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Repeated Games: Perfect Monitoring

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Introduction

- The theory of repeated games provides a central underpinning for our understanding of social, political, and economic institutions, both formal and informal.
- A key ingredient in understanding institutions and other long run relationships is the role of
  - shared expectations about behavioral norms (cultural beliefs), and
  - sanctions in ensuring that people follow the "rules."
- Repeated games allow for a clean description of both the myopic incentives that agents have to not follow the rules and, via appropriate specifications of future behavior (and so rewards and punishments), the incentives that deter such opportunistic behavior.

Examples of Long-Run Relationships and Opportunistic Behavior

- Buyer-seller.
  The seller selling an inferior good.
- Employer and employees.
  Employees shirking on the job, employer reneging on implicit terms of employment.
- A government and its citizens.
  Government expropriates (taxes) all profits from investments.
- World Trade Organization
  Imposing tariffs to protect a domestic industry.
- Cartels
  A firm exceeding its share of the monopolistic output.

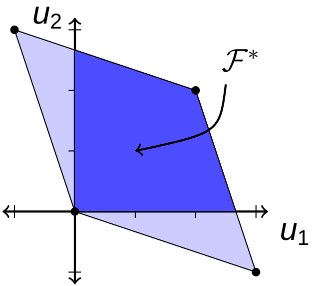
Two particularly interesting examples

1. Dispute Resolution.
   Ellickson (1991) presents evidence that neighbors in Shasta County, CA, resolve disputes arising from the damage created by escaped cattle in ways that both ignore legal liability and are supported by intertemporal incentives.
2. Traders selling goods on consignment.
   Grief (1994) documents how the Maghribi and Genoese merchants deterred their agents from misreporting that goods were damaged in transport, and so were worth less. These two communities of merchants did this differently, and in ways consistent with the different cultural characteristics of the communities and repeated game analysis.

The Leading Example

The prisoners' dilemma as a partnership

	E	S
E	$2, 2$	$-1, 3$
S	$3, -1$	$0, 0$



- Each player can guarantee herself a payoff of 0.
  A payoff profile is individually rational if each player receives at least their minmax payoff:
  $$\underline{v}_i^p := \min_{a_j} \max_{a_i} u_i(a_i, a_j).$$
- $\mathcal{F}^*$ is the set of feasible and individually rational payoffs.

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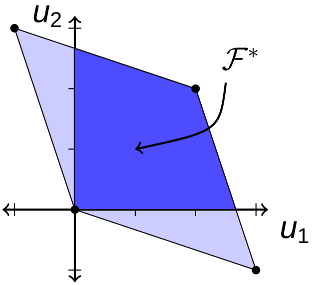


- Each player can guarantee herself a payoff of 0.
  A payoff profile is individually rational if each player receives at least their minmax payoff.
- $\mathcal{F}^*$ is the set of feasible and individually rational payoffs.
- In the static (one shot) play, each player will play S, resulting in SS.

Intertemporal Incentives

- Suppose the game is repeated (once), and payoffs are added.
- We "know" *SS* will be played in last period, so no intertemporal incentives.
- Infinite horizon—relationship never ends.
  The infinite stream of payoffs $(u_i^0, u_i^1, u_i^2, \ldots)$ is evaluated as the (average) discounted sum
  $$\sum_{t \geq 0} (1 - \delta)\delta^t u_i^t.$$
- Individual i is indifferent between $0, 1, 0, \ldots$ and $\delta, 0, 0 \ldots$.
- The normalization $(1 - \delta)$ implies that repeated game payoffs are comparable to stage game payoffs.
  The infinite constant stream of 1 utils has a value of 1.

- A strategy σ_i for individual i describes how that individual behaves (at each point of time and after any possible history).
- A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ describes how everyone behaves (at each point of...).

Definition

The profile σ^* is a Nash equilibrium if for all i, when everyone else is behaving according to σ^*_{-i}, then i is also willing to behave as described by σ^*_i.
The profile σ^* is a subgame perfect equilibrium if for all histories of play, the behavior described (induced) by the profile is a Nash equilibrium.

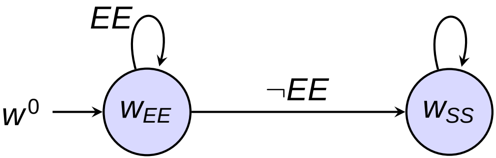
- Useful to think of social norms as equilibria: shared expectations over behavior that provide appropriate sanctions to deter deviations.

