2018 Delhi Winter School

Repeated Games: Imperfect Public Monitoring

George J. Mailath

University of Pennsylvania
and
Australian National University

December 2018

What we learned from perfect monitoring

- Multiplicity of equilibria is to be expected.
- In general, efficiency requires being able to reward and punish individuals independently.
- Histories coordinate behavior to provide intertemporal incentives by punishing deviations. This requires monitoring (communication networks) and a future.

What we learned from perfect monitoring

- Multiplicity of equilibria is to be expected.
- In general, efficiency requires being able to reward and punish individuals independently.
- Histories coordinate behavior to provide intertemporal incentives by punishing deviations. This requires monitoring (communication networks) and a future.

But suppose deviations are not observed?
Suppose instead actions are only imperfectly observed.

Collusion in Oligopoly

Perfect Monitoring

- In each period, firms $i = 1, \ldots, n$ simultaneously choose quantities q_i.
- Firm i profits

$$\pi_i(q_1, \ldots, q_n) = pq_i - c(q_i),$$

  where p is market clearing price, and $c(q_i)$ is the cost of q_i.
- Suppose $p = P(\sum_i q_i)$ and P is a strictly decreasing function of $Q := \sum_i q_i$.
- If firms are patient, there is a subgame perfect equilibrium in which the each firm sells Q^m/n, where Q^m is monopoly output, supported by the threat that any deviation results in perpetual Cournot (static Nash) competition.

Collusion in Oligopoly

Imperfect Monitoring

- In each period, firms $i = 1, \ldots, n$ simultaneously choose quantities q_i.
- Firm i profits
  $$\pi_i(q_1, \ldots, q_n) = pq_i - c(q_i),$$
  where p is market clearing price, and $c(q_i)$ is the cost of q_i.
- Suppose $p = P(\sum_i q_i)$ and P is a strictly decreasing function of $Q := \sum_i q_i$.
- Suppose now $q_1, \ldots, q_n$ are not public, but the market clearing price p still is (so each firm knows its profit).
  Nothing changes! A deviation is still necessarily detected, since the market clearing price changes.

Collusion in Oligopoly

Noisy Imperfect Monitoring–Green and Porter (1984)

- In each period, firms $i = 1, \ldots, n$ simultaneously choose quantities q_i.
- Firm i profits

$$\pi_i(q_1, \ldots, q_n) = pq_i - c(q_i),$$

where p is market clearing price, and $c(q_i)$ is the cost of q_i.

- But suppose demand is random, so that the market clearing price p is a function of$Q$ and a demand shock η. Moreover, suppose p has full support for all Q.

$\Longrightarrow$ no deviation is detected.

Repeated Games with Noisy Imperfect Monitoring

- In a setting with noisy imperfect monitoring where it is impossible to detect deviations, are there still intertemporal incentives?
- If so, what is their nature?
- And, how effective are these intemporal incentives?

Repeated Games with Noisy Imperfect Monitoring

- In a setting with noisy imperfect monitoring where it is impossible to detect deviations, are there still intertemporal incentives? Yes
- If so, what is their nature?
- And, how effective are these intemporal incentives?

Repeated Games with Noisy Imperfect Monitoring

- In a setting with noisy imperfect monitoring where it is impossible to detect deviations, are there still intertemporal incentives?
- If so, what is their nature?
- And, how effective are these intemporal incentives? Surprisingly strong!

Repeated Games with Imperfect Public Monitoring

Structure 1

- Action space for i is A_i, with typical action $a_i \in A_i$.
- Profile a is not observed.
- All players observe a public signal $y \in Y$, $|Y| < \infty$, with

$$\Pr\{y \mid (a_1, \dots, a_n)\} =: \rho(y \mid a).$$

- Since y is a possibly noisy signal of the action profile a in that period, the actions are imperfectly monitored.
- Since the signal is public (observed by all players), the game is said to have public monitoring.
- Assume Y is finite.
- $u_i^* : A_i \times Y \to \mathbb{R}$, i's ex post or realized payoff.
- Stage game (ex ante) payoffs:

$$u_i(a) := \sum_{y \in Y} u_i^*(a_i, y)\rho(y \mid a).$$

Ex post payoffs

Oligopoly with imperfect monitoring

- Ex post payoffs are given by realized profits,

$$u_i^*(q_i, p) = pq_i - c(q_i),$$

  where p is the public signal.
- Ex ante payoffs are given by expected profits,

$$\begin{aligned} u_i(q_1, \dots, q_n) &= E[pq_i - c(q_i) \mid q_1, \dots q_n] \\ &= E[p \mid q_1, \dots q_n]q_i - c(q_i). \end{aligned}$$

