Correction to the proof of Lemma 9.4.2 in *Repeated Games and Reputations* by George J. Mailath and Larry Samuelson (OUP 2006)

September 21, 2008

Claim 9.4.3 (on page 308) is false. We thank Songzi Du for bringing the error in the proof to our attention and providing a counterexample to the claim. There does not appear to be an easy way to patch the proof presented in the text of Lemma 9.4.2 (based on Fudenberg, Levine, and Maskin (1994)). The following is an alternative proof, also due to Fudenberg, Levine, and Maskin.

Lemma 9.4.2 *If an action profile with pure long-lived players actions is enforceable and pairwise identifiable for long-lived players i and j, then it is orthogonally enforceable in all ij-pairwise directions, λ^{ij}.*

Proof. Suppose $\alpha^* = (a_1^*, \ldots, a_n^*, \alpha_{SL}^*)$ is an action profile with pure long-lived players actions enforced by some normalized continuations $\hat{x}$. Let $v_i = u_i(\alpha^*) + E[\hat{x}_i(y) \mid \alpha^*]$, and define $g_i(a_i) \equiv u_i(a_i, \alpha^*) - v_i$ for all $a_i \in A_i$. We first prove a preliminary claim.

Claim 1 *For all i, and all $\mu_i \in \mathbb{R}^{A_i}$ satisfying*

$$\sum_{a_i \in A_i} \mu_i(a_i)\rho(y \mid (a_i, \alpha_{-i}^*)) = 0, \quad \forall y \tag{1}$$

$$\textit{and} \qquad \mu_i(a_i) \geq 0, \quad \forall a_i \neq a_i^*, \tag{2}$$

we have

$$\sum_{a_i \in A_i} \mu_i(a_i) g_i(a_i) \leq 0. \tag{3}$$

Proof. By hypothesis, the normalized continuations $\hat{x}_i : Y \to \mathbb{R}$ satisfy (8.1.2) and (8.1.3) at α^*. We rewrite (8.1.2) and (8.1.3) as

$$\sum_{y \in Y} \rho(y \mid (a_i^*, \alpha_{-i}^*)) x_i(y) = -g_i(a_i^*), \tag{4}$$

$$\text{and} \qquad \sum_{y \in Y} \rho(y \mid (a_i, \alpha_{-i}^*)) x_i(y) \leq -g_i(a_i), \quad \forall a_i \in A_i \setminus \{a_i^*\}. \tag{5}$$

These linear constraints can be viewed as the constraints of a linear program, with an objective function identically equal to 0. That is, since α^* is enforceable, the following linear program has a solution (for all i):

$$\max_{x_i \in \mathbb{R}^Y} \sum_{y \in Y} 0 \cdot x_i(y) \quad \text{subject to (4) and (5).} \tag{LP$_i$}$$

Since the continuations are not constrained to be nonnegative, the dual of this linear program is

$$\max_{\mu_i \in \mathbb{R}^{A_i}} \sum_{a_i \in A_i} \mu_i(a_i) g_i(a_i) \quad \text{subject to}$$

$$\sum_{a_i \in A_i} \mu_i(a_i) \rho(y \mid (a_i, \alpha_{-i}^*)) = 0, \quad \forall y \tag{6}$$

$$\text{and} \quad \mu_i(a_i) \geq 0, \quad \forall a_i \neq a_i^*. \tag{7}$$

Note that (6) coincides with (1), and (7) coincides with (2). The dual has at least one feasible feasible vector of dual variables (given by $\mu_i(a_i) = 0$ for all a_i), and so the primal program LP$_i$ has a solution if, and only if, the dual is bounded.[1] Since the primal program has a solution, the dual is bounded.

Suppose $\mu_i \neq 0$ is a feasible vector of dual variables. Then, $\mu_i(a_i^*) < 0$ (if not, the dual constraints imply $\rho(y \mid (a_i, \alpha_{-i}^*) = 0$ for all y and any a_i satisfying $\mu_i(a_i) > 0$, which is impossible). Rewrite (6) as (where $B_i \equiv \{a_i \in A_i : \mu_i(a_i) \geq 0\} = A_i \setminus \{a_i^*\}$),

$$\sum_{a_i \in B_i} \mu_i(a_i) \rho(y \mid (a_i, \alpha_{-i}^*)) = -\mu_i(a_i^*) \rho(y \mid \alpha^*). \tag{8}$$

Since $\mu_i \neq 0$, we have $\sum_{a_i \in B_i} \mu_i(a_i) > 0$. Defining $\tilde{\mu}_i(a_i) \equiv \mu_i(a_i) / \sum_{a_i \in B_i} \mu_i(a_i)$, we have a well-defined probability distribution over actions in B_i. Moreover,

$$\begin{aligned} 1 &= \sum_y \sum_{a_i \in B_i} \tilde{\mu}_i(a_i) \rho(y \mid (a_i, \alpha^*)) \\ &= -\sum_y \tilde{\mu}_i(a_i^*) \rho(y \mid \alpha^*) = -\tilde{\mu}_i(a_i^*). \end{aligned}$$

Equation (8) thus implies that the mixed action induced by $\tilde{\mu}_i$ on B_i implies the *same* distribution on the public signal as a_i^*.

[1] For a compact introduction to linear programming, see Vohra (2005).

