenous. Second, for some bargaining functions, there may be no pure strategy equilibrium. This nonexistence reflects an inconsistency between an exogenously specified bargaining function and the given data of an investment problem, I, J, B, S, ψ, c, and v. The way in which bargaining resolves indeterminacy must be endogenously determined in concert with agents' investment choices.

We impose further restrictions on weak ex post contracting equilibria, in an equilibrium-selection spirit. As defined, for a given set of attribute investments, the outcome selected by the bargaining function can depend on the identity of the individuals who have chosen particular attributes. We focus on the case in which bargaining is anonymous in the sense that it depends only on attributes, independent of the identities of the agents choosing those attributes.

Even with anonymity, the definition of weak ex post contracting equilibrium allows for bargaining functions that embody a substantial amount of arbitrariness. Consider, for example, a bargaining function that selects the stable outcome that is most favorable to buyers as long as all buyers choose attributes that are consistent with maximizing aggregate net value, and selecting the stable outcome that is most favorable to sellers otherwise.[6] A bargaining function that utilizes

[6] In our setting, all buyers agree on the best and worst stable payoff vectors (and all sellers have the reverse ranking). Moreover, with a finite set of buyers, even in an anonymous equilibrium any deviation is detected, since any deviation results in a different empirical distribution over

a trigger specification of this type would break any link between the marginal social return from an investment and its private return: A single buyer's change in attribute would alter the payoffs to all agents. Clearly, such bargaining functions will not provide agents with incentives to invest efficiently, but the reasons underlying the inefficiency are economically uninteresting. We restrict attention to a subset of weak ex post contracting equilibria that reduce the arbitrariness of the bargaining function by fixing the split at the bottom pair.

Definition 5. *An* ***ex post contracting equilibrium (EPCE)*** *is a weak ex post contracting equilibrium,* $\{g^*, (\mathbf{b}^*, \mathbf{s}^*)\}$*, that is anonymous and, if for any two attribute vectors* $(\mathbf{b}', \mathbf{s}')$ *and* $(\mathbf{b}'', \mathbf{s}'')$*, there exists* $i \in I$ *such that* $b_i' = \min_{\ell \in I}\{b_\ell'\} = b_i'' = \min_{\ell \in I}\{b_\ell''\}$ *and there exists* $j \in J$ *such that* $s_j' = \min_{\ell \in J}\{s_\ell'\} = s_j'' = \min_{\ell \in J}\{s_\ell''\}$*, then*

$$\begin{aligned} x_i^*(\mathbf{b}', \mathbf{s}') &= x_i^*(\mathbf{b}'', \mathbf{s}'') \text{ and} \\ p_j^*(\mathbf{b}', \mathbf{s}') &= p_j^*(\mathbf{b}'', \mathbf{s}''). \end{aligned}$$

If there is an imbalance between the number of buyers and sellers, then we (along the lines of footnote 2) add enough dummy agents to equalize the number of buyers and sellers. In this case, the bottom pair necessarily receives a payoff of zero, and consequently, the stable outcome necessarily favors agents on the short side of the market.

3. Characterization of stable allocations for a finite population

Much is known about the properties of stable allocations (see, e.g., Roth and Sotomayor [17] and Becker [2]). The lemma below summarizes several properties that we will need: For any attribute vector (b, s), any stable outcome matches attributes positively assortatively; all buyers with equal attributes receive equal payoffs, and similarly for sellers; and finally, in checking stability, one need not examine all unmatched pairs, but only those unmatched pairs for which the partners have attributes which are "close" to those of their matches. Before stating the lemma we make the following definition:

Definition 6. *A matching* m *is* ***positively assortative*** *if* $m(I) = J$ *and for any* $i, j \in I, b_i > b_j \Rightarrow s_{m(i)} \geq s_{m(j)}$*. A labeling of agents is* ***positively assortative*** *if* $I, J = \{1, ..., n\}$ *and attributes are weakly increasing in index.*

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attributes.

Lemma 1. *Given a vector of attributes* $(\mathbf{b}, \mathbf{s})$ *and a positively assortative labeling of agents,*

1. *every stable matching is positively assortative on attributes;*
2. *every stable payoff exhibits equal treatment:* $b_i = b_{i'} = b \Rightarrow x_i = x_{i'} \equiv x_b$ *and* $s_j = s_{j'} = s \Rightarrow p_j = p_{j'} \equiv p_s$*; and*
3. *a payoff* $(\mathbf{x}, \mathbf{p})$ *is stable if and only if for all* i*,*

$$\begin{aligned} x_i + p_i &= v(b_i, s_i), \\ x_i + p_{i+1} &\geq v(b_i, s_{i+1}), \text{ and} \\ x_{i+1} + p_i &\geq v(b_{i+1}, s_i). \end{aligned}$$

Proof. The first two statements are straightforward. Without loss of generality, the stable matching can be taken to be by index, yielding $x_i + p_i = v(b_i, s_i)$. The two inequalities are immediate implications of stability.

In order to show sufficiency, we argue to a contradiction. Suppose there exists a $k > 1$ such that $x_i + p_{i+k} < v(b_i, s_{i+k})$. Then

$$\begin{aligned} x_{i+1} + p_{i+k} &< x_{i+1} + v(b_i, s_{i+k}) - x_i \\ &\leq x_{i+1} + v(b_i, s_{i+k}) - v(b_i, s_{i+1}) + p_{i+1} \\ &= v(b_{i+1}, s_{i+1}) + v(b_i, s_{i+k}) - v(b_i, s_{i+1}) < v(b_{i+1}, s_{i+k}), \end{aligned}$$

where the last inequality holds because v is strictly supermodular. Induction then yields $x_{i+k-1} + p_{i+k} < v(b_{i+k-1}, s_{i+k})$, a contradiction. ■

The third part of this lemma implies that in order to check the stability of a payoff vector, we need only check adjacent pairs in a positively assortative matching. If no buyer (or seller) can block when matched with a partner adjacent to his or her current partner, the payoff vector is stable.

Since stable payoffs exhibit equal treatment, we sometimes refer to the payoffs to an attribute rather than the payoffs to an individual, and we often will not distinguish between the two.

Lemma 1 states that equal treatment is necessary for stability; in some cases, it is sufficient for stability as well. Consider the allocation in Figure 3.1 with $v(b, s) = b \cdot s$. As before, matches should be read by columns.

Once the bottom (left-most) pair's shares have been determined in this example, all other agents' payoffs are uniquely determined by equal treatment because of the "overlap" in the players' attributes.

x	2	2	4	4	4	7	7	7	15
b	2	2	3	3	3	4	4	4	6
s	2	2	2	3	3	3	4	4	4
p	2	2	2	5	5	5	9	9	9

Figure 3.1: Equal treatment can imply stability.

The next proposition and corollary formalize the intuition illustrated by this example. If we order the *values* of chosen attributes of the buyers from low to high, we denote by $b_{(\kappa)}$ the κ-th value and, similarly, by $s_{(\kappa)}$ the κ-th value for the seller.[7]

Definition 7. *The pair of attribute vectors* $(\mathbf{b}, \mathbf{s})$ *is* ***overlapping*** *if, for a positively assortative matching* m *and any* κ*, there exists* i, i' *such that* $b_i = b_{(\kappa)}$, $b_{i'} = b_{(\kappa+1)}$, $s_{m(i)} = s_{m(i')}$.

Overlapping attribute vectors have the following more transparent formulation. Suppose we index the buyers and sellers by the integers 1 through n so that attributes are weakly increasing in index. Matching by index (i.e., $i = m(i)$) is then positively assortative on attributes. The attribute vectors are overlapping if $b_{j-1} \neq b_j \Rightarrow s_{j-1} = s_j$. Note that the notion is symmetric, since $b_{i-1} \neq b_i \Rightarrow s_{i-1} = s_i$ implies $s_{i-1} \neq s_i \Rightarrow b_{i-1} = b_i$.

Proposition 1. *Suppose the attribute vectors are overlapping, the labeling of agents is positively assortative, and* $(\mathbf{x}, \mathbf{p})$ *is a payoff vector for a positively assortative matching that satisfies*

1. *equal treatment, and*
2. *no waste:* $x_i + p_i = v(b_i, s_i)$.

Then $(\mathbf{x}, \mathbf{p})$ *is stable.*

Proof. Since we have assumed no waste, we need only check to see that for adjacent pairs, if the matching is switched, neither of the new pairs can block. But since the vectors of attributes are overlapping, either both buyers have the

[7] Note that we are ordering distinct values of attribute choices, not individual agent choices. Consequently, $b_{(\kappa-1)} < b_{(\kappa)} < b_{(\kappa+1)}$, even if two buyers have attribute $b_{(\kappa)}$. In particular, $b_{(\kappa)}$ is not the κ^{th} order statistic.

same attribute or both sellers have the same attribute, and the assumption that $x_i + p_i = v(b_i, s_i)$ ensures that neither of the new pairs can block. ■

In what follows, we denote the payoff to attribute $b_{(\kappa)}$ ($s_{(\kappa)}$) by $x_{(\kappa)}$ ($p_{(\kappa)}$, respectively).