Ex post payoffs II

Prisoners' Dilemma with Noisy Monitoring

- There is a noisy signal of actions (output), $y \in \{\underline{y}, \bar{y}\} =: Y$,

$$\Pr(\bar{y} \mid a) := \rho(\bar{y} \mid a) = \begin{cases} p, & \text{if } a = EE, \\ q, & \text{if } a = SE \text{ or } SE, \text{ and} \\ r, & \text{if } a = SS. \end{cases}$$

- Player i's ex post payoffs

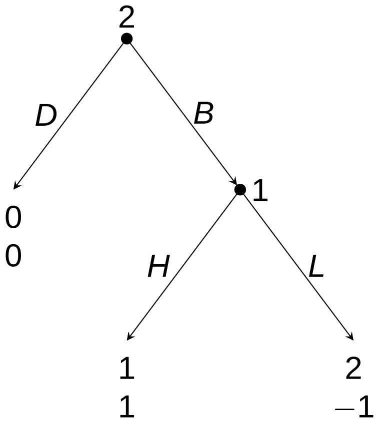
	$\bar{y}$	$\underline{y}$
E	$\frac{(3-p-2q)}{(p-q)}$	$-\frac{(p+2q)}{(p-q)}$
S	$\frac{3(1-r)}{(q-r)}$	$-\frac{3r}{(q-r)}$

ex ante payoffs

	E	S
E	$2, 2$	$-1, 3$
S	$3, -1$	$0, 0$

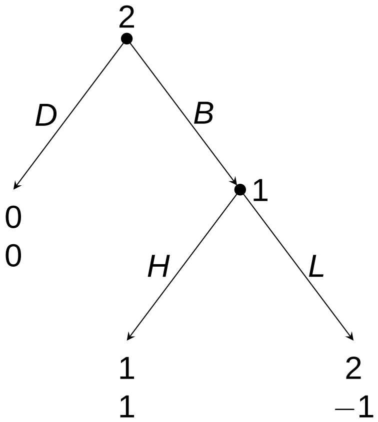
Ex post payoffs III

The purchase game



Ex post payoffs III

The purchase game



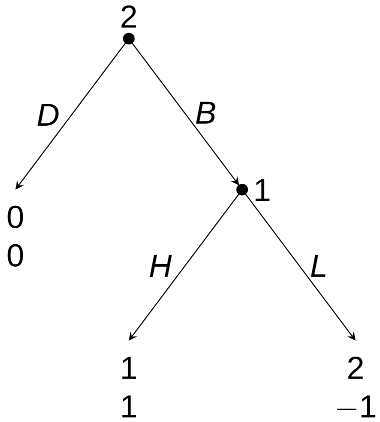
Terminal nodes are the signals

$$\Longrightarrow$$

game has imperfect monitoring: *DH* and *DL* generate the same terminal node.

Ex post payoffs III

The purchase game



Terminal nodes are the signals

$$\Longrightarrow$$

game has imperfect monitoring: *DH* and *DL* generate the same terminal node.

Any nontrivial repeated dynamic game is a repeated game with imperfect monitoring!

Ex post payoffs IV

Oligopoly with incomplete information

- In each period, firms $i = 1, \ldots, n$ simultaneously choose quantities q_i.
- Firm i profits
$$\pi_i(q_1, \ldots, q_n) = pq_i - c_i q_i,$$
where $p = a - \sum_i q_i$ is market clearing price, and $c_i \in [\underline{c}, \bar{c}]$ is a privately known (only to firm i) constant marginal cost.
- Quantities q_i are public.
- Ex post outcome is a realization of a cost for each firm, and an associated quantity for each firm.
- Imperfect monitoring: the ex ante action is $\tilde{q}_i : [\underline{c}, \bar{c}] \to \mathbb{R}_+$.

Repeated Games with Imperfect Public Monitoring

Structure 2

- Public histories:

$$H \equiv \cup_{t=0}^{\infty} Y^t,$$

with $h^t \equiv (y^0, \ldots, y^{t-1})$ a t period history of public signals ($Y^0 \equiv \{\varnothing\}$).

