Endogenous Inequality in Integrated Labor Markets with Two-sided Search*—Omitted Calculations

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Note that equations numbered (P.x) are the same as equation (x) of the published paper.

1. Symmetric Equilibrium

This section provides alternative calculations for the value functions of a symmetric equilibrium and calculates the equilibrium flow payoffs. It initially reproduces some calculations of Section II in order to be self-contained.

Let V_W denote the value of skills to a worker. The fraction of new workers who become skilled is

$$h_W = C(V_W).$$

Since there is a death rate of δ, the change in the stock of skilled workers is given by the difference between the measure of newly entering workers acquiring skills, or δh_W, and the proportion of skilled workers who die, or δH_W, giving

$$\dot{H}_W = \delta(h_W - H_W).$$

In a steady state equilibrium, $\dot{H}_W = 0$ and so

$$h_W = H_W = C(V_W). \tag{P.1}$$

We turn now to the determination of the vacancy rate among firms. There are two ways that additional jobs appear in the vacancy pool: an occupied firm dies, and is replaced by a new

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firm without a worker, or an employed worker dies. Thus, new jobs join the vacancy pool at rate $(1-\rho_F)\delta + (1-\rho_F)\delta = 2(1-\rho_F)\delta$. Vacancies are filled by successful searches on the part of both firms and workers. There are ρ_F firms searching at rate $\lambda_F \rho_W H_W$, so that the rate at which vacancies are filled as a result of firm search is $\rho_F \lambda_F \rho_W H_W$. There are also $\rho_W H_W$ workers searching at rate $\lambda_W \rho_F$, so that the rate at which vacancies are filled as a result of worker search is $\rho_W H_W \lambda_W \rho_F$. Thus, the change in the number of vacancies is given by

$$\dot{\rho}_F = 2(1-\rho_F)\delta - \rho_F \rho_W H_W (\lambda_F + \lambda_W),$$

giving, for a steady state equilibrium,

$$2\delta(1-\rho_F) = \rho_F \rho_W H_W (\lambda_F + \lambda_W). \tag{P.2}$$

Next, consider the unemployment rate among workers. This differs from the case of firms because a newborn worker is unemployed only if she chooses to acquire skills, with unskilled workers opting into the unskilled sector of the economy. If an occupied firm dies, the skilled worker is now unemployed, which occurs at rate $(1-\rho_F)\delta$. There is an inflow of δh_W into the unemployed skilled worker pool from newborns and an outflow of $\delta \rho_W H_W$ from death. Unemployed workers are hired at the same rate as vacancies are filled (given by $\rho_F \rho_W H_W(\lambda_W + \lambda_R)$). This yields

$$\frac{d}{dt}(\rho_W H_W) = \dot{\rho}_W H_W + \rho_W \dot{H}_W = (1-\rho_F)\delta + (h_W - \rho_W H_W)\delta - \rho_F \rho_W H_W (\lambda_F + \lambda_W).$$

Since the number of employed workers and filled jobs is the same, we have

$$H_W(1-\rho_W) = 1 - \rho_F,$$

so that

$$\dot{\rho}_W H_W + \rho_W \dot{H}_W = (H_W(1-\rho_W) + h_W - \rho_W H_W)\delta - \rho_F \rho_W H_W (\lambda_F + \lambda_W).$$

Since in a steady state, $H_W = h_W$, $\dot{\rho}_W = 0$, and $\dot{H}_W = 0$, we have

$$2\delta(1-\rho_W) = \rho_F \rho_W (\lambda_F + \lambda_W). \tag{P.3}$$

We next calculate V_W, the value to a worker of entering the skilled market. Since we are constructing a symmetric equilibrium in which both red and green workers are searched, their value functions, and hence skill decisions, will be identical, and so the firms will be indifferent over all their search possibilities.

Let $Z_W(s)$ denote the expected value of an employed worker at time s. Then we have:

$$\begin{aligned} Z_W(s) \;&=\; \int_s^\infty \left\{ \int_s^\omega \left\{ \int_s^\upsilon w(\tau) e^{-r(\tau - s)} d\tau + e^{-r(\upsilon - s)} V_W(\upsilon|\omega) \right\} \delta e^{-\delta(\upsilon - s)}\, d\upsilon \right. \\ &\quad \left. +\; e^{-\delta(\omega - s)} \int_s^\omega w(\tau) e^{-r(\tau - s)}\, d\tau \right\} \delta e^{-\delta(\omega - s)}\, d\omega, \end{aligned}$$

where ω is the date of the worker's death, υ is the date of the firm's death, $w(\tau)$ is the expected flow payoff of an employed worker at date τ, and $V_W(\upsilon|\omega)$ is the value of being an unemployed worker at date υ, conditional on death at date ω. The first line of this expression captures the payoff in the event that the firm dies before the worker, and consists of the sum of the wage payments received from the firm and the value of being pushed back into unemployment. The second line captures the payoff in the event the worker dies first. Since the value of being unemployed at time υ is given by