Characterizing Equilibria

- Difficult problem: many possible deviations after many different histories.
- But repeated games are recursive, and the one shot deviation principle (from dynamic programming) holds.
- Simple penal codes (Abreu, 1988): use i's worst eq to punish any (and all) deviation by i.

Prisoners' Dilemma

Grim Trigger



- This is an equilibrium if

$$(1-\delta)\times 2+\delta\times 2 = 2 \geq (1-\delta)\times 3+\delta\times 0$$
$$\Rightarrow \delta \geq \frac{1}{3}.$$

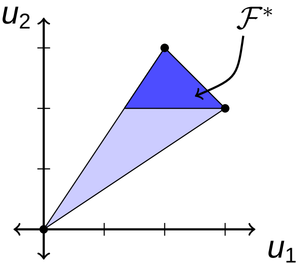
- Grim trigger is subgame perfect: always S is a Nash eq (because SS is an eq of the stage game and in w_{SS} behavior is history independent).

The need for credibility of punishments

The Purchase Game

A buyer and seller:

	Buy	Don't buy
High effort	2,3	0, 0
Low effort	3,2	0, 0



- The seller can guarantee payoff 0, while the buyer can guarantee 2.

The need for credibility of punishments

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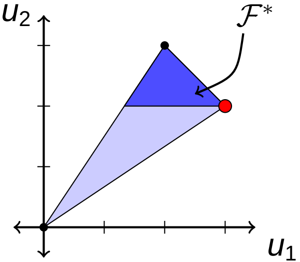

- The seller can guarantee payoff 0, while the buyer can guarantee 2.
- Seller always chooses low effort and buyer always buys is an eq.

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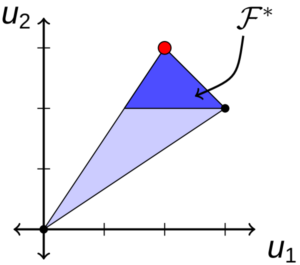
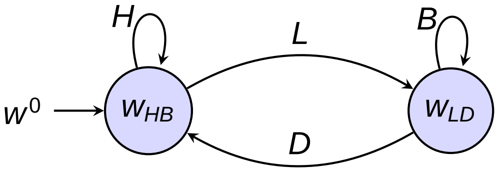


- The seller can guarantee payoff 0, while the buyer can guarantee 2.
- Seller always chooses low effort and buyer always buys is an eq.
- Is there a social norm in which the buyer threatens not to buy unless the seller chooses high effort? Need to provide incentives for buyer.

Why the buyer is willing to punish

Suppose, after the seller "cheats" the buyer by choosing low effort, the buyer expects the seller to continue to choose low effort until the buyer punishes him by not buying.

	B	D
H	$2, 3$	$0, 0$
L	$3, 2$	$0, 0$



- The seller chooses high effort as long as $\delta \geq \frac{1}{2}$.
- The buyer is willing to punish as long as $\delta \geq \frac{2}{3}$.

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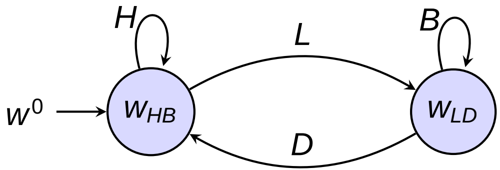


- The seller chooses high effort as long as $\delta \geq \frac{1}{2}$.
- The buyer is willing to punish as long as $\delta \geq \frac{2}{3}$.
- This is a carrot and stick punishment (Abreu, 1986).

The Game with Perfect Monitoring

- Action space for i is A_i, with typical action $a_i \in A_i$.
- An action profile is $a = (a_1, \ldots, a_n)$, with associated flow payoffs $u_i(a)$.
- Infinite stream of payoffs $(u_i^0, u_i^1, u_i^2, \ldots)$ is evaluated as the (average) discounted sum

  $$\sum_{t \geq 0} (1 - \delta) \delta^t u_i^t,$$

  where $\delta \in [0, 1)$ is the discount factor.
- Perfect monitoring: At the end of each period, all players observe the action profile a chosen.
- History to date t: $h^t \equiv (a^0, \ldots, a^{t-1}) \in A^t \equiv H^t$; $H^0 \equiv \{\varnothing\}$.
- Set of all possible histories: $H \equiv \cup_{t=0}^{\infty} H^t$.
- Strategy for player i is denoted $s_i : H \to A_i$.