Corollary 1. *Suppose* $(\mathbf{b}, \mathbf{s})$ *is overlapping and the pair* $(x_{(1)}, p_{(1)})$ *satisfies* $x_{(1)} \geq 0$, $p_{(1)} \geq 0$, *and* $x_{(1)} + p_{(1)} = v(b_{(1)}, s_{(1)})$. *Define* (x, p) *recursively as follows:*

$$x_{(\kappa+1)} = x_{(\kappa)} + [v(b_{(\kappa+1)}, s) - v(b_{(\kappa)}, s)],$$

where $s = s_{m(i)} = s_{m(i')}$ *and* $b_i = b_{(\kappa)}$, $b_{i'} = b_{(\kappa+1)}$ *for some positive assortative matching* m *and* $i, i' \in I$, *and similarly for the sellers.*[8] *Then the payoffs* (x, p) *are stable, and every stable payoff can be constructed in this way.*

Proof. Since there is a unique positive assortative matching of attributes, there is a unique seller attribute that satisfies, for any positively assortative matching of agents, $s = s_{m(i)} = s_{m(i')}$ and $b_i = b_{(\kappa)}$, $b_{i'} = b_{(\kappa+1)}$ for some $i, i' \in I$. Moreover, the hypothesis of overlapping attribute vectors ensures that s exists and that for all matched attributes (b, s), $x_b + p_s = v(b, s)$. Hence, we have equal treatment and no waste, and Proposition 1 applies.

Equal treatment in stable payoffs guarantees that every stable payoff has this property. ■

Corollary 1 provides a complete characterization of stable outcomes when attribute vectors are overlapping. When attribute vectors don't overlap, there is a degree of indeterminacy in stable payoffs, even fixing the division of the value for the bottom pair. One can, however, construct stable payoffs for a vector of positively assortative, nonoverlapping attributes in a straightforward way: Fix the share for the bottom pair. For the largest overlapping subset of attributes containing this bottom pair of attributes,[9] use equal treatment to determine the payoffs to those attributes. Where there is a gap between the attributes for this subset of agents and those higher, Lemma 1 puts constraints on how the surplus for the pair above the gap can be divided. Choose an arbitrary distribution of surplus for that pair, subject to those constraints. Allocate the surplus for the adjoining pairs so long as there is overlap, and each time a gap is encountered, proceed as above.

We now formalize this idea and provide bounds on the indeterminacy of stable payoffs. Note that, without loss of generality, given an attribute vector $(\mathbf{b}, \mathbf{s})$, a

[8] This defines the payoffs to attributes. Every agent with the same attribute receives that payoff.

[9] This subset may consist of only the bottom pair.

positively assortative labeling of agents, and a stable matching m, we can assume buyer i is matched with seller $m(i) = i$. Let $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ denote the vector of attributes for a population of agents $(I^\dagger, J^\dagger)$, $I \subset I^\dagger$ and $J \subset J^\dagger$, with overlap constructed as follows: if there exists i such that $b_i \neq b_{i+1}$ and $s_i \neq s_{i+1}$, then in the extended population, there is an additional buyer (with index $i + \frac{1}{2}$) with attribute b_i and an additional seller (also with index $i + \frac{1}{2}$) with attribute s_{i+1}. We refer to $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ as the *buyer-favored extension* of $(\mathbf{b}, \mathbf{s})$. Note that a stable matching for $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ is given by $m^\dagger(i) = i$ for all i. This maintains the original matching on I and extends it to the new agents by matching any new buyer $i + \frac{1}{2}$ with the new seller $i + \frac{1}{2}$. Similarly, let $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$ denote the vector of attributes for the population $(I^\ddagger, J^\ddagger)$ obtained from $(\mathbf{b}, \mathbf{s})$ by giving attribute b_{i+1} to buyer $i + \frac{1}{2}$ and attribute s_i to seller $i + \frac{1}{2}$. We refer to $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$ as the *seller-favored extension* of $(\mathbf{b}, \mathbf{s})$. Note that $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$ also satisfies overlap and that $I^\ddagger = I^\dagger$ and $J^\ddagger = J^\dagger$. Note also that for any stable payoff for either extension, the restriction of the payoff to the original agents, $I \cup J$, is stable.

The attribute vectors $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ and $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$ are minimal extensions of $(\mathbf{b}, \mathbf{s})$ that yield overlapping attribute vectors by adding just enough of the "right" attributes. Note that the bottom pair of matched attributes is unaffected by the extension, so that $b_1^\dagger = b_1^\ddagger = \min b_i \equiv \underline{b}$ and $s_1^\dagger = s_1^\ddagger = \min s_j \equiv \underline{s}$. Since $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ is overlapping, by Corollary 1, there is a unique stable payoff corresponding to each value of $x_{\underline{b}}^\dagger$, which we call a *buyer-favored* payoff, and similarly for $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$.

The following proposition shows that the vector $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ uniformly favors buyers in the sense that it gives the maximal payoff to buyers over stable payoffs given $x_{\underline{b}}^\dagger$. Suppose that buyers and sellers are positively assortatively matched and that there are adjacent pairs for which both the buyers' and sellers' attributes differ. The buyer-favored extension maximizes the buyer's payoff by having a seller with the same attribute as his partner match with a buyer with a lower attribute, which minimizes the payoff to that attribute (by Lemma 1). The buyer then receives the remainder. Analogously, the seller-favored extension $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$ gives the maximum payoff to the seller subject to the bound.

Proposition 2. *Suppose* $(\mathbf{b}, \mathbf{s})$ *is a vector of attributes and* $(\mathbf{x}, \mathbf{p})$ *is a stable payoff. Let* $(\mathbf{x}^\dagger, \mathbf{p}^\dagger)$ *be the unique stable payoff for the buyer-favored extension of* $(\mathbf{b}, \mathbf{s})$ *satisfying* $x_{\underline{b}}^\dagger = x_{\underline{b}}$, *and let* $(\mathbf{x}^\ddagger, \mathbf{p}^\ddagger)$ *be the unique stable payoff for the seller-favored extension of* $(\mathbf{b}, \mathbf{s})$ *satisfying* $x_{\underline{b}}^\ddagger = x_{\underline{b}}$. *Then,* $(\mathbf{x}^\dagger, \mathbf{p}^\dagger)$ *and* $(\mathbf{x}^\ddagger, \mathbf{p}^\ddagger)$ *are stable payoffs for* $(\mathbf{b}, \mathbf{s})$. *Moreover,*

$$x_b^\ddagger \leq x_b \leq x_b^\dagger, \ \ \forall b, \tag{3.1}$$

and

$$p_s^\dagger \leq p_s \leq p_s^\ddagger, \ \ \forall s. \tag{3.2}$$

Finally, for any attribute in $(\mathbf{b}, \mathbf{s})$*, any share* x_b *satisfying (3.1) or* p_s *satisfying (3.2), there is a stable payoff for* $(\mathbf{b}, \mathbf{s})$ *giving shares* x_b *to* b *or* p_s *to* s*.*

Proof. It is immediate from Lemma 1 that $(\mathbf{x}^\dagger, \mathbf{p}^\dagger)$ and $(\mathbf{x}^\ddagger, \mathbf{p}^\ddagger)$ are stable payoffs for $(\mathbf{b}, \mathbf{s})$. Since no new attributes are introduced in $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ or $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$, and every pair of attributes in $(\mathbf{b}, \mathbf{s})$ matched in a stable matching remains matched when the attribute vector is $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$ or $(\mathbf{b}^\ddagger, \mathbf{s}^\ddagger)$, it is enough to show that $x_b \leq x_b^\dagger \ \forall b$ to verify (3.1) and (3.2).

Let $b_{(\kappa)}$ be the first buyer attribute at which there is no overlap, and note that $b_{(\kappa)} = b_{(\kappa)}^\dagger$. The attribute $b_{(\kappa)}$'s stable payoff is at a maximum when the stable payoff of the sellers with attribute s^κ is at a minimum, where s^κ is the smallest seller attribute matched with the buyer attribute $b_{(\kappa)}$. This occurs when $x_{(\kappa-1)} + p_{s^\kappa} = v(b_{(\kappa-1)}, s^\kappa)$. Thus,

$$x_{_{(\kappa)}} \leq x_{(\kappa-1)} + v(b_{(\kappa)}, s^\kappa) - v(b_{(\kappa-1)}, s^\kappa) \equiv x_{(\kappa)}^\dagger(x_{(\kappa-1)}),$$

with equality yielding a payoff that is consistent with stability. Moreover, $x_{(\kappa)}^\dagger(x_{(\kappa-1)})$ is the payoff of attribute $b_{(\kappa)}$ when the population attribute vector is $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$, since attribute $b_{(\kappa-1)}$ receives a payoff of $x_{(\kappa-1)}$. Note also that $x_{(\kappa)}^\dagger(x_{(\kappa-1)})$ is increasing in $x_{(\kappa-1)}$. Proceeding recursively up buyer and seller attributes shows that buyer attribute $b_{(\kappa)}$'s maximum stable payoff is calculated as if there is the pattern of overlap of $(\mathbf{b}^\dagger, \mathbf{s}^\dagger)$.