$$V_W(\upsilon) = E_{\omega \geq \upsilon}\left[V_W(\upsilon|\omega)\right] = \int_\upsilon^\infty V_W(\upsilon|\omega)\delta e^{-\delta(\omega-\upsilon)}\,d\omega,$$

we can simplify to obtain

$$\begin{aligned} Z_W(s) = & \int_s^\infty \int_s^\omega \int_s^\upsilon w(\tau)e^{-r(\tau-s)}\delta e^{-\delta(\upsilon-s)}\delta e^{-\delta(\omega-s)}\,d\tau\,d\upsilon\,d\omega \\ & + \int_s^\infty e^{-r(\upsilon-s)}V_W(\upsilon)\delta e^{-2\delta(\upsilon-s)}\,d\upsilon \\ & + \int_s^\infty \int_s^\omega w(\tau)e^{-r(\tau-s)}\delta e^{-2\delta(\omega-s)}\,d\tau\,d\omega. \end{aligned}$$

In a steady state, $w(\tau) = w$, $V_W(\upsilon) = V_W$, and $Z_W(s) = Z_W$, and so we can perform the integration to find

$$Z_W = \frac{w + \delta V_W}{r + 2\delta}.$$

If s is the time that a match arrives, then

$$V_W = E_s\left[e^{-\delta s}e^{-rs}Z_W\right],$$

where s is the time that a match with a vacant firm occurs, $e^{-\delta s}$ is the probability that the worker is still alive at time s, and e^{-rs} provides the appropriate discounting of the value $Z_W(s)$ to the present. The time s at which the worker meets a vacant firm has density $\rho_F(\lambda_F + \lambda_W)e^{-\rho_F(\lambda_F+\lambda_W)s}$.[1] We have an expected value of:

$$\begin{aligned} V_W &= \frac{(w + \delta V_W)}{(r + 2\delta)} \int_0^\infty \rho_F(\lambda_F + \lambda_W)e^{-(\rho_F(\lambda_F+\lambda_W)+r+\delta)s}\,ds \\ &= \frac{(w + \delta V_W)}{(r + 2\delta)} \frac{\rho_F(\lambda_F + \lambda_W)}{(\rho_F(\lambda_F + \lambda_W) + r + \delta)}. \end{aligned}$$

Solving this equation for V_W yields

$$V_W = \frac{\rho_F(\lambda_F + \lambda_W)w}{(r + \delta)(\rho_F(\lambda_F + \lambda_W) + r + 2\delta)}.$$

A similar calculation gives the value to a firm or participating in the market:

$$V_F = \frac{\rho_W H_W(\lambda_F + \lambda_W)f}{(r + \delta)(\rho_W H_W(\lambda_F + \lambda_W) + r + 2\delta)},$$

where f is the expected steady-state flow payoff of an occupied firm and V_F is the steady-state value of a vacant firm. We now determine the expected flow payoffs w and f. Firms and workers bargain over the surplus created in a match by making wage proposals with equal probability, and any such proposal will make the responding agent indifferent between accepting the proposal and rejecting. Suppose the firm is chosen to make a proposal, and offers a wage of $\underline{w}$ to the worker. Accepting this offer gives an expected payoff of $(\underline{w} + \delta V_W)/(r + 2\delta)$. which is the value of Z_W

[1] The arrival rate of matches is the sum of matches from worker search ($\lambda_W\rho_F$) and firm search ($\lambda_F\rho_F$).

calculated at the wage $\underline{w}$. If the worker rejects this offer, her continuation value is V_W. The firm will choose $\underline{w}$ so as to make the worker indifferent between accepting and rejecting, giving:

$$\underline{w} = (r+\delta)V_W.$$

Since the firm must similarly be made indifferent by an offer received from the worker, should the worker be called upon to make the offer, the worker will offer $(r+\delta)V_f$ to the firm. We then have, in the steady state

$$\begin{aligned} w &= \frac{1}{2}\underline{w} + \frac{1}{2}\{x-(r+\delta)V_F\} \\ &= \frac{x}{2} + \frac{1}{2}(r+\delta)(V_W - V_F), \end{aligned}$$

and symmetrically for the firm,

$$f = \frac{x}{2} + \frac{1}{2}(r+\delta)(V_F - V_W).$$

Inserting the expressions for the value of a firm V_F and worker V_W, we then solve for:

$$\begin{aligned} w &= \frac{\rho_F(\lambda_F+\lambda_W)+r+2\delta}{(\rho_F+\rho_W H_W)(\lambda_F+\lambda_W)+2(r+2\delta)}x, \\ f &= \frac{\rho_W H_W(\lambda_F+\lambda_W)+r+2\delta}{(\rho_F+\rho_W H_W)(\lambda_F+\lambda_W)+2(r+2\delta)}x, \end{aligned}$$

giving

$$V_W = \frac{\rho_F(\lambda_F+\lambda_W)}{(r+\delta)\left[(\rho_F+\rho_W H_W)(\lambda_F+\lambda_W)+2(r+2\delta)\right]}x, \tag{P.6}$$

and

$$V_F = \frac{\rho_W H_W(\lambda_F+\lambda_W)}{(r+\delta)\left[(\rho_F+\rho_W H_W)(\lambda_F+\lambda_W)+2(r+2\delta)\right]}x. \tag{P.7}$$