Automaton Representation of Behavior

An automaton is the tuple $(\mathcal{W}, w^0, f, \tau)$, where

- $\mathcal{W}$ is set of states,
- w^0 is initial state,
- $f : \mathcal{W} \to A$ is output function (decision rule), and
- $\tau : \mathcal{W} \times A \to \mathcal{W}$ is transition function.

Any automaton $(\mathcal{W}, w^0, f, \tau)$ induces a strategy profile. Define

$$\tau(w, h^t) := \tau(\tau(w, h^{t-1}), a^{t-1}).$$

The induced strategy s is given by $s(\varnothing) = f(w^0)$ and

$$s(h^t) = f(\tau(w^0, h^t)), \quad \forall h^t \in H \backslash \{\varnothing\}.$$

Every profile can be represented by an automaton (set $\mathcal{W} = H$).

Nash Equilibrium

Definition

An automaton is a Nash equilibrium if the strategy profile s represented by the automaton is a Nash equilibrium.

Subgames and Continuation Play

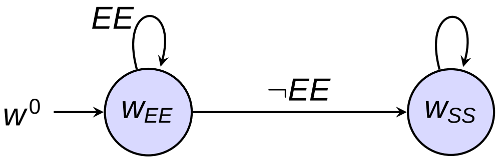
- Each history h^t reaches ("indexes") a distinct subgame.
- Suppose s is represented by $(\mathcal{W}, w^0, f, \tau)$. Recall that

$$\tau(w^0, h^t) := \tau(\tau(w^0, h^{t-1}), a^{t-1}).$$

- The continuation strategy profile after a history h^t, $s|_{h^t}$ is represented by the automaton $(\mathcal{W}, w^t, f, \tau)$, where

$$w^t := \tau(w^0, h^t).$$

- Grim Trigger after any $h^t = (EE)^t$:



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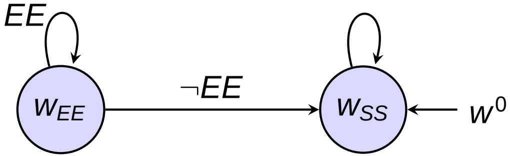
$$w^t := \tau(w^0, h^t).$$

- Grim Trigger after h^t with an S (equivalent to always SS):



Subgame Perfection

Definition

The state $w \in \mathcal{W}$ of an automaton $(\mathcal{W}, w^0, f, \tau)$ is reachable from w^0 if $w = \tau(w^0, h^t)$ for some history $h^t \in H$. Denote the set of states reachable from w^0 by $\mathcal{W}(w^0)$.

Definition

The automaton $(\mathcal{W}, w^0, f, \tau)$ is a subgame perfect equilibrium if for all states $w \in \mathcal{W}(w^0)$, the automaton $(\mathcal{W}, w, f, \tau)$ is a Nash equilibrium.

The automaton $(\mathcal{W}, w, f, \tau)$ induces the sequences

$$\hat{w}^0 := w, \qquad a^0 := f(\hat{w}^0)$$
$$\hat{w}^1 := \tau(\hat{w}^0, a^0), \qquad a^1 := f(\hat{w}^1),$$
$$\hat{w}^2 := \tau(\hat{w}^1, a^1), \qquad a^2 := f(\hat{w}^2),$$
$$\vdots \qquad \vdots$$

Given an automaton $(\mathcal{W}, w^0, f, \tau)$, let $V_i(w)$ be i's value from being in the state $w \in \mathcal{W}$, i.e.,

$$\begin{aligned} V_i(w) &= (1-\delta)u_i(f(\hat{w}^0)) + \delta V_i(\tau(\hat{w}^0, f(\hat{w}^0))) \\ &= (1-\delta)u_i(a^0) + \delta\{(1-\delta)u_i(a^1) + \delta V_i(\hat{w}^2)\} \\ &\vdots \\ &= (1-\delta)\sum_{t=0}^{\infty} \delta^t u_i(a^t). \end{aligned}$$

Principle of No Profitable One-Shot Deviations

Definition

Player i has a profitable one-shot deviation from $(\mathcal{W}, w^0, f, \tau)$, if there is a state $w \in \mathcal{W}(w^0)$ and some action $a_i \in A_i$ such that

$$V_i(w) < (1-\delta)u_i(a_i, f_{-i}(w)) + \delta V_i(\tau(w, (a_i, f_{-i}(w)))).$$

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Theorem

The automaton $(\mathcal{W}, w^0, f, \tau)$ is subgame perfect if, and only if, there are no profitable one-shot deviations.