- Public strategies:

$$s_i : H \to A_i.$$

Automaton Representation of Public Strategies

An automaton is the tuple $(\mathcal{W}, w^0, f, \tau)$, where

- $\mathcal{W}$ is set of states,
- w^0 is initial state,
- $f : \mathcal{W} \to A$ is output function (decision rule), and
- $\tau : \mathcal{W} \times Y \to \mathcal{W}$ is transition function.

The automaton is strongly symmetric if $f_i(w) = f_j(w) \quad \forall i, j, w$.

Any automaton $(\mathcal{W}, w^0, f, \tau)$ induces a public strategy profile. Define

$$\tau(w, h^t) := \tau(\tau(w, h^{t-1}), y^{t-1}).$$

The induced strategy s is given by $s(\varnothing) = f(w^0)$ and

$$s(h^t) = f(\tau(w^0, h^t)), \quad \forall h^t \in H \backslash \{\varnothing\}.$$

Automaton Representation of Public Strategies

An automaton is the tuple $(\mathcal{W}, w^0, f, \tau)$, where

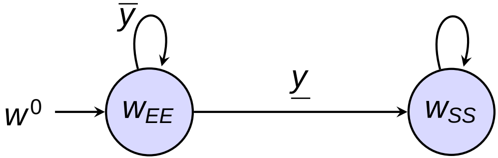
- $\mathcal{W}$ is set of states,
- w^0 is initial state,
- $f : \mathcal{W} \to A$ is output function (decision rule), and
- $\tau : \mathcal{W} \times Y \to \mathcal{W}$ is transition function.

The automaton is strongly symmetric if $f_i(w) = f_j(w) \quad \forall i, j, w$.

Every public profile can be represented by an automaton (set $\mathcal{W} = H$).

Prisoners' Dilemma with Noisy Monitoring

Grim Trigger



This is an eq if

$$\begin{aligned} V &= (1-\delta)2 + \delta[pV + (1-p)\times 0] \\ &\geq (1-\delta)3 + \delta[qV + (1-q)\times 0] \\ &\Rightarrow \frac{2\delta(p-q)}{(1-\delta p)} \geq 1 \qquad \iff \quad \delta \geq \frac{1}{3p-2q}. \end{aligned}$$

Note that

$$V = \frac{2(1-\delta)}{(1-\delta p)},$$

and so $\lim_{\delta\to 1} V = 0$.

Equilibrium Notion

- Game has no proper subgames, so how to usefully capture sequential rationality?

Equilibrium Notion

- Game has no proper subgames, so how to usefully capture sequential rationality?
- A public strategy for an individual ignores that individual's private actions, so that behavior only depends on public information. Every player has a public strategy best response when all other players are playing public strategies.

Definition

The automaton $(\mathcal{W}, w^0, f, \tau)$ is a perfect public equilibrium (PPE) if for all states $w \in \mathcal{W}(w^0)$, the automaton $(\mathcal{W}, w, f, \tau)$ is a Nash equilibrium.

Principle of No Profitable One-Shot Deviations

Definition

Player i has a profitable one-shot deviation from $(\mathcal{W}, w^0, f, \tau)$, if there is a state $w \in \mathcal{W}(w^0)$ and some action $a_i \in A_i$ such that

$$V_i(w) < (1-\delta)u_i(a_i, f_{-i}(w)) + \delta \sum_y V_i(\tau(w, y))\rho(y \mid (a_i, f_{-i}(w))).$$

Principle of No Profitable One-Shot Deviations

Definition

Player i has a profitable one-shot deviation from $(\mathcal{W}, w^0, f, \tau)$, if there is a state $w \in \mathcal{W}(w^0)$ and some action $a_i \in A_i$ such that

$$V_i(w) < (1-\delta)u_i(a_i, f_{-i}(w)) + \delta \sum_y V_i(\tau(w,y))\rho(y \mid (a_i, f_{-i}(w))).$$