2. Asymmetric Equilibria

This section presents the calculations leading to equations (13)–(15). Define $\lambda_G = 2\lambda_F + \lambda_W$, $\tilde{\rho}_G = \rho_G H_G$, $\tilde{\rho}_R = \rho_R H_R$, and

$$Z_F = \frac{\lambda_G\tilde{\rho}_G Z_{F,G} + \lambda_W\tilde{\rho}_R Z_{F,R}}{(\lambda_G\tilde{\rho}_G + \lambda_W\tilde{\rho}_R)}.$$

The value Z_F is the expected value of an occupied firm, where the probabilities reflect the relative likelihood of meeting a red and green worker. This allows us to simplify the system of value functions to:

$$\begin{aligned} Z_R &= \frac{x}{2(r+2\delta)} + \frac{1}{2(r+2\delta)}\left((r+3\delta)V_R - (r+\delta)V_F\right), \\ V_R &= \frac{\rho_F\lambda_W Z_R}{\rho_F\lambda_W + r+\delta}, \\ Z_G &= \frac{x}{2(r+2\delta)} + \frac{1}{2(r+2\delta)}\left((r+3\delta)V_G - (r+\delta)V_F\right), \\ V_G &= \frac{\rho_F\lambda_G Z_G}{\rho_F\lambda_G + r+\delta}, \\ Z_F &= \frac{x}{2(r+2\delta)} + \frac{1}{2(r+2\delta)}\left((r+3\delta)V_F - \frac{(r+\delta)}{(\lambda_G\tilde{\rho}_G + \lambda_W\tilde{\rho}_R)}(\lambda_G\tilde{\rho}_G V_G + \lambda_W\tilde{\rho}_R V_R)\right), \end{aligned}$$

and

$$V_F = \frac{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right) Z_F}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)}.$$

Eliminating the unmatched value functions gives

$$\begin{aligned}
2(r+2\delta)Z_R &= x + \left((r+3\delta)\frac{\rho_F \lambda_W Z_R}{\rho_F \lambda_W + r + \delta} - (r+\delta)\frac{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right) Z_F}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)} \right) \\
2(r+2\delta)Z_G &= x + \left((r+3\delta)\frac{\rho_F \lambda_G Z_G}{\rho_F \lambda_G + r + \delta} - (r+\delta)\frac{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right) Z_F}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)} \right) \\
2(r+2\delta)Z_F &= x + \left\{ (r+3\delta)\frac{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right) Z_F}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)} \right. \\
&\quad \left. - \frac{(r+\delta)}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)} \left(\lambda_G \tilde{\rho}_G \frac{\rho_F \lambda_G Z_G}{\rho_F \lambda_G + r + \delta} + \lambda_W \tilde{\rho}_R \frac{\rho_F \lambda_W Z_R}{\rho_F \lambda_W + r + \delta} \right) \right\}.
\end{aligned}$$

Simplifying,

$$\begin{aligned}
Z_R &= \frac{\left(\rho_F \lambda_W + r + \delta\right)}{\left(\rho_F \lambda_W + 2(r+2\delta)\right)} \left\{ \frac{x}{(r+\delta)} - \frac{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right) Z_F}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)} \right\} \\
Z_G &= \frac{\left(\rho_F \lambda_G + r + \delta\right)}{\left(\rho_F \lambda_G + 2\left(r+2\delta\right)\right)} \left\{ \frac{x}{(r+\delta)} - \frac{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right) Z_F}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)} \right\} \\
Z_F &= \frac{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + 2\left(r+2\delta\right)\right)} \left\{ \frac{x}{(r+\delta)} - \frac{1}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)} \right. \\
&\quad \left. \times \left(\lambda_G \tilde{\rho}_G \frac{\rho_F \lambda_G Z_G}{\left(\rho_F \lambda_G + r + \delta\right)} + \lambda_W \tilde{\rho}_R \frac{\rho_F \lambda_W Z_R}{\left(\rho_F \lambda_W + r + \delta\right)} \right) \right\}.
\end{aligned}$$

Substituting for Z_R and Z_G into the expression for Z_F gives

$$\begin{aligned}
Z_F &= \frac{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + 2\left(r+2\delta\right)\right)} \left\{ \frac{x}{(r+\delta)} - \frac{1}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)} \right. \\
&\quad \times \left[\lambda_G \tilde{\rho}_G \frac{\rho_F \lambda_G}{\left(\rho_F \lambda_G + 2\left(r+2\delta\right)\right)} \left(\frac{x}{(r+\delta)} - \frac{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right) Z_F}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)} \right) \right. \\
&\quad \left. \left. + \lambda_W \tilde{\rho}_R \frac{\rho_F \lambda_W}{\left(\rho_F \lambda_W + 2(r+2\delta)\right)} \left(\frac{x}{(r+\delta)} - \frac{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right) Z_F}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)} \right) \right] \right\}.
\end{aligned}$$