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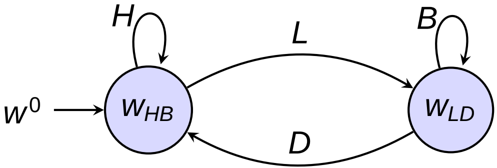
Theorem

The automaton $(\mathcal{W}, w^0, f, \tau)$ is subgame perfect if, and only if, there are no profitable one-shot deviations, that is, iff for all $w \in \mathcal{W}(w^0)$, $f(w)$ is a Nash eq of the normal form game with payoff function $g^w : A \to \mathbb{R}^n$, where

$$g_i^w(a) = (1-\delta)u_i(a) + \delta V_i(\tau(w, a)).$$

Return to the purchase game I

	B	D
H	$2, 3$	$0, 0$
L	$3, 2$	$0, 0$



The value to each player of being in states w_{BH} and w_{LD} are

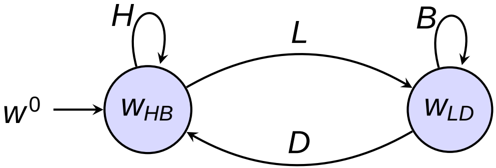
$$V_1(w_{HB}) = 2, \quad V_2(w_{HB}) = 3,$$

and

$$V_1(w_{LD}) = 2\delta, \quad V_2(w_{LD}) = 3\delta.$$

Return to the purchase game II

	B	D
H	$2, 3$	$0, 0$
L	$3, 2$	$0, 0$



The auxiliary game for w_{HB}:

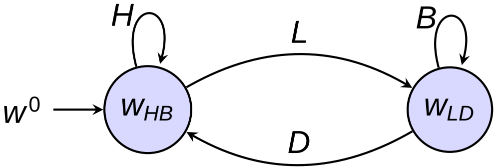
	B	D
H	$2, 3$	$2\delta, 3\delta$
L	$3(1-\delta)+2\delta^2, 2(1-\delta)+3\delta^2$	$2\delta^2, 3\delta^2$

H is a BR to B if $2 \geq 3(1-\delta)+2\delta^2$

$$\iff 2(1-\delta^2) \geq 3(1-\delta) \iff 2(1+\delta) \geq 1 \iff \delta \geq \tfrac{1}{2}.$$

Return to the purchase game III

	B	D
H	$2, 3$	$0, 0$
L	$3, 2$	$0, 0$



The auxiliary game for w_{LD}:

	B	D
H	$2(1-\delta)+2\delta^2, 3(1-\delta)+3\delta^2$	$2\delta, 3\delta$
L	$3(1-\delta)+2\delta^2, 2(1-\delta)+3\delta^2$	$2\delta, 3\delta$

D is a BR to L if $3\delta \geq 2(1-\delta)+3\delta^2$

$$\iff 3\delta(1-\delta) \geq 2(1-\delta) \iff 3\delta \geq 2 \iff \delta \geq \tfrac{2}{3}.$$

SPE if No Profitable One-Shot Deviations

Proof I

- Let $\widetilde{V}_i(w)$ be player i's payoff from the best response to $(\mathcal{W}, w, f_{-i}, \tau)$ (i.e., the strategy profile for the other players specified by the automaton with initial state w). Then

$$\widetilde{V}_i(w) = \max_{a_i \in A_i} \left\{ (1-\delta) u_i(a_i, f_{-i}(w)) + \delta \widetilde{V}_i(\tau(w, (a_i, f_{-i}(w)))) \right\}.$$

- Note that $\widetilde{V}_i(w) \geq V_i(w)$ for all w. Denote by $\bar{w}_i$, the state that maximizes $\widetilde{V}_i(w) - V_i(w)$ (if there is more than one, choose one arbitrarily).
- If $(\mathcal{W}, w^0, f, \tau))$ is not SGP, then for some player i,

$$\widetilde{V}_i(\bar{w}_i) - V_i(\bar{w}_i) > 0.$$

SPE iff No Profitable One-Shot Deviations

Proof II

Then, for all w,

$$\widetilde{V}_i(\bar{w}_i) - V_i(\bar{w}_i) > \delta[\widetilde{V}_i(w) - V_i(w)],$$

and so (where $a_i^{\bar{w}_i}$ yields $\widetilde{V}_i(\bar{w}_i)$)

$$\widetilde{V}_i(\bar{w}_i) - V_i(\bar{w}_i) > \delta[\widetilde{V}_i(\tau(\bar{w}_i, (a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i)))) - V_i(\tau(\bar{w}_i, (a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i))))]$$