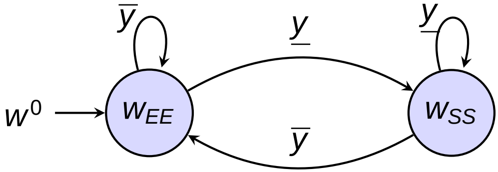
Theorem

The automaton $(\mathcal{W}, w^0, f, \tau)$ is a PPE iff there are no profitable one-shot deviations, i.e, for all $w \in \mathcal{W}(w^0)$, $f(w)$ is a Nash eq of the normal form game with payoff function $g^w : A \to \mathbb{R}^n$, where

$$g_i^w(a) = (1-\delta)u_i(a) + \delta \sum_y V_i(\tau(w,y))\rho(y \mid a).$$

Prisoners' Dilemma with Noisy Monitoring

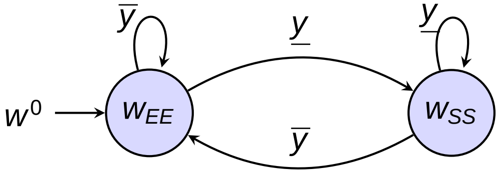
Bounded Recall



- $V(w_{EE}) = (1-\delta)2 + \delta\{pV(w_{EE}) + (1-p)V(w_{SS})\}$
  $V(w_{SS}) = \delta\{rV(w_{EE}) + (1-r)V(w_{SS})\}$
- $V(w_{EE}) > V(w_{SS})$, but $V(w_{EE}) - V(W_{SS}) \to 0$ as $\delta \to 1$.
- At w_{EE}, EE is a Nash eq of $g^{w_{EE}}$ if $\delta \geq (3p - 2q - r)^{-1}$.
- At w_{SS}, SS is a Nash eq of $g^{w_{SS}}$ if $\delta \leq (p + 2q - 3r)^{-1}$.

Prisoners' Dilemma with Noisy Monitoring

Bounded Recall



- $V(w_{EE}) = (1-\delta)2 + \delta\{pV(w_{EE}) + (1-p)V(w_{SS})\}$
  $V(w_{SS}) = \delta\{rV(w_{EE}) + (1-r)V(w_{SS})\}$
- $V(w_{EE}) > V(w_{SS})$, but $V(w_{EE}) - V(W_{SS}) \to 0$ as $\delta \to 1$.
- At w_{EE}, EE is a Nash eq of $g^{w_{EE}}$ if $\delta \geq (3p - 2q - r)^{-1}$.
- At w_{SS}, SS is a Nash eq of $g^{w_{SS}}$ if $\delta \leq (p + 2q - 3r)^{-1}$.
- PPE if $(3p - 2q - r)^{-1} \leq \delta \leq (p + 2q - 3r)^{-1}$.

Characterizing PPE

- A major conceptual breakthrough was to focus on continuation values in the description of equilibrium, rather than focusing on behavior directly.
- This yields a more transparent description of incentives, and an informative characterization of equilibrium payoffs.
- The cost is that we know little about the details of behavior underlying most of the equilibria, and so have little sense which of these equilibria are plausible descriptions of behavior.

Enforceability and Decomposability

Definition

An action profile $a' \in A$ is enforced by the continuation promises $\gamma : Y \to \mathbb{R}^n$ if a' is a Nash eq of the normal form game with payoff function $g^\gamma : A \to \mathbb{R}^n$, where

$$g_i^\gamma(a) = (1-\delta)u_i(a) + \delta \sum_y \gamma_i(y)\rho(y \mid a).$$

Enforceability and Decomposability

Definition

An action profile $a' \in A$ is enforced by the continuation promises $\gamma : Y \to \mathbb{R}^n$ if a' is a Nash eq of the normal form game with payoff function $g^\gamma : A \to \mathbb{R}^n$, where

$$g_i^\gamma(a) = (1-\delta)u_i(a) + \delta \sum_y \gamma_i(y)\rho(y \mid a).$$

Definition

A payoff v is decomposable on a set of payoffs $\mathcal{V}$ if there exists an action profile a' enforced by some continuation promises $\gamma : Y \to \mathcal{V}$ satisfying, for all i,

$$v_i = (1-\delta)u_i(a') + \delta \sum_y \gamma_i(y)\rho(y \mid a').$$

Characterizing PPE

The Role of Continuation Values

- Let $\mathcal{E}^p(\delta) \subset \mathcal{F}^*$ be the set of (pure strategy) PPE.
- If $v \in \mathcal{E}^p(\delta)$, then there exists $a' \in A$ and $\gamma : Y \to \mathcal{E}^p(\delta)$ so that, for all i,

$$\begin{aligned} v_i &= (1-\delta)u_i(a') + \delta \sum\nolimits_y \gamma_i(y)\rho(y \mid a') \\ &\geq (1-\delta)u_i(a_i, a'_{-i}) + \delta \sum\nolimits_y \gamma_i(y)\rho(y \mid a_i, a'_{-i}) \quad \forall a_i \in A_i. \end{aligned}$$

That is, v is decomposed on $\mathcal{E}^p(\delta)$.