In other words, for any feasible nonzero vector μ_i, the objective function can be written as

$$\sum_{a_i \in B_i} \mu_i(a_i) \left\{ \sum_{a_i \in B_i} \tilde{\mu}_i(a_i) g_i(a_i) - g_i(a_i^*) \right\}.$$

Note that if μ_i is a feasible vector, then so is $k\mu_i$ for any positive k. The requirement that the dual be bounded over the set of feasible dual variables is hence equivalent to the requirement that for all dual variables satisfying (6) and (7), we have (3). ■

The action profile α^* is orthogonally enforceable in the pairwise direction λ^{ij} if there exist normalized continuations $x : Y \to \mathbb{R}^n$ satisfying (4) and (5) for all long-lived players, and $\lambda_i^{ij} x_i(y) + \lambda_j^{ij} x_j(y) = 0$ for all $y \in Y$. As usual, since the pairwise direction λ^{ij} imposes no further restrictions on the normalized continuations for the long-lived players other than i and j, we may ignore them.

We now argue that the normalized continuations $\hat{x}_i$ and $\hat{x}_j$ that enforce α^* (and yield payoffs v_i and v_j) can be adjusted to obtain orthogonal enforceability without affecting the values v_i and v_j. As in the proof of the claim, the existences of the desired normalized continuations for i and j is equivalent to the following linear program having a solution (where we have eliminated x_j using the orthogonality condition):

$$\max_{x_i \in \mathbb{R}^Y} \sum_{y \in Y} 0 \cdot x_i(y) \quad \text{subject to}$$

$$\begin{aligned}
&\sum_{y \in Y} \rho(y \mid (a_i^*, \alpha_{-i}^*)) x_i(y) = -g_i(a_i^*), \\
&\sum_{y \in Y} \rho(y \mid (a_i, \alpha_{-i}^*)) x_i(y) \leq -g_i(a_i), \quad \forall a_i \in A_i \setminus \{a_i^*\}, \\
&\sum_{y \in Y} \rho(y \mid (a_j^*, \alpha_{-j}^*)) (-\lambda_i^{ij}/\lambda_j^{ij}) x_i(y) = -g_j(a_j^*), \\
\text{and} \quad &\sum_{y \in Y} \rho(y \mid (a_j, \alpha_{-j}^*)) (-\lambda_i^{ij}/\lambda_j^{ij}) x_i(y) \leq -g_j(a_j), \quad \forall a_j \in A_j \setminus \{a_j^*\}.
\end{aligned}$$

The dual of this linear program is

$$\max_{\mu_i \in \mathbb{R}^{A_i}, \mu_j \in \mathbb{R}^{A_j}} \sum_{a_i \in A_i} \mu_i(a_i) g_i(a_i) + \sum_{a_j \in A_j} \mu_j(a_j) g_j(a_j)$$

subject to

$$\sum_{a_i \in A_i} \mu_i(a_i)\rho(y \mid (a_i, \alpha^*_{-i})) + \mu_j(a_j)\rho(y \mid (a_j, \alpha^*_{-j}))(-\lambda_i^{ij}/\lambda_j^{ij}) = 0, \quad \forall y, \tag{9}$$

$$\mu_i(a_i) \geq 0, \quad \forall a_i \neq a_i^*, \tag{10}$$

$$\text{and} \quad \mu_j(a_j) \geq 0, \quad \forall a_j \neq a_j^*. \tag{11}$$

As in the proof of the claim, there exist orthogonally enforcing continuations if the dual problem is bounded for feasible dual variables. We now argue that this pairwise problem can be effectively separated into distinct individual problems.

The system (9) can be written as

$$(\mu_i,\ \hat{\mu}_j) \cdot R_{ij}(\alpha^*) = 0, \tag{12}$$

where $\hat{\mu}_j = -\lambda_i^{ij}\mu_j/\lambda_j^{ij}$. Note that $\hat{\mu}_j$ satisfies (1) if, and only if, μ_j does.

For an arbitrary matrix R, denote its "left" null space by $N(R)$, that is, $N(R) = \{\xi : \xi \cdot R = 0\}$. Thus, (1) is the requirement that $\mu_i \in N(R_i(\alpha^*_{-i}))$, while (12) is the requirement that $(\mu_i,\ \hat{\mu}_j) \in N(R_{ij}(\alpha^*))$. Let $\dim(N(R))$ denote the dimension of the left null space of R. Pairwise identifiability implies

$$\dim(N(R_{ij}(\alpha^*))) = \dim(N(R_i(\alpha^*_{-i}))) + \dim(N(R_j(\alpha^*_{-j}))) + 1.$$

Since $(\alpha_i^*, -\alpha_j^*) \in N(R_{ij}(\alpha^*))$ (recall the comment just after Definition 9.2.2), every vector in $N(R_{ij}(\alpha^*))$ is a linear combination of $(\alpha_i^*, -\alpha_j^*)$ and some (μ_i, μ_j), where $\mu_i \in N(R_i(\alpha^*_{-i}))$ and $\mu_j \in N(R_j(\alpha^*_{-j}))$. Note that the value of the objective function at $(\alpha_i^*, -\alpha_j^*)$ is zero (since $g_i(a_i^*) = g_j(a_j^*) = 0$), and so the weight placed on that vector can be ignored.

Since (10) and (11) are each (2) for players i and j, the claim implies that for any (μ_i, μ_j), where $\mu_i \in N(R_i(\alpha^*_{-i}))$ and $\mu_j \in N(R_j(\alpha^*_{-j}))$, (3) holds for i and j, i.e., the dual for the pairwise problem is bounded above by 0. Hence, the primal for the pairwise problem has a solution and so α^* is orthogonally enforceable in all ij-pairwise directions. ■

References

FUDENBERG, D., D. K. LEVINE, AND E. MASKIN (1994): "The Folk Theorem with Imperfect Public Information," *Econometrica*, 62(5), 997–1039.

VOHRA, R. V. (2005): *Advanced Mathematical Economics*. Routledge, London and New York.