Collecting terms, the term involving x is

$$\begin{aligned}
&\frac{x}{(r+\delta)} \frac{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + 2\left(r+2\delta\right)\right)} \left\{ 1 - \frac{1}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)} \left(\frac{\lambda_G \tilde{\rho}_G \rho_F \lambda_G}{\left(\rho_F \lambda_G + 2\left(r+2\delta\right)\right)} + \frac{\lambda_W \tilde{\rho}_R \rho_F \lambda_W}{\left(\rho_F \lambda_W + 2(r+2\delta)\right)} \right) \right\} \\
&= \frac{x}{(r+\delta)} \frac{\left(\tilde{\rho}_G \lambda_G + \lambda_W \tilde{\rho}_R + r + \delta\right) 2\left(r+2\delta\right)}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + 2\left(r+2\delta\right)\right)} \frac{\left(\lambda_G \rho_F \lambda_W \left(\tilde{\rho}_G + \tilde{\rho}_R\right) + 2\left(r+2\delta\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)\right)}{\left(\tilde{\rho}_G \lambda_G + \lambda_W \tilde{\rho}_R\right)\left(\rho_F \lambda_G + 2r + 4\delta\right)\left(\rho_F \lambda_W + 2r + 4\delta\right)}.
\end{aligned}$$

The coefficient of Z_F on the right hand side is

$$\begin{aligned}
&\frac{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)}{\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + 2\left(r+2\delta\right)\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)} \\
&\times \left(\frac{\lambda_G \tilde{\rho}_G \rho_F \lambda_G \left(\rho_F \lambda_G + r + \delta\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)}{\left(\rho_F \lambda_G + r + \delta\right)\left(\rho_F \lambda_G + 2\left(r+2\delta\right)\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)} \right. \\
&\left. + \frac{\lambda_W \tilde{\rho}_R \rho_F \lambda_W \left(\rho_F \lambda_W + r + \delta\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)}{\left(\rho_F \lambda_W + r + \delta\right)\left(\rho_F \lambda_W + 2(r+2\delta)\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + r + \delta\right)} \right)
\end{aligned}$$

$$= \frac{\rho_F \left(2\lambda_G^2 \tilde{\rho}_G r + 4\lambda_G^2 \tilde{\rho}_G \delta + \lambda_G^2 \tilde{\rho}_G \rho_F \lambda_W + 2\lambda_W^2 \tilde{\rho}_R r + 4\lambda_W^2 \tilde{\rho}_R \delta + \lambda_W^2 \tilde{\rho}_R \rho_F \lambda_G\right)}{\left(\rho_F \lambda_W + 2r + 4\delta\right)\left(\rho_F \lambda_G + 2r + 4\delta\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + 2r + 4\delta\right)}.$$

So, the equation determining Z_F can be written as

$$\left(1 - \frac{\rho_F \left(2\lambda_G^2 \tilde{\rho}_G r + 4\lambda_G^2 \tilde{\rho}_G \delta + \lambda_G^2 \tilde{\rho}_G \rho_F \lambda_W + 2\lambda_W^2 \tilde{\rho}_R r + 4\lambda_W^2 \tilde{\rho}_R \delta + \lambda_W^2 \tilde{\rho}_R \rho_F \lambda_G\right)}{\left(\rho_F \lambda_W + 2r + 4\delta\right)\left(\rho_F \lambda_G + 2r + 4\delta\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + 2r + 4\delta\right)}\right) Z_F$$

$$= \frac{x\left(\tilde{\rho}_G \lambda_G + \lambda_W \tilde{\rho}_R + r + \delta\right) 2\left(r + 2\delta\right)}{\left(r+\delta\right)\left(\tilde{\rho}_G \lambda_G + \lambda_W \tilde{\rho}_R + 2r + 4\delta\right)} \frac{\left(\lambda_G \rho_F \lambda_W \left(\tilde{\rho}_G + \tilde{\rho}_R\right) + 2\left(r + 2\delta\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)\right)}{\left(\tilde{\rho}_G \lambda_G + \lambda_W \tilde{\rho}_R\right)\left(\rho_F \lambda_G + 2r + 4\delta\right)\left(\rho_F \lambda_W + 2r + 4\delta\right)},$$

or

$$\begin{aligned}&\{\left(\rho_F \lambda_W + 2r + 4\delta\right)\left(\rho_F \lambda_G + 2r + 4\delta\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R + 2r + 4\delta\right) \\ &-\rho_F \left(2\lambda_G^2 \tilde{\rho}_G r + 4\lambda_G^2 \tilde{\rho}_G \delta + \lambda_G^2 \tilde{\rho}_G \rho_F \lambda_W + 2\lambda_W^2 \tilde{\rho}_R r + 4\lambda_W^2 \tilde{\rho}_R \delta + \lambda_W^2 \tilde{\rho}_R \rho_F \lambda_G\right)\} Z_F\end{aligned}$$