Characterizing PPE

The Role of Continuation Values

- Let $\mathcal{E}^p(\delta) \subset \mathcal{F}^*$ be the set of (pure strategy) PPE.
- If $v \in \mathcal{E}^p(\delta)$, then there exists $a' \in A$ and $\gamma : Y \to \mathcal{E}^p(\delta)$ so that, for all i,

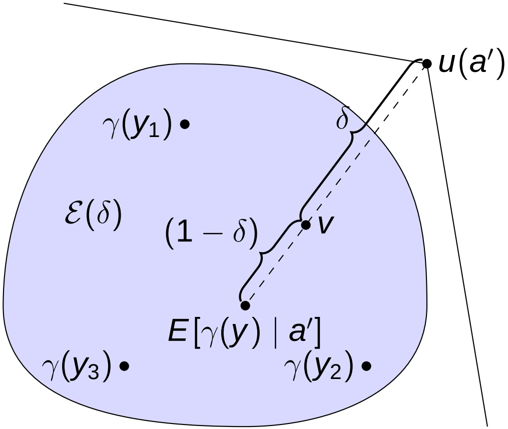
$$\begin{aligned} v_i &= (1-\delta)u_i(a') + \delta \sum\nolimits_y \gamma_i(y)\rho(y \mid a') \\ &\geq (1-\delta)u_i(a_i, a'_{-i}) + \delta \sum\nolimits_y \gamma_i(y)\rho(y \mid a_i, a'_{-i}) \quad \forall a_i \in A_i. \end{aligned}$$

Theorem (Self-generation, Abreu, Pearce, Stacchetti, 1990)

$B \subset \mathcal{E}^p(\delta)$ if and only if for all $v \in B$, B bounded, there exists $a' \in A$ and $\gamma : Y \to B$ so that, for all i,

$$\begin{aligned} v_i &= (1-\delta)u_i(a') + \delta \sum\nolimits_y \gamma_i(y)\rho(y \mid a') \\ &\geq (1-\delta)u_i(a_i, a'_{-i}) + \delta \sum\nolimits_y \gamma_i(y)\rho(y \mid a_i, a'_{-i}) \quad \forall a_i \in A_i. \end{aligned}$$

Decomposability



$$v = (1-\delta)u(a') + \delta E[\gamma(y) \mid a']$$
$$\Longrightarrow$$
$$v - E[\gamma(y) \mid a']$$
$$= (1-\delta)(u(a') - E[\gamma(y) \mid a'])$$

and

$$u(a') - v = \delta(u(a') - E[\gamma(y) \mid a']).$$

Impact of Increased Precision

- Let R be the $|A| \times |Y|$-matrix, $[R]_{ay} := \rho(y \mid a)$.
- (Y, ρ') is a garbling of (Y, ρ) if there exists a stochastic matrix Q such that

  $$R' = RQ.$$

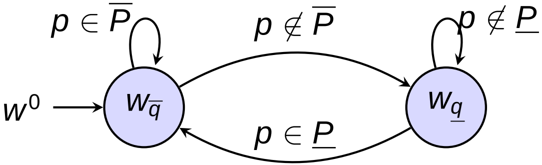
  That is, the "experiment" (Y, ρ') is obtained from (Y, ρ) by first drawing y according to ρ, and then adding noise.
- If $\mathcal{W}$ can be decomposed on $\mathcal{W}'$ under ρ', then $\mathcal{W}$ can be decomposed on the convex hull of $\mathcal{W}'$ under ρ. And so the set of PPE payoffs is weakly increasing as the monitoring becomes more precise.