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$$\begin{aligned}\widetilde{V}_i(\bar{w}_i) - V_i(\bar{w}_i) > \delta[&\widetilde{V}_i(\tau(\bar{w}_i,(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i)))) - V_i(\tau(\bar{w}_i,(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i))))] \\ &+ [(1-\delta)u_i(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i)) - (1-\delta)u_i(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i))]\end{aligned}$$

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Then, for all w,

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$$\begin{aligned}
\widetilde{V}_i(\bar{w}_i) - V_i(\bar{w}_i) > {} & \delta[\widetilde{V}_i(\tau(\bar{w}_i,(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i)))) - V_i(\tau(\bar{w}_i,(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i))))] \\
& +[(1-\delta)u_i(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i)) - (1-\delta)u_i(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i))] \\
= {} & \widetilde{V}_i(\bar{w}_i) \\
- {} & \left\{(1-\delta)u_i(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i)) + \delta V_i(\tau(\bar{w}_i,(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i))))\right\}.
\end{aligned}$$

SPE iff No Profitable One-Shot Deviations

Proof II

Then, for all w,

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$$\begin{aligned}\widetilde{V}_i(\bar{w}_i) - V_i(\bar{w}_i) &> \delta[\widetilde{V}_i(\tau(\bar{w}_i,(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i)))) - V_i(\tau(\bar{w}_i,(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i))))] \\ &\quad + [(1-\delta)u_i(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i)) - (1-\delta)u_i(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i))] \\ &= \widetilde{V}_i(\bar{w}_i) \\ &\quad - \left\{(1-\delta)u_i(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i)) + \delta V_i(\tau(\bar{w}_i,(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i))))\right\}.\end{aligned}$$

Thus,

$$(1-\delta)u_i(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i)) + \delta V_i(\tau(\bar{w}_i,(a_i^{\bar{w}_i}, f_{-i}(\bar{w}_i)))) > V_i(w_i),$$

that is, player i has a profitable one-shot deviation at $\bar{w}_i$.

Enforceability and Decomposability

Definition

An action profile $a' \in A$ is enforced by the continuation promises $\gamma : A \to \mathbb{R}^n$ if a' is a Nash eq of the normal form game with payoff function $g^\gamma : A \to \mathbb{R}^n$, where

$$g_i^\gamma(a) = (1 - \delta)u_i(a) + \delta\gamma_i(a).$$

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Definition

A payoff v is decomposable on a set of payoffs $\mathcal{V}$ if there exists an action profile a' enforced by some continuation promises $\gamma : A \to \mathcal{V}$ satisfying, for all i,

$$v_i = (1-\delta)u_i(a') + \delta\gamma_i(a').$$

The payoff v is decomposed by a' on $\mathcal{V}$.

The Purchase Game 1

	Buy	Don't buy
High effort	2,3	$0,0$
Low effort	3,2	$0,0$

- Only *LB* can be enforced by constant continuation promises, and so
- only $(3,2)$ can be decomposed on a singleton set, and that set is $\{(3,2)\}$.

The Purchase Game 2

	Buy	Don't buy
High effort	2,3	$0, 0$
Low effort	3,2	$0, 0$

Suppose $\mathcal{V} = \{(2\delta, 3\delta), (2, 3)\}$, and $\delta > \frac{2}{3}$.

- $(2, 3)$ is decomposed on $\mathcal{V}$ by *HB* and promises

$$\gamma(a) = \begin{cases} (2, 3), & \text{if } a_1 = H, \\ (2\delta, 3\delta), & \text{if } a_1 = L. \end{cases}$$

- $(2\delta, 3\delta)$ is decomposed on $\mathcal{V}$ by *LD* and promises

$$\gamma(a) = \begin{cases} (2, 3), & \text{if } a_2 = D, \\ (2\delta, 3\delta), & \text{if } a_2 = B. \end{cases}$$

- No one-shot deviation principle $\implies$ every payoff in $\mathcal{V}$ is SPE payoff.

The Purchase Game 3

	Buy	Don't buy
High effort	2,3	0, 0
Low effort	3,2	0, 0

Suppose $\mathcal{V} = \{(2\delta, 3\delta), (2, 3)\}$, and $\delta > \frac{2}{3}$.