Now consider the sufficiency of (3.1) for a single buyer attribute's share to be stable. Fix some share satisfying (3.1) for an attribute b. We now proceed inductively to fill in shares to the other attributes above and below. For attributes above b, apply the procedure described just after Corollary 1. The same procedure can also be applied for attributes below b, starting at b and working down. The bounds (3.1) guarantee that each step will be feasible and result in the bottom pair receiving the split $(x_{\underline{b}}, p_{\underline{s}})$. ■

Note that the proposition does not assert that any vector of shares that satisfies (3.1) for all attributes can be achieved in a single stable payoff. Propositions 1 and 2 characterize the stable outcomes associated with any attribute vector $(\mathbf{b}, \mathbf{s})$. These propositions provide the tools we use in the next section to analyze the incentives agents have in making investment decisions.

4. Incentives for efficient investment

The investment inefficiency that can result from the hold-up problem is illustrated by the following bargaining function. Given a vector of attributes $(\mathbf{b}, \mathbf{s})$, suppose the labeling of agents is positively assortative and buyer i matches with seller i. The bottom pair divides the surplus equally, so that $x_1 = p_1 = \frac{1}{2}v(b_1, s_1)$. We then proceed recursively, dividing the surplus net of outside options equally. Given a sharing of the surplus for pair $i-1$, $v(b_{i-1}, s_{i-1}) = x_{i-1} + p_{i-1}$, the surplus net of the outside options for pair i is $\Delta_i \equiv v(b_i, s_i) - (v(b_i, s_{i-1}) - p_{i-1}) - (v(b_{i-1}, s_i) - x_{i-1}) = v(b_i, s_i) + v(b_{i-1}, s_{i-1}) - v(b_i, s_{i-1}) - v(b_{i-1}, s_i)$, which is nonnegative, by supermodularity. Set $x_i = v(b_i, s_{i-1}) - p_{i-1} + \frac{1}{2}\Delta_i$, and $p_i = v(b_{i-1}, s_i) - x_{i-1} + \frac{1}{2}\Delta_i$. The resulting bargaining function, which yields stable payoffs for all attribute vectors, is the result of applying the symmetric Nash bargaining solution to each matched pair in order, with the disagreement point given by the outside option of matching with the preceding pair. Accordingly, we refer to this bargaining function as the Nash bargaining function.

Under the Nash bargaining function, an agent typically shares the social value of any change in attribute with his or her partner, and so in general we should not expect such an function to yield incentives for efficient investment. Trivially, if there is only one buyer and one seller, there is underinvestment (this is the standard hold-up problem). Moreover, there is a sense in which the Nash bargaining function typically will not lead to efficiency. If efficiency requires each buyer to choose a distinct buyer attribute and each seller to choose a distinct seller attribute (as would be the case if each agent had a different cost of acquiring attributes and attributes were continuous), then outside options are not binding, and agents do not receive the full social return on their attribute choices under the Nash bargaining function.

The situation is very different when outside options bind. Suppose, for example, that in the example in Figure 3.1, a buyer with attribute $b = 2$ changed his attribute to $b = 5$. If we leave unchanged the bottom pair's division, the unique payoffs consistent with equal treatment are as in Figure 4.1 (an asterisk indicates a seller for whom the matched buyer has a different attribute level as a result of the change).

The share to the buyer whose attribute changed increased by 9. In principle, this need not be the change in the social value. The change in the buyer's attribute from 2 to 3 alters the matching of buyers and sellers. A buyer who increases his attribute will "leapfrog" other buyers and match with a higher attribute seller. This will result in some of the other buyers being matched with lower attribute sellers than they had originally been matched with and some of the sellers being

x	2	4	4	4	7	7	7	11	15
b	2	3	3	3	4	4	4	5	6
s	2	2*	2	3	3*	3	4	4*	4
p	2	2	2	5	5	5	9	9	9

Figure 4.1: The result of a buyer's change in attribute in Figure 3.1. An asterisk indicates a seller for whom the matched buyer has a different attribute level as a result of the change.

matched with higher attribute buyers than before. In other words, when this buyer (or other buyers or sellers) chooses an attribute, he imposes an externality on other players simply because the matching is changed. While an increase in a buyer's attribute causes some of the other players to be in matches with higher total surplus and others to be in matches with lower total surpluses, it is unambiguous that the aggregate surplus is increased. When a buyer increases his or her attribute, a number of the sellers are matched with higher attribute buyers following the increase, while none is matched with a lower attribute buyer. Hence, the increase in the social value is the sum of the increases in the total surplus of those pairs with sellers matched with higher attribute buyers after the increase.

These externalities may lead individuals to either overinvest or underinvest from a social perspective. While it is true that, in general, changes in attribute can result in changes to the individual's payoff that differ from the change in social value, it is not the case in this example. The particular pattern of overlapping attributes for the vectors of attributes results in each of the players whose attribute is unchanged getting the same payoff after the specified player's change as before. Since no other agent's payoff is changed by the buyer increasing his attribute, it follows trivially that this buyer captures the full social value of the attribute change. The qualitative characteristics of this example are quite general as shown by the next proposition (which is proved in the appendix).

Proposition 3. *Let* $(\mathbf{b}, \mathbf{s})$ *and* $(\mathbf{b}', \mathbf{s})$ *denote two vectors of attributes satisfying* $b_i = b'_i$, $\forall i \neq \ell$. *Let* $\{(\mathbf{x}, \mathbf{p}), m\}$ *denote a stable payoff and matching for* $(\mathbf{b}, \mathbf{s})$, *and* $\{(\mathbf{x}', \mathbf{p}'), m'\}$ *a stable payoff and matching for the attributes* $(\mathbf{b}', \mathbf{s})$. *If* $p_{m(\ell)} = p'_{m(\ell)}$ and $p_{m'(\ell)} = p'_{m'(\ell)}$, *then*

$$x'_\ell = x_\ell + V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s}).$$

Definition 8. *The attribute vector* $(\mathbf{b}, \mathbf{s})$ *is* ***doubly overlapping*** *if* $(\mathbf{b}, \mathbf{s})$ *is overlapping and each matched pair of attributes appears at least twice.*

Corollary 2. *Let* $(\mathbf{b}, \mathbf{s})$ *and* $(\mathbf{b}', \mathbf{s})$ *denote two vectors of attributes satisfying* $b_i = b_i'$, $\forall i \neq \ell$. *Let* $\{(\mathbf{x}, \mathbf{p}), m\}$ *denote a stable payoff and matching for* $(\mathbf{b}, \mathbf{s})$, *and* $\{(\mathbf{x}', \mathbf{p}'), m'\}$ *a stable payoff and matching for the attributes* $(\mathbf{b}', \mathbf{s})$ *satisfying* $x_{\underline{b}}^{'} = x_{\underline{b}}$. *If* $(\mathbf{b}, \mathbf{s})$ *is doubly overlapping, then*

$$x_\ell' = x_\ell + V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s}).$$

Proof. If $(\mathbf{b}, \mathbf{s})$ is doubly overlapping, then the vector of attributes following any single agent's change of attribute is overlapping. It is straightforward to see that if $x_{\underline{b}}^{'} = x_{\underline{b}}$, the construction in Corollary 1 results in $p_{m(\ell)} = p'_{m(\ell)}$ and $p_{m'(\ell)} = p'_{m'(\ell)}$. Hence, the proposition applies. ■

The proposition and corollary provide sufficient conditions that rule out one source of inefficiency in investments. If the attribute choice vector is doubly overlapping, each agent captures exactly the incremental aggregate surplus that results from his attribute choice. Competition among future potential partners eliminates any "holdup problem" that might arise due to the investment choice being made prior to matching and bargaining over the surplus.

Double overlap is not necessary for agents to receive the correct incentives for efficient attribute choice. There are trivial examples for which double overlap may fail, yet Proposition 3 still holds. There are, however, trivial examples for which there are equilibria for which an agent will not capture the change in surplus that results from a change in his attribute when the conditions for Proposition 3 fail.