Bang-Bang

- Suppose A is finite and the signals y are distributed absolutely continuously with respect to Lebesgue measure on a subset of $\mathbb{R}^k$. Every pure strategy eq payoff can be achieved by $(\mathcal{W}, w^0, f, \tau)$ with the bang-bang property:

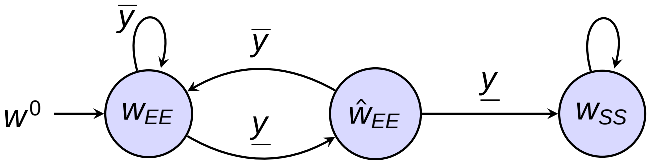
$$V(w) \in \text{ext}\, \mathcal{E}^p(\delta) \quad \forall w \neq w^0,$$

  where ext $\mathcal{E}^p(\delta)$ is the set of extreme points of $\mathcal{E}^p(\delta)$.
- (Green-Porter) If $(\mathcal{W}, w^0, f, \tau)$ is strongly symmetric, then ext $\mathcal{E}^p(\delta) = \{\underline{V}, \overline{V}\}$, where $\underline{V} := \min \mathcal{E}^p(\delta)$, $\overline{V} := \max \mathcal{E}^p(\delta)$.



Prisoners' Dilemma with Noisy Monitoring

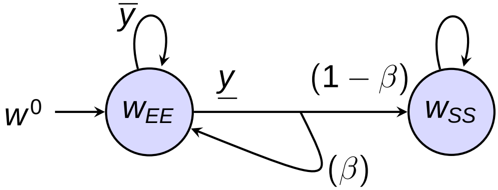
The value of "forgiveness" I



- This has a higher value than grim trigger, since permanent *SS* is only triggered after two consecutive $\underline{y}$.
- But the limiting value (as $\delta \to 1$) is still zero. As players become more patient, the future becomes more important, and smaller variations in continuation values suffice to enforce *EE*.
- *EE* can be enforced by more forgiving specifications as $\delta \to 1$.

Prisoners' Dilemma with Noisy Monitoring

The value of "forgiveness" II



- Public correlating device: β.
- This is an eq if

$$\begin{aligned} V &= (1-\delta)2 + \delta(p + (1-p)\beta)V \\ &\geq (1-\delta)3 + \delta(q + (1-q)\beta)V \end{aligned}$$

- In the efficient eq (requires $p > q$ and $\delta(3p-2q) > 1$),

$$\beta = \frac{\delta(3p-2q)-1}{\delta(3p-2q-1)} \quad \text{and} \quad V = 2 - \frac{1-p}{p-q} < 2.$$

Prisoners' Dilemma with Noisy Monitoring

The value of "forgiveness" III

- Public correlating device is not necessary: Every pure strategy strongly symmetric PPE has payoff no larger than

$$2 - \frac{1-p}{p-q} =: \overline{\gamma}.$$

Prisoners' Dilemma with Noisy Monitoring

The value of "forgiveness" III

- Public correlating device is not necessary: Every pure strategy strongly symmetric PPE has payoff no larger than

$$2 - \frac{1-p}{p-q} =: \overline{\gamma}.$$

- Moreover, the upper bound is achieved: For sufficiently large δ, both $[0, \overline{\gamma}]$ and $(0, \overline{\gamma}]$ are self-generating.

Prisoners' Dilemma with Noisy Monitoring

The value of "forgiveness" III

- Public correlating device is not necessary: Every pure strategy strongly symmetric PPE has payoff no larger than

$$2 - \frac{1-p}{p-q} =: \overline{\gamma}.$$

- Moreover, the upper bound is achieved: For sufficiently large δ, both $[0, \overline{\gamma}]$ and $(0, \overline{\gamma}]$ are self-generating.
- The use of payoff 0 is Nash reversion.

Prisoners' Dilemma with Noisy Monitoring

The value of "forgiveness" III

- Public correlating device is not necessary: Every pure strategy strongly symmetric PPE has payoff no larger than

$$2 - \frac{1-p}{p-q} =: \overline{\gamma}.$$

- Moreover, the upper bound is achieved: For sufficiently large δ, both $[0, \overline{\gamma}]$ and $(0, \overline{\gamma}]$ are self-generating.
- The use of payoff 0 is Nash reversion.
- Forgiving grim trigger: the set $\mathcal{W} = \{0\} \cup [\underline{\gamma}, \overline{\gamma}]$, where