- $(3 - 3\delta + 2\delta^2, 2 - 2\delta + 3\delta^2) =: v^\dagger$ is decomposed on $\mathcal{V}$ by *LB* and the constant promises

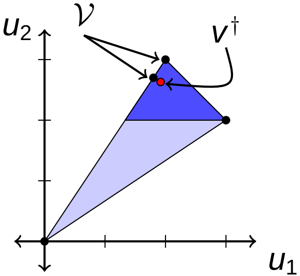
$$\gamma(a) = (2\delta, 3\delta).$$

- So, payoffs outside $\mathcal{V}$ can also be decomposed on $\mathcal{V}$.
- No one-shot deviation principle $\implies$

$$v^\dagger \text{ is a subgame perfect eq payoff.}$$

The Purchase Game 4

	Buy	Don't buy
High effort	2,3	0, 0
Low effort	3,2	0, 0



Subgame Perfection redux

Let $\mathcal{E}^p(\delta) \subset \mathcal{F}^{p*}$ be the set of pure strategy subgame perfect equilibrium payoffs.

Theorem

A payoff $v \in \mathbb{R}^n$ is decomposable on $\mathcal{E}^p(\delta)$ if, and only if, $v \in \mathcal{E}^p(\delta)$.

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Theorem

Suppose every payoff v in some bounded set $\mathcal{V} \subset \mathbb{R}^n$ is decomposable with respect to $\mathcal{V}$. Then, $\mathcal{V} \subset \mathcal{E}^p(\delta)$.

Any set of payoffs with the property described above is said to be self-generating.

A Folk Theorem

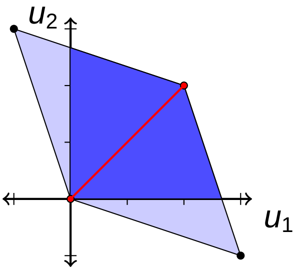
- Intertemporal incentives allow for efficient outcomes, but also for inefficient outcomes, as well as crazy outcomes.
- This is illustrated by the “Folk” Theorem, so called because results of this type have been part of game theory folklore since at least the late sixties.

The Discounted Folk Theorem (Fudenberg&Maskin 1986)

Suppose v is a feasible and strictly individually rational vector of payoffs. If the individuals are sufficiently patient (there exists $\underline{\delta} \in (0,1)$ such that for all $\delta \in (\underline{\delta}, 1)$), then there is a subgame perfect equilibrium with payoff v.

Symmetric Folk Theorem for PD I

	E	S
E	$2, 2$	$-1, 3$
S	$3, -1$	$0, 0$



- Strongly symmetric strategies: $\forall w \in \mathcal{W}$, $f_1(w) = f_2(w)$.
- When is $\{(v, v) : v \in [0, 2]\}$ a set of equilibrium payoffs?

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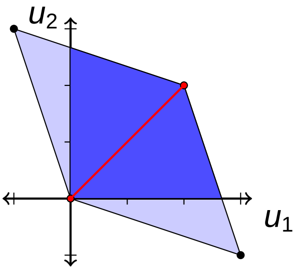

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- $\mathcal{W}^{EE} :=$ set of player 1 payoffs decomposed on $[0, 2]$ using EE.

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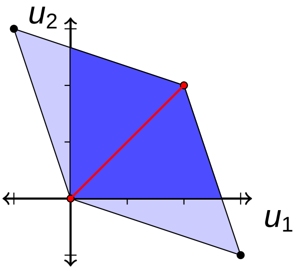

- Strongly symmetric strategies: $\forall w \in \mathcal{W}$, $f_1(w) = f_2(w)$.
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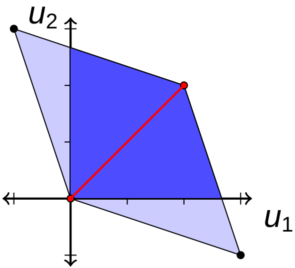

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- When is $\{(v, v) : v \in [0, 2]\}$ a set of equilibrium payoffs?
- $\mathcal{W}^{EE}$:= set of player 1 payoffs decomposed on $[0, 2]$ using EE.
- $\mathcal{W}^{SS}$:= set of player 1 payoffs decomposed on $[0, 2]$ using SS.
- Then $\mathcal{W}^{EE} \cup \mathcal{W}^{SS} \subset [0, 2]$.
- When do we have equality? And why is this a folk theorem?

Symmetric Folk Theorem for PD II

- $\mathcal{W}^{EE}$ is the set of player 1 payoffs that could be enforceably achieved by EE followed by appropriate symmetric continuations in $[0, 2]$:

$$\begin{aligned} v \in \mathcal{W}^{EE} \iff v &= 2(1-\delta) + \delta\gamma(EE) \\ &\geq 3(1-\delta) + \delta\gamma(SE), \end{aligned}$$

  for some $\gamma(EE), \gamma(SE) \in [0, 2]$.
- This implies $\gamma(EE) \in [(1-\delta)/\delta, 2]$.
- So $\mathcal{W}^{EE} = [3(1-\delta), 2]$.