It is important to note that Proposition 3 does *not* say that when the hypotheses of the proposition hold, the outcome is efficient. The proposition only guarantees that any inefficiency in the investment choices does not stem from a single person's decision, i.e., the proposition provides sufficient conditions for *constrained efficient* investment choices.[10] There remains the possibility of inefficiencies due to coordination failures resulting from the choices of multiple agents. For example, if we consider the surplus function that we have used in the examples above, $v(b, s) = b \cdot s$, it is clearly an equilibrium for all buyers and sellers to choose attribute 0 if the cost of choosing this attribute is 0, regardless of the cost of higher investment levels. The problem, of course, is that unilateral deviations from no investment have no value. We will show in the next section, however, that for any investment problem, there is always one equilibrium for which each agent will capture precisely the change in surplus that results from a change in

[10] Cole, Mailath, and Postlewaite [5] show that in a continuum version of this model all equilibria are constrained efficient. Intuitively, this is a consequence of the inability of a single agent to affect other agents' payoffs through his investment. Note that this property is stronger than the condition in Proposition 3.

attribute and, further, that no pair of agents can *jointly* change their attributes in a way that increases their surplus, net of investment cost (or other set of agents for that matter).

All stable bargaining functions are essentially equivalent when the attribute vector is double overlapping, since equal treatment (together with the bottom pair division) completely determines the returns to an attribute. On the other hand, when the attribute vector is not overlapping, different bargaining functions have different efficiency properties. We suggested at the beginning of this section that the Nash bargaining function can be expected to have poor efficiency properties (in particular, constrained inefficient equilibrium attribute choices are consistent with the Nash bargaining function). We illustrate this idea in the context of a symmetric investment problem, so that $I = \{1, \ldots, n\}$, $v(b, s) = v(s, b)$, and $\psi(b, i) = c(b, i)$. Suppose also that $B = S$ is a compact interval, and that ψ is $\mathcal{C}^2$, with $\partial \psi(b, i) / \partial b$ strictly increasing in i. In any symmetric ex post contracting equilibrium with the Nash bargaining function, each buyer chooses a distinct buyer attribute, and each seller chooses a distinct seller attribute. Moreover, every such equilibrium is inefficient. Since the Nash bargaining function is symmetric, in any symmetric equilibrium outcome, each matched pair is dividing the surplus equally.

Consider now the following bargaining function g'. For any attribute vector, give the agents a positively assortative labeling so that buyer i matches with seller i. If the attribute vector $(\mathbf{b}, \mathbf{s})$ is symmetric, i.e., $\mathbf{b} = \mathbf{s}$, g' divides the surplus equally for each pair. If the attribute vector $(\mathbf{b}, \mathbf{s})$ is not symmetric, let i' be the first pair for which $b_{i'} \neq s_{i'}$. Pairs with smaller attributes equally divide the surplus. For the pair i', the agent with the smaller attribute receives under g' half the surplus that would have resulted had he been matched with an agent with the same attribute as his own, and his partner receives the residual. For buyers and sellers with higher index, g' uses the buyer-favored extension if $b_{i'} > s_{i'}$, and uses the seller-favored extension otherwise. This bargaining function has the property that given a symmetric attribute vector, any agent who increases his attribute will receive the full social value of that increase.[11] As a consequence, for this

[11] Fix a symmetric attribute vector, and suppose each buyer chooses a distinct buyer attribute. Consider a replica economy in which each attribute is chosen by two agents, with a positively assortative labeling, so that $b_{(\kappa)} = b_{2\kappa}$, for $\kappa = 1, \ldots, n$, in the replica economy. Now consider the result of buyer i' changing attribute from $b_{i'}$ to $b'_{i'} > b_{i'}$. Let κ be the rank order of $b_{i'}$, i.e., $b_{(\kappa)} = b_{i'}$. If $b'_{i'} < b_{(\kappa+1)}$, then buyer i' is still matched with seller i', seller i''s share is unchanged, and so buyer i' captures the full social value of the change. If $b'_{i'} > b_{(\kappa+1)}$, then buyers and sellers are rematched, and because each other attribute is chosen by two agents, there is overlap between seller i' and the seller who is now matched with buyer i'. This pattern of overlap corresponds to the buyer-first extension for sellers $j > i'$. Since the payoff to seller i'

bargaining function, there are no symmetric equilibria with underinvestment that is constrained inefficient.

5. Comparison to ex ante contracting equilibrium

We now compare the investments taken in an ex post contracting equilibrium with the investments agents would make if buyers and sellers could contract with each other over matches, the investments to be undertaken, and the sharing of the resulting surplus. If a buyer i and seller j agree to match and make investments b and s respectively, then the total surplus so generated is $v(b,s)-\psi(b,i)-c(s,j)$. In a world of ex ante contracting, investments maximize this total surplus. Thus, if buyer i and seller j are considering matching, they are bargaining over the surplus $\varphi(i,j) = \max_{b,s} v(b,s) - \psi(b,i) - c(s,j)$. The *ex ante assignment game* is the assignment game with the population I of buyers, J of sellers, and value function φ. Just as we considered stable outcomes for the ex post assignment, we impose stability on outcomes of the ex ante assignment game. A stable outcome, together with the implied attribute investments, is an *ex ante contracting equilibrium*:

Definition 9. *The outcome of the ex ante assignment game* $\{m^*, (\mathbf{b}^*, \mathbf{s}^*), (\mathbf{x}^*, \mathbf{p}^*)\}$ *is an* ***ex ante contracting equilibrium (EACE)*** *if*

1. $(b_i^*, s_{m^*(i)}^*)$ *maximizes* $v(b,s) - \psi(b,i) - c(s, m^*(i))$ *if* $m^*(i) \in J$;
2. $(\mathbf{x}^*, \mathbf{p}^*)$ *is feasible for* m^*; *and*
3. *for all* $i \in I$ *and* $j \in J$,

$$x_i^* - \psi(b_i^*, i) + p_j^* - c(s_j^*, j) \geq \varphi(i,j).$$

Since the ex ante assignment game is a finite assignment game, ex ante contracting equilibria exist (see footnote 5). It is immediate that $(\mathbf{x}^*, \mathbf{p}^*)$ is a stable payoff of the ex post assignment game associated with the attribute vector $(\mathbf{b}^*, \mathbf{s}^*)$.

We pointed out in the previous section that investments could be inefficient. Given the bargaining function in the equilibrium, some agents might not be able to capture the incremental surpluses that would result from altering their investments in attributes. Further, regardless of the bargaining function, there may be

is unchanged, the proof of Lemma A applies here and so buyer i' again captures the full social value of the change in attribute.

coordination failures in which Pareto improvements are possible, but only if pairs of agents jointly change their attributes.

We should not be surprised that an inability to contract over investment choices in the presence of complementarities can lead to inefficiency. Indeed, one might expect that in such an environment inefficiency is inevitable, but this is not the case. The following proposition states that any outcome achievable under ex ante contracting is part of an ex post contracting equilibrium.[12]

Proposition 4. *Given an ex ante contracting equilibrium* $\{m^*, (\mathbf{b}^*, \mathbf{s}^*), (\mathbf{x}^*, \mathbf{p}^*)\}$, *there exists* g^* *such that* $(g^*, (\mathbf{b}^*, \mathbf{s}^*))$ *is an ex post contracting equilibrium.*

The idea behind the construction of g^* is to begin with the efficient attribute choices $(\mathbf{b}^*, \mathbf{s}^*)$ and surplus divisions $(\mathbf{x}^*, \mathbf{p}^*)$ in the ex ante contracting equilibrium. Define g^* for the efficiently matched pairs in the EACE to be the surplus division in the equilibrium. To extend the bargaining function to the case of a unilateral deviation by a buyer, we maintain the efficient surplus division for low attribute pairs whose matching is unaffected by the deviation. For those buyers and sellers whose matching is affected by the buyer's deviation, we essentially extend g^* as if it is the bargaining function for the seller-favored extension of the attribute vector. This minimizes the deviating buyer's payoff consistent with stability. We show that this is enough to deter deviations by any buyer. A similar idea applies to deviations by a seller.

Proof. If necessary, relabel buyers and sellers so that $I = J = \{1, \ldots, n\}$ and $m^*(i) = i$. Define $g^*(\mathbf{b}^*, \mathbf{s}^*) = (\mathbf{x}^*, \mathbf{p}^*)$. Since ex post contracting equilibria are Nash equilibria, we need only be concerned with unilateral deviations (any specification of g^* for multilateral deviations consistent with the definition of an ex post contracting equilibrium will work). Consider then an attribute vector $(\mathbf{b}^*_{-\ell}, b_\ell, \mathbf{s}^*)$ for some $b_\ell \in B$ and $\ell \in I$ (the extension of g^* to a deviation by a seller is identical). Denote the stable payoff we are defining by $(\mathbf{x}, \mathbf{p})$.