$$= \frac{x\left(\tilde{\rho}_G \lambda_G + \lambda_W \tilde{\rho}_R + r + \delta\right) 2\left(r + 2\delta\right)}{\left(r+\delta\right)} \frac{\left(\lambda_G \rho_F \lambda_W \left(\tilde{\rho}_G + \tilde{\rho}_R\right) + 2\left(r + 2\delta\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)\right)}{\left(\tilde{\rho}_G \lambda_G + \lambda_W \tilde{\rho}_R\right)},$$

and hence,

$$Z_F = \frac{x\left(\tilde{\rho}_G \lambda_G + \lambda_W \tilde{\rho}_R + r + \delta\right)}{\left(r+\delta\right)\left(\tilde{\rho}_G \lambda_G + \lambda_W \tilde{\rho}_R\right)} \frac{\left(\lambda_G \rho_F \lambda_W \left(\tilde{\rho}_G + \tilde{\rho}_R\right) + 2\left(r + 2\delta\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)\right)}{\Delta},$$

where $\Delta \equiv 2\left(r + 2\delta\right)\left(\rho_F \lambda_G + \lambda_G \tilde{\rho}_G + \rho_F \lambda_W + \lambda_W \tilde{\rho}_R + 2\left(r + 2\delta\right)\right) + \rho_F \lambda_W \lambda_G \left(\rho_F + \tilde{\rho}_R + \tilde{\rho}_G\right)$.

We can use this result to calculate:

$$V_F = \frac{x}{\left(r+\delta\right)} \frac{\left(\lambda_G \rho_F \lambda_W \left(\tilde{\rho}_G + \tilde{\rho}_R\right) + 2\left(r + 2\delta\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)\right)}{\Delta}. \tag{P.13}$$

Similarly,

$$\begin{aligned}Z_R &= \frac{x\left(\rho_F \lambda_W + r + \delta\right)}{\left(r+\delta\right)\left(\rho_F \lambda_W + 2(r + 2\delta)\right)} \left\{1 - \Delta^{-1}\left(\lambda_G \rho_F \lambda_W \left(\tilde{\rho}_G + \tilde{\rho}_R\right) + 2\left(r + 2\delta\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)\right)\right\} \\ &= \frac{x}{\left(r+\delta\right)\Delta}\left(\rho_F \lambda_W + r + \delta\right)\left(\rho_F \lambda_G + 2\left(r + 2\delta\right)\right),\end{aligned}$$

and

$$\begin{aligned}Z_G &= \frac{x\left(\rho_F \lambda_G + r + \delta\right)}{\left(r+\delta\right)\left(\rho_F \lambda_G + 2\left(r + 2\delta\right)\right)} \left\{1 - \Delta^{-1}\left(\lambda_G \rho_F \lambda_W \left(\tilde{\rho}_G + \tilde{\rho}_R\right) + 2\left(r + 2\delta\right)\left(\lambda_G \tilde{\rho}_G + \lambda_W \tilde{\rho}_R\right)\right)\right\} \\ &= \frac{x}{\left(r+\delta\right)\Delta} x\left(\rho_F \lambda_G + r + \delta\right)\left(\rho_F \lambda_W + 2\left(r + 2\delta\right)\right),\end{aligned}$$

giving

$$V_R = \frac{\rho_F \lambda_W \left(\rho_F \lambda_G + 2\left(r + 2\delta\right)\right)}{\left(r+\delta\right)\Delta} x \tag{P.14}$$

and

$$V_G = \frac{\rho_F \lambda_G \left(\rho_F \lambda_W + 2\left(r + 2\delta\right)\right)}{\left(r+\delta\right)\Delta} x. \tag{P.15}$$

Thus,

$$\begin{aligned} V_G &= \frac{\rho_F \lambda_G \rho_F \lambda_W + \rho_F \lambda_G 2\,(r+2\delta)}{(r+\delta)\,\Delta} x = V_R + \frac{\rho_F 2\,(r+2\delta)\,(\lambda_G - \lambda_W)}{(r+\delta)\,\Delta} x \\ &= V_R + \frac{\rho_F 2\,(r+2\delta)\,2\lambda_F}{(r+\delta)\,\Delta} x > V_R. \end{aligned}$$

3. The Extreme Asymmetric Steady State

This appendix verifies equation (17). In the extreme asymmetric steady state, $H_G = 1/2$ and $H_R = 0$, so the vacancies steady state condition is

$$4\delta(1-\rho_F) = \rho_F \rho_G (\lambda + \lambda_F). \tag{1}$$

Similarly, the green unemployment steady state condition is

$$2\delta(1-\rho_G) = \rho_G \rho_F (\lambda + \lambda_F). \tag{2}$$

These imply

$$\rho_G = 2\rho_F - 1. \tag{3}$$

From (3) and (1),

$$\frac{d\rho_F}{d\lambda_F} = \rho_F' = \frac{-\rho_F(2\rho_F - 1)}{\{4\delta + (4\rho_F - 1)(\lambda + \lambda_F)\}}. \tag{4}$$