$$\underline{\gamma} := \frac{2(1-\delta)}{1-\delta p},$$

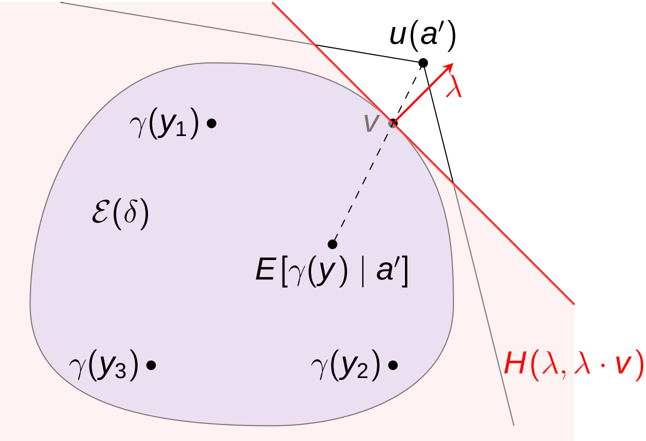
is, for large δ, self-generating with all payoffs > 0 decomposed using EE.

Implications

- Providing intertemporal incentives requires imposing punishments on the equilibrium path.
- These punishments may generate inefficiencies, and the greater the noise, the greater the inefficiency.
- How to impose punishments without creating inefficiencies: transfer value rather than destroying it.
- In PD example, impossible to distinguish ES from SE.
- Efficiency requires the monitoring be statistically sufficiently informative.
- Other examples reveal the need for asymmetric/ nonstationary behavior in symmetric stationary environments.

Bounding PPE Payoffs I

- Bounding convex sets is easier, so bound $\text{co}\mathcal{E}^p(\delta)$, convex hull of $\mathcal{E}^p(\delta)$.
- Every convex set can be written as the intersection of containing half spaces.



Bounding PPE Payoffs II

Decomposing on half spaces

- Given $\lambda \in \mathbb{R}^n \setminus \{0\}$,

$$H(\lambda, k) := \{v \in \mathbb{R}^n : \lambda \cdot v \leq k\}.$$

- Define $\mathcal{B}(\mathcal{W}; \delta, a)$ as the set of payoffs decomposed by a on $\mathcal{W}$.
- For fixed λ and a, set

$$k^*(a, \lambda, \delta) := \max_{v} \lambda \cdot v \quad \text{s.t. } v \in \mathcal{B}(H(\lambda, \lambda \cdot v); \delta, a).$$

- This is a linear program.

Bounding PPE Payoffs III

The LP

- $v \in \mathcal{B}(H(\lambda, \lambda \cdot v); \delta, a) \iff$ there exists $\gamma : Y \to \mathbb{R}^n$ satisfying

$$\begin{aligned} v_i &= (1-\delta)u_i(a) + \delta E[\gamma_i(y) \mid a], \quad \forall i, \\ v_i &\geq (1-\delta)u_i(a_i', a_{-i}) + \delta E[\gamma_i(y) \mid a_i', a_{-i}], \quad \forall a_i', \forall i, \\ \lambda \cdot v &\geq \lambda \cdot \gamma(y), \quad \forall y. \end{aligned}$$

Bounding PPE Payoffs III

The LP

- $v \in \mathcal{B}(H(\lambda, \lambda \cdot v); \delta, a) \iff$ there exists $\gamma : Y \to \mathbb{R}^n$ satisfying

$$\begin{aligned} v_i &= (1-\delta)u_i(a) + \delta E[\gamma_i(y) \mid a], \quad \forall i, \\ v_i &\geq (1-\delta)u_i(a_i', a_{-i}) + \delta E[\gamma_i(y) \mid a_i', a_{-i}], \quad \forall a_i', \forall i, \\ \lambda \cdot v &\geq \lambda \cdot \gamma(y), \quad \forall y. \end{aligned}$$

- subtract δv_i or v from both sides of constaints:

$$\begin{aligned} (1-\delta)v_i &= (1-\delta)u_i(a) + \delta E[\gamma_i(y) - v_i \mid a], \quad \forall i, \\ (1-\delta)v_i &\geq (1-\delta)u_i(a_i', a_{-i}) + \delta E[\gamma_i(y) - v_i \mid a_i', a_{-i}], \quad \forall a_i', \forall i, \\ 0 &\geq \lambda \cdot (\gamma(y) - v), \quad \forall y. \end{aligned}$$

Bounding PPE Payoffs III

The LP

- $v \in \mathcal{B}(H(\lambda, \lambda \cdot v); \delta, a) \iff$ there exists $\gamma : Y \to \mathbb{R}^n$ satisfying

$$v_i = (1-\delta)u_i(a) + \delta E[\gamma_i(y) \mid a], \quad \forall i,$$
$$v_i \geq (1-\delta)u_i(a_i', a_{-i}) + \delta E[\gamma_i(y) \mid a_i', a_{-i}], \quad \forall a_i', \forall i,$$
$$\lambda \cdot v \geq \lambda \cdot \gamma(y), \quad \forall y.$$