Suppose $b_\ell < b^*_\ell$, and let i' satisfy $b^*_{i'-1} < b_\ell \leq b^*_{i'}$ (where $b^*_0 \equiv -1$); clearly, $i' \leq \ell$. Since stable matchings are positively assortative in attributes, $m(i) = i$ for $i < i'$, $m(i) = i + 1$ for $i' \leq i < \ell$, $m(\ell) = i'$, and $m(i) = i$ for $i > \ell$ is a stable matching for $(\mathbf{b}^*_{-\ell}, b_\ell, \mathbf{s}^*)$. Since $m(i) = m^*(i)$ for $i < i'$, we can set $(x_i, p_i) = (x^*_i, p^*_i)$ for $i < i'$. Set

$$p_{i'} = v(b_\ell, s^*_{i'}) - v(b_\ell, s^*_{i'-1}) + p^*_{i'-1}$$

[12] Note that the task of inducing efficient investments here is simpler than that in typical team problems (see, e.g., Holmstrom [10]) since the surplus division can depend on the level of investments chosen.

(this is the most that seller i' can receive consistent with stability and $p_{i'-1}$—Lemma 1),[13] and then complete g^* consistent with Lemma 1. Before considering $b_\ell > b_\ell^*$, we show that $b_\ell < b_\ell^*$ is not a profitable choice with this specification. The difference in payoffs is

$$\begin{aligned} & x_\ell^* - \psi(b_\ell^*, \ell) - \{v(b_\ell, s_{i'}^*) - p_{i'} - \psi(b_\ell, \ell)\} \\ = \; & x_\ell^* - \psi(b_\ell^*, \ell) - \{v(b_\ell, s_{i'-1}^*) - p_{i'-1}^* - \psi(b_\ell, \ell)\} \\ = \; & x_\ell^* - \psi(b_\ell^*, \ell) + p_{i'-1}^* - \{v(b_\ell, s_{i'-1}^*) - \psi(b_\ell, \ell)\} \\ \geq \; & \varphi(\ell, i'-1) - \{v(b_\ell, s_{i'-1}^*) - \psi(b_\ell, \ell) - c(s_{i'-1}^*, i'-1)\} \geq 0, \end{aligned}$$

where the first inequality comes from the stability of ex ante contracting outcomes in the ex ante assignment game, and the second comes from the definition of φ.

Now, suppose $b_\ell > b_\ell^*$, and now let i' satisfy $b_{i'}^* < b_\ell \leq b_{i'+1}^*$ (where $b_{n+1}^* \equiv \infty$); clearly, $i' \geq \ell$. Set $m(i) = i$ for $i < \ell$, $m(i) = i - 1$ for $\ell < i \leq i'$, $m(\ell) = i'$, and $m(i) = i$ for $i \geq i' + 1$. As before, for $i < \ell$, we set $(x_i, p_i) = (x_i^*, p_i^*)$. Potentially all the matches between seller ℓ and seller i' (inclusive) involve the seller being matched with a different buyer attribute than under m^*. Moreover, all these sellers are matching, under m, with buyers whose attributes are at least as large as those under m^*. Then it is still feasible (and stable) to set $p_\ell = p_\ell^*$ and $x_{\ell+1} = v(b_{\ell+1}^*, s_\ell^*) - p_\ell$ (note that $x_{\ell+1} \leq x_{\ell+1}^*$). We now proceed inductively, setting $p_{i+1} = v(b_{i+2}^*, s_{i+1}^*) - (v(b_{i+2}^*, s_i^*) - p_i)$ and $x_{i+2} = v(b_{i+2}^*, s_i^*) - p_i$ for $\ell \leq i < i' - 2$, $p_{i'} = v(b_\ell, s_{i'}^*) - \Big(v(b_\ell, s_{i'-1}^*) - p_{i'-1}\Big)$ and $x_\ell = v(b_\ell, s_{i'-1}^*) - p_{i'-1}$, and then complete g^* as described above. By Lemma 1, we have described a stable outcome of the ex post assignment game associated with $(\mathbf{b}_{-\ell}^*, b_\ell, \mathbf{s}^*)$. Moreover, $p_{i'} \geq p_{i'}^*$. [The proof is by induction. Note that $p_\ell \geq p_\ell^*$, and suppose that $p_i \geq p_i^*$ for $\ell \leq i < i' - 1$. Then, $p_{i+1} = v(b_{i+2}^*, s_{i+1}^*) - v(b_{i+2}^*, s_i^*) + p_i \geq v(b_{i+2}^*, s_{i+1}^*) - v(b_{i+2}^*, s_i^*) + p_i^* \geq v(b_{i+1}^*, s_{i+1}^*) - v(b_{i+1}^*, s_i^*) + p_i^* \geq v(b_{i+1}^*, s_{i+1}^*) - x_{i+1}^* = p_{i+1}^*$ (where the first inequality follows from $p_i \geq p_i^*$, the second from the supermodularity of v, and the third from stability). Thus, $p_{i'-1} \geq p_{i'-1}^*$, and the same logic then implies $p_{i'} \geq p_{i'}^*$.] The difference in payoffs for buyer ℓ for the deviation to b_ℓ is then

$$\begin{aligned} & x_\ell^* - \psi(b_\ell^*, \ell) - \{v(b_\ell, s_{i'}^*) - p_{i'} - \psi(b_\ell, \ell)\} \\ \geq \; & x_\ell^* - \psi(b_\ell^*, \ell) - \{v(b_\ell, s_{i'}^*) - p_{i'}^* - \psi(b_\ell, \ell)\} \\ = \; & x_\ell^* - \psi(b_\ell^*, \ell) + p_{i'}^* - \{v(b_\ell, s_{i'}^*) - \psi(b_\ell, \ell)\} \\ \geq \; & \varphi(\ell, i') - \{v(b_\ell, s_{i'}^*) - \psi(b_\ell, \ell) - c(s_{i'}^*, i')\} \geq 0, \end{aligned}$$

[13] If $i' = \ell$, then there is no rematching as a result of the lower attribute choice, and p_i^* may be feasible in a stable outcome. If it is, then setting $p_i = p_i^*$ also works.

where the second inequality comes from the stability of ex ante contracting outcomes in the ex ante assignment game and the third from the definition of φ. ■

The result is trivial when the ex ante contracting attribute choices are doubly overlapping. The nontriviality comes from the possibility that there may be gaps in the attribute matchings (after a deviation), so that stability and the bottom pair do not uniquely determine attribute payoffs. This indeterminacy is important. It is because of this indeterminacy that we do not interpret Proposition 4 as a strong positive result. It is true that for any outcome that is supportable as part of an EACE, there is an EPCE yielding the same investments and payoffs. But the EPCE that does this depends crucially on the bargaining function. The issue is the interpretation of the bargaining function. We suggested above that we could think of it as generally determining how surplus is shared subject to the constraints of competition implicit in stability. For some problems sellers might capture most of this, and in others, it may be the buyers. But in Proposition 4, the bargaining function responds to changes in the underlying investment problem (e.g., changes in the costs of investment ψ or c), since it depends upon $(\mathbf{b}^*, \mathbf{s}^*)$.

While the indeterminacy is eliminated if the ex ante contracting attribute choices are doubly overlapping, there is good reason not to expect double overlap. Typically, if each agent has different costs of acquiring attributes and attributes are continuous variables, then the efficient attribute choices $(\mathbf{b}^*, \mathbf{s}^*)$ will not be doubly overlapping. Proposition 2, on the other hand, provides bounds that suggest that as the set of chosen attributes become sufficiently rich (in the sense that the set of attributes looks like an interval), the indeterminacy in stable payoffs disappears. However, attributes are endogenous, and even if there are many agents, the set of chosen attributes may *not* be rich. The complementarity of attributes means that, in general (in particular, when the complementarity is strong), in the limit the set of efficient attributes may be a disconnected set. The case of a continuum of agents is analyzed in Cole, Mailath, and Postlewaite [5]; an example in which the set of efficient attributes is a disconnected set is described there.

5.1. Inefficient investment: underinvestment

We mentioned at the end of Section 4 that ex post contracting equilibrium outcomes might easily be inefficient (the example of $v(b,s) = b \cdot s$ and all buyers and sellers choosing attribute 0). While having all agents choose attribute 0 is a particularly simple way to illustrate the possibility of inefficiency, it isn't difficult to construct examples in which all agents are choosing positive attributes. In fact, we can modify any investment problem to generate inefficiency; moreover, this inefficiency cannot be eliminated by *any* restrictions on the bargaining function.