We first show that $\hat{V}_G$ is increasing in λ_F. The argument parallels our demonstration that $dV_W/d\lambda > 0$ in the symmetric steady state. First, we can write

$$\hat{V}_G = \frac{\rho_F(\lambda + \lambda_F)}{(r+d)[(\rho_F + \frac{1}{2}\rho_G)(\lambda + \lambda_F) + 2(r+2\delta)]}.$$

Notice that this is (P.6), with ρ_W replaced by ρ_G and H_W by $\frac{1}{2}$ (because only green workers enter, and all green workers enter), and with the total search intensity given by $2\lambda_F + \lambda_W = \lambda + \lambda_F$ (because the decision of firms to search only greens doubles the firm's effective search intensity). Alternatively, it is straightforward to verify that this expression equals (P.15). Using (3), we have

$$\hat{V}_G = \frac{\rho_F(\lambda + \lambda_F)}{(r+d)[(2\rho_F - 1)(\lambda + \lambda_F) + 2(r+2\delta)]}.$$

Suppose $d\hat{V}_G/d\lambda_F < 0$. Then, taking the derivative, it must be the case that

$$\frac{d(\rho_F(\lambda + \lambda_F))}{d\lambda_F}[(2\rho_F - \frac{1}{2})(\lambda + \lambda_F) + 2(r+2\delta)] - \rho_F(\lambda + \lambda_F)\frac{d((2\rho_F - \frac{1}{2})(\lambda + \lambda_F))}{d\lambda_F} < 0.$$

Simplifying, we must have

$$\frac{d(\rho_F(\lambda + \lambda_F))}{d\lambda_F}[(-\frac{1}{2})(\lambda + \lambda_F) + 2(r+2\delta)] - \rho_F(\lambda + \lambda_F)\frac{d((-\frac{1}{2})(\lambda + \lambda_F))}{d\lambda_F} < 0.$$

It is immediate from (1)–(2) that $d(\rho_F(\lambda+\lambda_F))/d\lambda_F > 0$. We can then delete the (positive) term involving $2(r+2\delta)$ and divide by $\lambda+\lambda_F$ to find that a necessary condition is

$$\frac{d(\rho_F(\lambda+\lambda_F))}{d\lambda_F} > \rho_F.$$

But this implies that $d\rho_F/d\lambda_F > 0$, which contradicts (1)–(2). Hence, $\hat{V}_G$ is increasing in λ_F.

We next show that $\hat{V}_R$ is decreasing in λ_F. From (P.14),

$$\hat{V}_R = \frac{\rho_F^2(\lambda^2-\lambda_F^2)+\rho_F(\lambda-\lambda_F)2\,(r+2\delta)\,x}{(r+\delta)\,\hat{\Delta}},$$

where $\hat{\Delta} \equiv 2\,(r+2\delta)\,\{\rho_F(\lambda+\lambda_F)+(\lambda+\lambda_F)\rho_G/2+\rho_F(\lambda-\lambda_F)+2\,(r+2\delta)\}+(\lambda^2-\lambda_F^2)\left(\rho_F^2+\rho_F\rho_G/2\right) = \rho_F^2(\lambda^2-\lambda_F^2)+\rho_F(\lambda-\lambda_F)2\,(r+2\delta)+4\,(r+2\delta)^2+2\,(r+2\delta)\,(\lambda+\lambda_F)\{\rho_F+\rho_G/2\}+(\lambda^2-\lambda_F^2)\rho_F\rho_G/2$. Now, $\hat{V}_R$ is decreasing in λ_F if and only if $((r+\delta)\hat{V}_R/x)^{-1}$ is increasing in λ_F. Substituting,

$$\frac{x}{(r+\delta)\hat{V}_R} = 1+\frac{4\,(r+2\delta)^2+2\,(r+2\delta)\,(\lambda+\lambda_F)\{\rho_F+\rho_G/2\}+(\lambda^2-\lambda_F^2)\rho_F\rho_G/2}{\rho_F^2(\lambda^2-\lambda_F^2)+\rho_F(\lambda-\lambda_F)2\,(r+2\delta)}.$$

Differentiating the denominator with respect to λ_F yields

$$2\rho_F\rho_F'(\lambda^2-\lambda_F^2)-2\rho_F^2\lambda_F+\rho_F'(\lambda-\lambda_F)2\,(r+2\delta)-\rho_F 2\,(r+2\delta) < 0,$$

since $\lambda_F \leq \lambda$.