Illustration of $\mathcal{W}^{EE}$

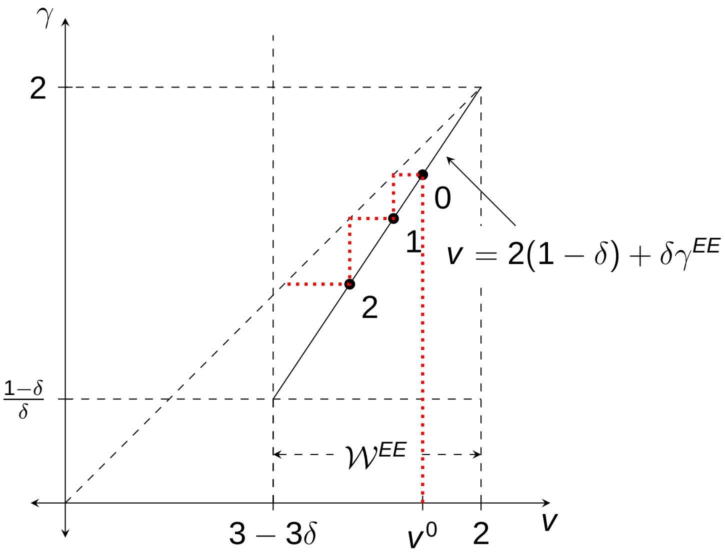
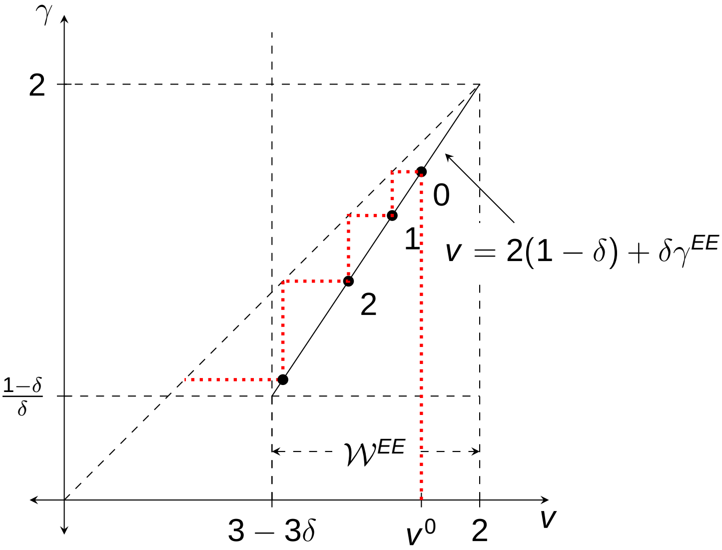


Illustration of $\mathcal{W}^{EE}$



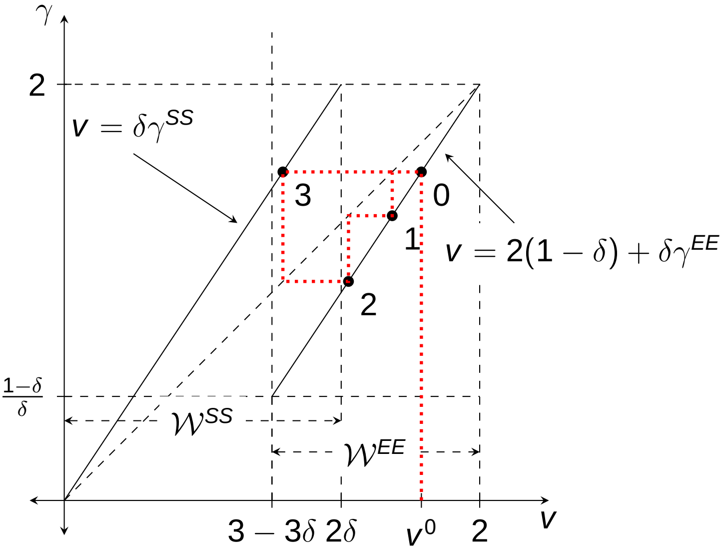
Symmetric Folk Theorem for PD III

- $\mathcal{W}^{SS}$ is the set of player 1 payoffs that could be enforceably achieved by SS followed by appropriate symmetric continuations in $[0, 2]$:

$$v \in \mathcal{W}^{SS} \iff v = 0 \times (1-\delta) + \delta\gamma(SS) \\ \geq (-1)(1-\delta) + \delta\gamma(ES),$$

  for some $\gamma(SS), \gamma(ES) \in [0, 2]$.
- Inequality satisfied by $\gamma(SS) = \gamma(ES)$.
- So $\mathcal{W}^{SS} = [0, 2\delta]$.