Fix an investment problem $\Gamma = \{I, J, B, S, \psi, c, v\}$, and define $B' \equiv B \cup \{b'\}$ and $S' \equiv S \cup \{s'\}$, where $b' > \overline{b} \equiv \max B$ and $s' > \overline{s} \equiv \max S$. Extend the definition of v to $B' \times S'$ by setting $v(b, s') = v(b, \bar{s})$ for all $b \in B$ and $v(b', s) = v(\overline{b}, s)$ for all $s \in S$ and setting $v(b', s') = v(\overline{b}, \overline{s}) + \max_i \psi(\overline{b}, i) + \max_j c(\overline{s}, j) + 2a + 1$, where $a > v(\overline{b}, \overline{s})$. Extend the cost functions by setting $\psi(b', i) = \psi(\overline{b}, i) + a$ for all $i \in I$ and $c(s', j) = c(\overline{s}, j) + a$ for all $j \in J$. Note that, unless both the buyer and the seller in a pair choose the added elements b' and s', the new attributes are simply expensive substitutes for $\overline{b}$ and $\overline{s}$.

The only efficient outcome in the investment problem $\Gamma' = \{I, J, B', S', \psi, c, v\}$ is for every buyer to choose b' and every seller s' (since $v(b', s') - \psi(b', i) - c(s', j) = v(\overline{b}, \overline{s}) + \max_i \psi(\overline{b}, i) + \max_j c(\overline{s}, j) + 2a + 1 - \psi(\overline{b}, i) - a - c(s', j) = c(\overline{s}, j) - a \geq v(\overline{b}, \overline{s}) + 1)$.

Fix an ex post contracting equilibrium of Γ', and denote its bargaining function by g. We claim that there is another ex post contracting equilibrium of Γ' with the *same* bargaining function g that involves inefficient attribute choices. Consider the strategic form game implied by g on the attribute sets B and S. This has an equilibrium (perhaps in mixed strategies). Moreover, this will be an ex post contracting equilibrium of Γ': If all other agents are choosing attributes in B and S, then no matter how the bargaining function divides the surplus, since $a > v(\overline{b}, \overline{s})$, there is insufficient total surplus to justify choosing the added attribute.

5.2. Inefficient investment: overinvestment

The previous subsection illustrated an ex post contracting equilibrium outcome with agents making inefficiently low investment in attributes. There is a similar possibility of overinvestment, but with an important difference. We first give a simple example with overinvestment.

There are two buyers, $\{1, 2\}$, and two sellers, $\{1, 2\}$. The possible characteristics for buyers and sellers are $B = S = \{4, 6\}$. The surplus function is $v(b, s) = b \cdot s$. The cost functions are $\psi(4, i) = c(4, j) = 5$, $i, j = 1, 2$; $\psi(6, i) = c(6, j) = 16$, $i, j = 1, 2$. The efficient attribute choices are for all buyers and sellers to choose attribute level 4. These efficient choices can, of course, be part of an EPCE. Suppose that when all agents choose attribute 4, the surpluses are shared as in the left side of Figure 5.1 and as in the right side of Figure 5.1 if a single agent (here, a buyer) deviates and chooses attribute 6.

Since a single agent switching from attribute 4 to attribute 6 decreases his

$x_i - \psi$	3	3
x_i	8	8
b_i	4	4
i	1	2
j	1	2
s_j	4	4
p_j	8	8
$p_j - c$	3	3

$x_i - \psi$	3	0
x_i	8	16
b_i	4	6
i	1	2
j	1	2
s_j	4	4
p_j	8	8
$p_j - c$	3	3

Figure 5.1: The efficient equilibrium.

$x_i - \psi$	2	2
x_i	18	18
b_i	6	6
i	1	2
j	1	2
s_j	6	6
p_j	18	18
$p_j - c$	2	2

$x_i - \psi$	1	2
x_i	6	18
b_i	4	6
i	1	2
j	1	2
s_j	6	6
p_j	18	18
$p_j - c$	2	2

Figure 5.2: The overinvestment equilibrium.

net payoff from 3 to 0, the efficient choice of attribute level 4 for all agents is an EPCE. However, there may be another EPCE in which all agents overinvest; that is, all agents choose the high attribute level 6. Suppose that the payoffs resulting from all agents choosing attribute level 6 and those following a single agent deviating and choosing level 4 are as given in Figure 5.2.

These figures make clear that it is an EPCE for all agents to choose the inefficient attribute level 6. Note that there is a common bargaining function g that supports (that is, is part of) both equilibria.

This illustrates that we can get inefficient overinvestment as well as inefficient underinvestment, but as we stated above, there is a difference between the two cases. For the example in the previous section illustrating an equilibrium with underinvestment, we pointed out that the inefficiency could arise regardless of the bargaining rule g (that is, there was no g for which the underinvestment outcomes would not be an equilibrium).

We conjecture that there are bargaining functions g that might preclude overinvestment for many investment problems. For example, for finite symmetric

x_i	2	2	6
b_i	2	2	4
i	1	2	3
j	1	2	3
s_j	2	2	4
p_j	2	2	10
$p_j - c$	2	2	3

Figure 5.3: The efficient equilibrium.

investment problems with the net surplus function $v(b,s) - \psi(b,i) - c(s,j)$ concave in attributes, it can be shown that overinvestment cannot occur in any EPCE with the bargaining investment function g' defined at the end of Section 4. This illustrates an important distinction between the finite and large (continuum) economies, where a similar result is false. It is not possible to rule out overinvestment in ex post contracting equilibria through an appropriate restriction on the bargaining function in a continuum economy. In a continuum economy, every reasonable bargaining function has the property that a single agent changing his or her attribute does not affect the payoff of any other agent.[14] This rules out bargaining functions like g'. We believe a similar bargaining function will also work for nonsymmetric (finite) problems even without concavity of the net surplus function; an investigation of this is beyond the scope of the present paper however.

5.3. Inefficient investment: coordination failure

There is a final source of inefficiency due to the finite number of agents, identified by Felli and Roberts [7]. We illustrate it in a simple example with three buyers and three sellers. The set of possible attributes is $B = S = \{2,3,4\}$. We treat the buyers' attributes as fixed, with $b_1 = b_2 = 2$ and $b_3 = 4$. The sellers' cost function is given by: $c(2,j) = 0$ for all j; $c(3,1) = c(4,1) = c(4,2) = 8$, $c(3,2) = 3\frac{3}{4}$, $c(3,3) = 3\frac{1}{2}$, and $c(4,3) = 7$. The cost function was chosen so that seller 1 will always choose $s_1 = 2$, and the cost to seller 3 of attribute level 3 or 4 is less than to seller 2. We suppose that the bargaining function is that implied by the seller-favored extension with equal division at the bottom. The efficient attribute choices are illustrated in the matrix of Figure 5.3. It is straightforward to verify that this is an equilibrium.

[14] We discuss this issue in Cole, Mailath, and Postlewaite [5].

x_i	2	2	6
b_i	2	2	4
i	1	2	3
j	1	3	2
s_j	2	2	3
p_j	2	2	6
$p_j - c$	2	2	$2\frac{1}{4}$

x_i	2	2	8
b_i	2	2	4
i	1	2	3
j	1	2	3
s_j	2	3	4
p_j	2	4	8
$p_j - c$	2	$\frac{1}{4}$	1

Figure 5.4: An equilibrium illustrating coordination failure.

There is however the possibility of a coordination failure between sellers. In particular, the configuration on the left in Figure 5.4 is also an equilibrium. In this equilibrium, seller 2 (who has higher costs than seller 3) chooses a larger attribute and so matches with buyer 3—a clearly inefficient outcome. Under the seller-favored bargaining function, seller 3 cannot profitably deviate to the attribute 4 (his choice in the efficient attribute vector) because buyer 3 can demand a payoff of 8, reducing the return to seller 3 (see the configuration on the right).

Note that this example illustrates a qualitatively different type of inefficiency than that due to the coordination failures between buyers and sellers illustrated in the earlier subsections. In particular, this inefficiency is mitigated as the numbers of agents increase, and cannot arise in a model with a continuum of agents.