Turning to the numerator, its derivative is

$$2\,(r+2\delta)\,\{\rho_F+\rho_G/2\}+2\,(r+2\delta)\,(\lambda+\lambda_F)\{\rho_F'+\rho_G'/2\}-\lambda_F\rho_F\rho_G+(\lambda^2-\lambda_F^2)(\rho_F'\rho_G/2+\rho_F\rho_G'/2).$$

Using $\rho_G' = 2\rho_F'$ and (3), we can rewrite this as

$$\begin{aligned} & (r+2\delta)\,(4\rho_F-1)+4\,(r+2\delta)\,(\lambda+\lambda_F)\rho_F'-\lambda_F\rho_F(2\rho_F-1)+(\lambda^2-\lambda_F^2)(\rho_F'(\rho_F-1/2)+\rho_F\rho_F') \\ = \; & (r+2\delta)\,(4\rho_F-1)+4\,(r+2\delta)\,(\lambda+\lambda_F)\rho_F'-\lambda_F\rho_F(2\rho_F-1)+(\lambda^2-\lambda_F^2)\rho_F'(2\rho_F-1/2), \end{aligned}$$

which has the same sign as, substituting (4) and ignoring the positive denominator $\{4\delta+(4\rho_F-1)(\lambda+\lambda_F)\}$,

$$\begin{gathered} [(r+2\delta)\,(4\rho_F-1)-\lambda_F\rho_F(2\rho_F-1)]\,\{4\delta+(4\rho_F-1)(\lambda+\lambda_F)\} \\ -\rho_F(2\rho_F-1)4\,(r+2\delta)\,(\lambda+\lambda_F)-\rho_F(2\rho_F-1)(\lambda^2-\lambda_F^2)(2\rho_F-1/2) \\[1ex] = (r+2\delta)\,(4\rho_F-1)\,\{4\delta+(4\rho_F-1)(\lambda+\lambda_F)\}-\lambda_F\rho_F(2\rho_F-1)4\delta \\ -\rho_F(2\rho_F-1)(\lambda+\lambda_F)\lambda_F(4\rho_F-1)-\rho_F(2\rho_F-1)4\,(r+2\delta)\,(\lambda+\lambda_F) \\ -\rho_F(2\rho_F-1)(\lambda^2-\lambda_F^2)(2\rho_F-1/2) \\[1ex] = (r+2\delta)\,(4\rho_F-1)\,\{4\delta+(4\rho_F-1)(\lambda+\lambda_F)\}-\lambda_F\rho_F(2\rho_F-1)4\delta \\ -\rho_F(2\rho_F-1)(\lambda+\lambda_F)\,\{\lambda_F(4\rho_F-1)+4\,(r+2\delta)+(\lambda-\lambda_F)(2\rho_F-1/2)\} \equiv \Theta. \end{gathered}$$

It suffices to show that $\Theta > 0$. We first observe that variations in the interest rate r affect none of the other variables appearing in Θ. We can accordingly take the derivative

$$\frac{d\Theta}{dr} = (4\rho_F-1)[4\delta+(4\rho_F-1)(\lambda+\lambda_F)]-4\rho_F(2\rho_F-1)(\lambda+\lambda_F),$$

which will be positive if

$$(4\rho_F-1)^2-4\rho_F(2\rho_F-1) = 16\rho_F^2-8\rho_F+1-8\rho_F^2-4\rho_F = 8\rho_F^2-4\rho_F+1 > 0,$$

which holds for all $\rho_F \in (\frac{1}{2}, 1]$. As a result, it suffices to examine the value of r that minimizes Θ, namely $r = 0$, giving

$$2\delta(4\rho_F - 1)\left\{4\delta + (4\rho_F - 1)(\lambda + \lambda_F)\right\} - \lambda_F \rho_F (2\rho_F - 1)4\delta$$
$$-\rho_F(2\rho_F - 1)(\lambda + \lambda_F)\left\{\lambda_F(4\rho_F - 1) + 8\delta + (\lambda - \lambda_F)(2\rho_F - 1/2)\right\} \equiv \Lambda.$$

Next, we note that, from (1)–(2), the vacancy rate ρ_F depends only upon the sum $\lambda + \lambda_F$. We can accordingly take a derivative of Λ with respect to λ_F, letting $d\lambda/d\lambda_F = -1$ to as to preserve the sum $\lambda + \lambda_F$, to obtain

$$\begin{aligned} \frac{d\Lambda}{d\lambda_F} &= -\rho_F(2\rho_F - 1)4\delta - \rho_F(2\rho_F - 1)(\lambda + \lambda_F)[4\rho_F - 1 - 2(2\rho_F - \frac{1}{2})] \\ &= -\rho_F(2\rho_F - 1)4\delta < 0. \end{aligned}$$

We can then again confine attention to the worst case, namely the value of λ_F that minimizes Λ, or $\lambda_F = \lambda$. Our task is then to show

$$2\delta(4\rho_F - 1)\left\{4\delta + (4\rho_F - 1)(2\lambda)\right\} - \lambda\rho_F(2\rho_F - 1)4\delta - \rho_F(2\rho_F - 1)(2\lambda)\left\{\lambda(4\rho_F - 1) + 8\delta\right\} > 0.$$