Illustration of $\mathcal{W}^{EE} \cup \mathcal{W}^{SS}$



Symmetric Folk Theorem for PD IV

- $\mathcal{W}^{EE} = [3(1-\delta), 2]$
- $\mathcal{W}^{SS} = [0, 2\delta]$
- So
$$[0,2] \supset \mathcal{W}^{SS} \cup \mathcal{W}^{EE} = [0, 2\delta] \cup [3(1-\delta), 2].$$
- Folk theorem holds if
$$2\delta \geq 3(1-\delta) \iff \delta \geq \frac{3}{5}.$$
- Recall grim trigger is an equilibrium if $\delta \geq \frac{1}{3}$.

Interpretation

- While efficient payoffs are consistent with equilibrium, so are many other payoffs, and associated behaviors. (Consistent with experimental evidence.)
- Moreover, multiple equilibria are consistent with the same payoff.
- The theorem does not justify restricting attention to efficient payoffs.

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- While efficient payoffs are consistent with equilibrium, so are many other payoffs, and associated behaviors. (Consistent with experimental evidence.)
- Moreover, multiple equilibria are consistent with the same payoff.
- The theorem does not justify restricting attention to efficient payoffs.

Nonetheless:

- In many situations, understanding the potential scope of equilibrium incentives helps us to understand possible plausible behaviors.
- Understanding what it takes to achieve efficiency gives us important insights into the nature of equilibrium incentives.
- It is sometimes argued that the punishments imposed are too severe. But this does simplify the analysis.

Renegotiation-Proof Equilibria

- Are Pareto inefficient equilibria plausible threats?

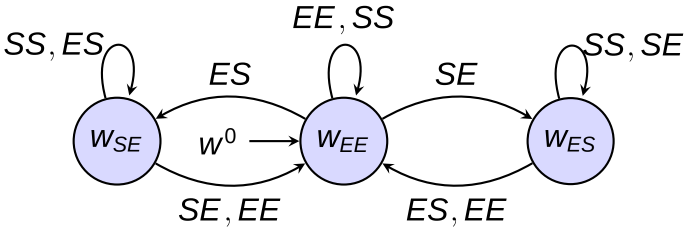
Definition (Farrell and Maskin, 1989)

The subgame perfect automaton $(\mathcal{W}, w^0, f, \tau)$ is weakly renegotiation proof if for all $w, w' \in \mathcal{W}(w_0)$ and i,

$$V_i(w) > V_i(w') \implies V_j(w) \leq V_j(w') \text{ for some } j.$$

- A notion of internal dominance. Does not preclude there being a Pareto dominating equilibrium played by an unrelated automaton.
- Grim trigger is not weakly renegotiation proof.

Weakly Renegotiation Proof in PD



- $V_1(w_{SE}) = 3(1-\delta) + 2\delta = 3 - \delta$; $V_2(w_{SE}) = -(1-\delta) + 2\delta = -1 + 3\delta$; $V_1(w_{EE}) = V_2(w_{EE}) = 2$.
- $V_1(w_{SE}) > V_1(w_{EE}) > V_2(w_{SE})$
- state w_{SE} punishes player 2:

$$3(1-\delta) + \delta(-1+3\delta) \le 2 \iff \delta \ge \tfrac{1}{3}, \text{ and}$$

$$0 \times (1-\delta) + \delta(-1+3\delta) \le -1 + 3\delta \iff \delta \ge \tfrac{1}{3}.$$

What we learn from perfect monitoring

- Multiplicity of equilibria is to be expected.
  - This is necessary for repeated games to serve as a building block for any theory of institutions.
  - Selection of equilibrium can (should) be part of modelling.
- In general, efficiency requires being able to reward and punish individuals independently (this is the role of the full dimensionality assumption).
- Histories coordinate behavior to provide intertemporal incentives by punishing deviations. This requires monitoring (communication networks) and a future.
  - Intertemporal incentives require that individuals have something at stake: “Freedom’s just another word for nothin’ left to lose.”—Kris Kristofferson