6. Discussion

Remark 1. The first welfare theorem tells us that under quite general circumstances, a competitive outcome is Pareto efficient. If we consider a market in which prices are quoted ex ante for different attributes, we would expect agents to make efficient attribute choices. One might expect, then, that we could simply consider a bargaining function that split the surplus according to the equilibrium prices for attributes. Suppose, for example, that one considers the commodities to be the attributes that buyers and sellers choose. One could then think of a market in which all buyers and sellers produce an attribute, and one side—say the buyer side—purchases a seller attribute and generates surplus according to the surplus function v.[15] A market equilibrium would then be a pair of price vectors x and p that determine the prices of attributes for the buyers and sellers respectively, and a pair of attribute vectors β and σ that are respectively the buyers' and sellers' attribute choices. Market clearing would entail that buyers

[15] See Acemoglu [1] for a discussion of a model along these lines.

purchase seller attributes from distinct sellers.[16]

There will be a market equilibrium of this type when prices are quoted ex ante for different attributes, and that equilibrium will entail efficient attribute choices. However, the prices in such an equilibrium cannot, in general, be used to construct a bargaining function as in Proposition 4. This is easy to see by considering a simple example with one buyer and one seller, with the surplus function $v(b, s) = b \cdot s$. Suppose that both agents choose attribute level 4, and that this is the competitive level. The competitive prices (x, p) for these the attributes $(b, s) = (4, 4)$ must sum to 16, the surplus generated by the attributes, and so either x or p (or both) is at least 8. But now suppose that the buyer chooses attribute 1 instead of 4 (the seller still choosing 4); the maximal feasible payoff to the seller is the entire surplus, 4. Analogously, if the buyer chooses attribute 4 and the seller chooses attribute 1, the maximal feasible payoff to the buyer is 4. Trivially then, one cannot use only the competitive prices x and p in specifying how the two will share the surplus for each possible set of attribute choices they could make. In other words, one cannot construct a bargaining function for this problem by naively using the market equilibrium prices to determine how the surplus will shared in all circumstances.[17] We should note that there is

[16] More specifically, we can think of an equilibrium as being (β, σ, x, p, a) where $\beta : I \to B$ and $\sigma : J \to S$ specify the attribute choices for the buyers and sellers respectively, $x : B \to \Re_+$ and $p : S \to \Re_+$ are the prices of buyers' and sellers' attributes, and $a : I \to B \times S$ indicates the choice of attributes purchased by each buyer. The equilibrium conditions are:

1. for each buyer i, $\beta(i)$ is a solution to $\arg\max_{b \in B} x(b) - \psi(b, i)$ (buyer optimization in choice of own attribute),
2. for each seller j, $\sigma(j)$ is a solution to $\arg\max_{s \in S} p(s) - c(s, j)$ (seller optimization in choice of attribute),
3. for each buyer i, $a(i)$ is a solution to $\arg\max_{(b,s) \in B \times S} v(b, s) - x(b) - p(s)$ (buyer optimization in choice of purchased attributes), and
4. $a_1(i) = \beta(i)$ and there is a matching function $m : I \to J$ such that $\sigma(m(i)) = a_2(i)$ (market clearing).

[17] The problem is not that not all attributes are chosen in equilibrium (or, in other words, that not all attribute markets are open in equilibrium). Suppose there were three buyers and three sellers, and the surplus function is again $v(b, s) = b \cdot s$. Suppose further that initially there is one buyer and one seller each choosing attribute 1, and two buyers and two sellers each choosing attribute 4. As above, either $x(4)$ or $p(4)$ is at least 8. If from this initial configuration, a single buyer with attribute 4 changes to attribute 1, one of the matched pairs will be a buyer with attribute 1 and seller with attribute 4. In this pair, the maximum feasible payoff to the seller is 4, and by equal treatment, both sellers with attribute 4 must receive this payoff. A similar argument can be made if a single seller with attribute 4 changes to attribute 1, and thus, we are lead to the same conclusion as in the two person example above: one cannot construct a bargaining outcome function using only equilibrium market prices.

one special case in which market equilibrium prices *can* be used to construct a bargaining function, and that is when the efficient attribute vector satisfies double overlap. In this case, efficient market equilibrium prices are market clearing even if a single agent deviates to an attribute that is not less than the smallest efficient attribute. However, as we discuss just after Proposition 4, double overlap is a quite restrictive condition.

The problem, of course, is that one individual changing his attribute choice affects the feasible returns to other agents. This is analogous to the standard problem in general equilibrium that while market prices determine returns to agents in equilibrium, those returns typically can be infeasible if an agent makes a disequilibrium choice. As we emphasized when we formulated our model, we introduced bargaining functions so that we could treat the attribute choice problem as a well-defined noncooperative game. For agents' problems to be well-defined, changes in an agent's attribute choice *necessarily* affect returns to other attributes in general.[18]

One alternative would be to consider a more sophisticated notion of market equilibrium along the lines of state contingent prices for attributes, where the state depends on agents' attribute choices. The notion of equilibrium would have agents choose attributes, with those attributes having market prices that depend on the vectors of chosen attributes. The problem with this approach is that an agent's choice of attribute affects the state, and hence, the price of attributes (so that there is a tension with price taking). While such a framework permits the responses in sharing to changes of attribute that feasibility demands, the analogue of the first welfare theorem doesn't hold there.[19]

Remark 2. We treat in this paper the case in which the relevant groups for production are pairs. We could easily have extended the analysis to cover the case in which production necessitated a group of people, one of each of a number of different types. With analogous assumptions on the surplus and cost functions, we would have had similar results regarding positive assortative matching, equal treatment, etc. An interesting extension that is not so direct is to treat the case

[18] This is not true in large economies. In Cole, Mailath, and Postlewaite [5], we require that the bargaining function have the property that a single agent cannot affect the payoff of any other agent. In principle, in large economies, competitive prices can be used to support efficient attribute choices. However, in that context, we still cannot simply appeal to the standard welfare theorems: the large economy with attribute choice is a nonconvex economy with indivisibilities. We provide there a direct proof that there are bargaining functions consistent with efficiency.

[19] Makowski and Ostroy [14] consider such a model and show that, with assumptions that essentially guaranteeing a sufficient amount of competition and rule out coordination failures, outcomes will be efficient.

in which groups need not have every type of agent with the surplus a group generates depending on the composition of that group.

Remark 3. In our model matching is frictionless, that is, there is no cost in agents' searching out appropriate partners. It is clear that frictionless matching drives some of the qualitative results; for example, we would not expect to see perfectly assortative matching if matching is accomplished through costly search.[20]

Remark 4. For many of the problems the model is meant to address—such as matching workers to firms—the process of matching and production is ongoing. That is, there is a sequence of periods in which matching may take place, and once matched, the pair may stay matched for several periods. A natural way to model such a problem would be with a new cohort of individuals on each side of the market entering each period, making investments in the first period of their lives and entering the matching market the next period. If the cost functions vary stochastically across cohorts, individuals who are looking for partners might find it profitable to defer matching until later periods in the hope of finding a better match. The static nature of our model clearly precludes an analysis of such behavior. Extending it to such an environment would be difficult, but potentially quite interesting.

7. Related literature

Our focus is on whether agents have the right incentives in terms of their investment decisions, given that a core allocation of the induced assignment game will result. Since the core in this case coincides with the set of Walrasian allocations, a question related to ours is whether in a competitive equilibrium, agents have incentives to make efficient ex ante investments. This question has been addressed by Hart [8, 9], Makowski [13], and Makowski and Ostroy [14]. See Cole, Mailath and Postlewaite [5] for a discussion of how these papers relate to our approach.

MacLeod and Malcomson [12] study the hold-up problem associated with investment decisions taken prior to contracting and provide, in a specific model, the idea that ex ante investments will be efficient, as long as the investments are general and there are outside options. That investments in their model are general leads to competition for the individual making the investment, assuring him of the incremental surplus that results from the investment. This is similar to

[20] See Burdett and Coles [3] for an analysis of such a model, although one in which attributes are given exogenously.

the effect of "local competition" in our overlap case above. Their model, however, doesn't give rise to the coordination inefficiencies in our model.

Subsequent to our work, there have been several other papers that study the case in which contracting at the time investments are made is ruled out. Felli and Roberts [7] analyze a finite agent, two-sided matching model in which one or the other side, but not both, can make investments, followed by Bertrand competition. Their focus on one-sided investment and a particular selection from the set of stable payoffs (which is either the buyer-favored or seller-favored payoff) allows a more specific analysis of inefficiencies. DeMeza and Lockwood [6] and Chatterjee and Chiu [4] analyze models in which both sides of a market can undertake investments prior to matching. Both, however, analyze models that are constructed to generate inefficient investment, with the aim to understanding how different ownership structures affect the inefficiency. Peters and Siow [15] analyze a model in which utility is not transferable between parties (the marriage problem) and demonstrate conditions under which investments will be efficient.

A. Proof of Proposition 3

Proposition 3 follows from the following 2 lemmas.

Lemma A. *Let* $(\mathbf{b}, \mathbf{s})$ *and* $(\mathbf{b}', \mathbf{s})$ *denote two vectors of attributes satisfying* $b_i = b'_i,\ \forall i \neq \ell$. *Let* $\{(\mathbf{x}, \mathbf{p}), \mathbf{m}\}$ *denote a stable payoff and matching for* $(\mathbf{b}, \mathbf{s})$ *and* $\{(\mathbf{x}', \mathbf{p}'), \mathbf{m}'\}$ *a stable payoff and matching for the attributes* $(\mathbf{b}', \mathbf{s})$. *If* $(\mathbf{b}', \mathbf{s})$ *are overlapping and* $p_{m(\ell)} = p'_{m(\ell)}$, *then*

$$x'_\ell = x_\ell + V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s}).$$

(A similar result holds for the sellers.)