Dividing by λ^2, this is

$$2\frac{\delta}{\lambda}(4\rho_F - 1)\left\{4\frac{\delta}{\lambda} + 2(4\rho_F - 1)\right\} - 4\rho_F(2\rho_F - 1)\frac{\delta}{\lambda} - 2\rho_F(2\rho_F - 1)\left\{(4\rho_F - 1) + 8\frac{\delta}{\lambda}\right\} > 0.$$

From (1) and (3), we have

$$\frac{\delta}{\lambda} = \frac{\rho_F(2\rho_F - 1)}{2(1 - \rho_F)}.$$

Substituting, we have

$$2\tfrac{\rho_F(2\rho_F-1)}{2(1-\rho_F)}(4\rho_F - 1)\left\{4\tfrac{\rho_F(2\rho_F-1)}{2(1-\rho_F)} + 2(4\rho_F - 1)\right\} - 4\rho_F(2\rho_F - 1)\tfrac{\rho_F(2\rho_F-1)}{2(1-\rho_F)}$$
$$-2\rho_F(2\rho_F - 1)\left\{(4\rho_F - 1) + 8\tfrac{\rho_F(2\rho_F-1)}{2(1-\rho_F)}\right\} > 0.$$

Extracting and deleting the positive factor $\rho_F(2\rho_F - 1)/(1 - \rho_F)$, we have

$$(4\rho_F - 1)\left\{\tfrac{2\rho_F(2\rho_F-1)+2(4\rho_F-1)(1-\rho_F)}{(1-\rho_F)}\right\} - 2\rho_F(2\rho_F - 1)$$
$$-2(1 - \rho_F)\left\{(4\rho_F - 1) + \tfrac{4\rho_F(2\rho_F-1)}{(1-\rho_F)}\right\} > 0.$$

Extracting and deleting the positive factor $(1 - \rho_F)$, we have

$$(4\rho_F - 1)\left\{4\rho_F^2 - 2\rho_F + 2(4\rho_F - 1 - 4\rho_F^2 + \rho_F\right\} - 2\rho_F(1 - \rho_F)(2\rho_F - 1)$$
$$-2(1 - \rho_F)\left\{(4\rho_F - 1)(1 - \rho_F) + 4\rho_F(2\rho_F - 1)\right\} > 0.$$

A series of simplifications now gives:

$$(4\rho_F - 1)\left\{4\rho_F^2 - 2\rho_F + 8\rho_F - 2 - 8\rho_F^2 + 2\rho_F\right\} - 2\rho_F(2\rho_F - 1 - 2\rho_F^2 + \rho_F)$$
$$-2(1 - \rho_F)\left\{(4\rho_F - 1)(1 - \rho_F) + 4\rho_F(2\rho_F - 1)\right\} > 0.$$

$$(4\rho_F - 1)\left\{-4\rho_F^2 + 8\rho_F - 2\right\} - 2\rho_F(-2\rho_F^2 + 3\rho_F - 1)$$
$$-2(1 - \rho_F)\left\{4\rho_F - 1 - 4\rho_F^2 + \rho_F + 8\rho_F^2 - 4\rho_F)\right\} > 0.$$

$$-16\rho_F^3 + 32\rho_F^2 - 8\rho_F + 4\rho_F^2 - 8\rho_F + 2 + 4\rho_F^3 - 6\rho_F^2 + 2\rho_F - 2(1 - \rho_F)\left\{4\rho_F^2 + \rho_F - 1)\right\} > 0.$$

$$12\rho_F^3 + 30\rho_F^2 - 14\rho_F + 2 - 2[4\rho_F^2 + \rho_F - 1 - 4\rho_F^3 - \rho_F^2 + \rho_F] > 0.$$
$$12\rho_F^3 + 30\rho_F^2 - 14\rho_F + 2 - 8\rho_F^2 - 2\rho_F + 2 + 8\rho_F^3 + 2\rho_F^2 - 2\rho_F] > 0.$$
$$-4\rho_F^3 + 24\rho_F^2 - 18\rho_F + 4 > 0.$$
$$-2\rho_F^3 + 12\rho_F^2 - 9\rho_F + 2 \equiv \Phi > 0.$$

We now examine the cubic equation Φ. It is straightforward to calculate:

$$\begin{aligned} \lim_{\rho_F \to -\infty} \Phi(\rho_F) &> 0, \quad \Phi(0) = 2, \quad \Phi(\frac{1}{2}) = \frac{1}{4}, \\ \Phi(1) &= 3, \quad \text{and} \quad \lim_{\rho_F \to \infty} \Phi(\rho_F) < 0. \end{aligned}$$

In addition,

$$\frac{d\Phi(\rho_F)}{d\rho_F} = -6\rho_F^2 + 24\rho_F - 9.$$

This derivative is positive for all $\rho_F \in [\frac{1}{2}, 1]$, and hence $\Phi(\rho_F) > 0$ for all $\rho_F \in [\frac{1}{2}, 1]$. ■