Proof. Suppose $b'_\ell > b_\ell$ (the same argument applies, mutatis mutandis, to the case $b'_\ell < b_\ell$). Let κ' denote the rank order of b'_ℓ in $\mathbf{b}'$, i.e., $b'_\ell = b'_{(\kappa')}$, and let κ'' denote the rank order of $\min\{b_i : b_i > b_\ell\}$ in $\mathbf{b}'$. Since $(\mathbf{b}', \mathbf{s})$ has no gaps,

$$x'_\ell = x'_{(\kappa')} = x'_{(\kappa'')} + \sum_{k=\kappa''}^{\kappa'} [v(b'_{(k)}, s^k) - v(b'_{(k-1)}, s^k)], \qquad \text{(A.1)}$$

where, for each k, $s^k = s_{m'(i)} = s_{m'(i')}$ and $b'_i = b'_{(k)}$, $b'_{i'} = b'_{(\kappa-1)}$ for some positive assortative matching m' and $i, i' \in I$.

Since the only difference between the attribute vector $(\mathbf{b}, \mathbf{s})$ and $(\mathbf{b}', \mathbf{s})$ is that one worker has a higher attribute, the only *attribute* matchings that are different involve exactly one matching for each of the attributes $\{s^k : k = \kappa'', \ldots, \kappa'\}$. For each $k = \kappa'' + 1, \ldots, \kappa'$, one seller of attribute s^k matches with a worker with attribute $b'_{(k-1)}$ under $(\mathbf{b}, \mathbf{s})$ and matches with a worker with the next higher attribute $b'_{(k)}$ under $(\mathbf{b}', \mathbf{s})$. For $k = \kappa''$, one seller of attribute $s^{\kappa''}$ matches with a worker who has the same attribute (b_ℓ) as worker ℓ under $(\mathbf{b}, \mathbf{s})$ and matches with a worker with attribute $b'_{(\kappa'')}$ under $(\mathbf{b}', \mathbf{s})$. Thus,

$$V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s}) = v(b'_{(\kappa'')}, s^{\kappa''}) - v(b_\ell, s^{\kappa''}) + \sum_{k=\kappa''+1}^{\kappa'} [v(b'_{(k)}, s^k) - v(b'_{(k-1)}, s^k)].$$

Now, using $x'_{(\kappa'')} + p'_{m(\ell)} = v(b'_{(\kappa'')}, s^{\kappa''})$ and $p'_{m(\ell)} = p_{m(\ell)}$, equation (A.1) can be rewritten as

$$\begin{aligned} x'_\ell &= v(b'_{(\kappa'')}, s^{\kappa''}) - p_{m(\ell)} + \sum_{k=\kappa''}^{\kappa'} \left[v(b'_{(k)}, s^k) - v(b'_{(k-1)}, s^k) \right] \\ &= v(b'_{(\kappa'')}, s^{\kappa''}) - \left[v(b_\ell, s^{\kappa''}) - x_\ell \right] + \sum_{k=\kappa''}^{\kappa'} \left[v(b'_{(k)}, s^k) - v(b'_{(k-1)}, s^k) \right] \\ &= x_\ell + V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s}). \end{aligned}$$

■

Lemma B. *Let $(\mathbf{b}, \mathbf{s})$ and $(\mathbf{b}', \mathbf{s})$ denote two vectors of attributes satisfying $b_i = b'_i$, $\forall i \neq \ell$. Let $\{(\mathbf{x}, \mathbf{p}), \mathbf{m}\}$ denote a stable payoff and matching for $(\mathbf{b}, \mathbf{s})$, and let $\{(\mathbf{x}', \mathbf{p}'), \mathbf{m}'\}$ and $\{(\mathbf{x}'', \mathbf{p}''), \mathbf{m}'\}$ be two stable payoffs and matchings for the attributes $(\mathbf{b}', \mathbf{s})$. If $p_{m(\ell)} = p'_{m(\ell)}$ and $p_{m'(\ell)} = p''_{m'(\ell)}$, then*

$$x'_\ell - x_\ell \leq V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s}) \leq x''_\ell - x_\ell.$$

(A similar result holds for the sellers.)

Proof. The bound on x''_ℓ is immediate, given the bound on x'_ℓ (reverse $\mathbf{b}$ and $\mathbf{b}'$). If $(\mathbf{b}', \mathbf{s})$ has no gaps, the value of x'_ℓ is determined uniquely once $p'_{m(\ell)}$ is fixed, and by Lemma A, the bound holds with equality.

Suppose now that $(\mathbf{b}', \mathbf{s})$ has gaps and $b'_\ell > b_\ell$ (the same argument applies, mutatis mutandis, to the case $b'_\ell < b_\ell$).

Consider the impact of buyer ℓ's attribute change in a related collection of buyers and sellers that is a combination of the buyer and seller attributes that are rematched. Let $I' = \{i : b_\ell \leq b_i \leq b'_\ell\}$, $J' = m(I')$ and $J'' = m'(I')$.

Consider an economy $(\tilde{I}, \tilde{J}, (\tilde{\mathbf{b}}, \tilde{\mathbf{s}}))$ with $|\tilde{I}| = |\tilde{J}| = 2 \cdot |I'|$ buyers and sellers, $\tilde{\mathbf{b}} = ((b_i)_{i \in I'}, (b_i)_{i \in I'})$, and $\tilde{\mathbf{s}} = ((s_j)_{j \in J'}, (s_j)_{j \in J''})$. (Note that $\{s : s = s_j,\ j \in J'\} = \{s : s = s_j,\ j \in J''\}$.) The attribute vector of buyers after buyer ℓ changes attribute is $\tilde{\mathbf{b}}' = ((b_i)_{i \in I'}, (b'_i)_{i \in I'})$. Observe that $(\tilde{\mathbf{b}}', \tilde{\mathbf{s}})$ has no gaps and that $(\tilde{\mathbf{b}}', \tilde{\mathbf{s}})$ is the buyer-favored extension of $(\mathbf{b}', \mathbf{s})$, apart from the bottom matched pair (but the seller's attribute in that pair is the same as in $(\mathbf{b}', \mathbf{s})$) and some repeated matched pairs. By Lemma A and Proposition 2,

$$x'_\ell \leq x_\ell + V(\tilde{\mathbf{b}}', \tilde{\mathbf{s}}) - V(\tilde{\mathbf{b}}, \tilde{\mathbf{s}}),$$

which yields the desired upper bound, because $V(\tilde{\mathbf{b}}', \tilde{\mathbf{s}}) - V(\tilde{\mathbf{b}}, \tilde{\mathbf{s}}) = V(\mathbf{b}', \mathbf{s}) - V(\mathbf{b}, \mathbf{s})$. ■

References

[1] D. Acemoglu. A microfoundation for social increasing returns in human capital. *Quarterly Journal of Economics*, 111(3):779–804, August 1996.

[2] G. S. Becker. *A Treatise on the Family.* Harvard University Press, Cambridge, MA, 1981.

[3] K. Burdett and M. G. Coles. Marriage and class. *Quarterly Journal of Economics*, 112:141–168, 1997.

[4] K. Chatterjee and Y. S. Chiu. When does competition lead to efficient investments? mimeo, 1999.

[5] H. Cole, G. Mailath, and A. Postlewaite. Efficient non-contractible investments in large economies. Technical report, University of Pennsylvania, 2000.

[6] D. DeMeza and B. Lockwood. The property-rights theory of the firm. Technical report, mimeo, 1998.

[7] L. Felli and K. Roberts. Does competition solve the hold-up problem? Technical report, London School of Economics, 2000.

[8] O. D. Hart. Monopolistic competition in a large economy with differentiated commodities. *Review of Economic Studies*, 46:1–30, 1979.

[9] O. D. Hart. Perfect competition and optimal product differentiation. *Journal of Economic Theory*, 22:279–312, 1980.

[10] B. Holmstrom. Moral hazard in teams. *Bell Journal of Economics*, 13:324–340, 1982.

[11] M. Kaneko. The central assignment game and the assignment markets. *Journal of Mathematical Economics*, 10:205–232, 1982.

[12] W. B. MacLeod and J. M. Malcomson. Investments, holdup, and the form of market contracts. *American Economic Review*, 83:811–837, 1993.

[13] L. Makowski. Perfect competition, the profit criterion, and the organization of economic activity. *Journal of Economic Theory*, 22:222–242, 1980.

[14] L. Makowski and J. M. Ostroy. Appropriation and efficiency: A revision of the first theorem of welfare economics. *American Economic Review*, 85:808–827, 1995.

[15] M. Peters and A. Siow. Competing pre-marital investments. mimeo, University of Toronto, 2000.

[16] M. Quinzii. Core and competitive equilibria with indivisibilities. *International Journal of Game Theory*, 13:41–60, 1984.

[17] A. Roth and M. A. O. Sotomayer. *Two-Sided Matching*. Cambridge University Press, Cambridge, 1990.