- dividing by $1-\delta$ and set $x_i(y) = \delta(\gamma_i(y) - v_i)/(1-\delta)$:

$$v_i = u_i(a) + E[x_i(y) \mid a], \quad \forall i,$$
$$v_i \geq u_i(a_i', a_{-i}) + E[x_i(y) \mid a_i', a_{-i}], \quad \forall a_i', \forall i,$$
$$0 \geq \lambda \cdot x(y), \quad \forall y.$$

$x : Y \to \mathbb{R}^n$ are the normalized continuations.

Bounding PPE Payoffs III

The LP

- $v \in \mathcal{B}(H(\lambda, \lambda \cdot v); \delta, a) \iff$ there exists $\gamma : Y \to \mathbb{R}^n$ satisfying

$$\begin{aligned} v_i &= (1-\delta)u_i(a) + \delta E[\gamma_i(y) \mid a], \quad \forall i, \\ v_i &\geq (1-\delta)u_i(a_i', a_{-i}) + \delta E[\gamma_i(y) \mid a_i', a_{-i}], \quad \forall a_i', \forall i, \\ \lambda \cdot v &\geq \lambda \cdot \gamma(y), \quad \forall y. \end{aligned}$$

- dividing by $1-\delta$ and set $x_i(y) = \delta(\gamma_i(y) - v_i)/(1-\delta)$:

$$\begin{aligned} v_i &= u_i(a) + E[x_i(y) \mid a], \quad \forall i, \\ v_i &\geq u_i(a_i', a_{-i}) + E[x_i(y) \mid a_i', a_{-i}], \quad \forall a_i', \forall i, \\ 0 &\geq \lambda \cdot x(y), \quad \forall y. \end{aligned}$$

  $x : Y \to \mathbb{R}^n$ are the normalized continuations.
- And so $k^*(a, \lambda, \delta)$ is independent of δ and can be written as $k^*(a, \lambda)$.

Bounding PPE payoffs IV

- x orthogonally enforces a in the direction λ if all constraints are satisfied and $\lambda \cdot x(y) = 0$ for all y.
- $k^*(a, \lambda) \leq \lambda \cdot u(a)$, with equality if orthogonal enforcement.
- pairwise orthogonal enforcement is sufficient for orthogonal enforcement in all noncoordinate directions.
- Set $k^*(\lambda) := \max_a k^*(a, \lambda)$ and $H^*(\lambda) := H(\lambda, k^*(\lambda))$. Then,

$$\mathcal{E}^p(\delta) \subset \cap_\lambda H^*(\lambda) \subset \mathcal{F}^p.$$

  (Negative directions, particularly coordinate ones, are key, as we will see on the next slide.)
- Interpret $k^*(\lambda)$ as a bound on the average utility (according to λ) of providing appropriate incentives. If the enforcement is orthogonal there is no aggregate cost, in that $\lambda \cdot x(y) = 0$ for all y.

Bounding PPE payoffs V

Coordinate directions

- Suppose $\lambda = -e_j$, where e_j is the jth coordinate vector. Then, $\lambda \cdot v = -v_j$ and $0 \geq \lambda \cdot x(y) = -x_j(y)$ (i.e., $x_j(y) \geq 0$).
- Then LP (for fixed a) is choosing v and x to minimize v_j subject to

$$\begin{aligned} v_i &= u_i(a) + E[x_i(y) \mid a], \quad \forall i, \\ v_i &\geq u_i(a_i', a_{-i}) + E[x_i(y) \mid a_i', a_{-i}], \quad \forall a_i', \forall i, \\ 0 &\geq -x_j(y), \quad \forall y. \end{aligned}$$

  But the last constraint does not apply to $i \neq j$, and $x_i(y)$ for $i \neq j$ can be freely chosen.
- If x_i can be chosen to enforce a_{-j}, and if a_j is BR to a_{-j}, then a can be orthogonally enforced with $x_j(y) = 0$.
- If, for all a_{-j}, x_i can be chosen to enforce a_{-j}, then

$$k^*(-e_j) = -\underline{v}_j^p = -\min_{a_j} \max_{a_j} u_j(a).$$

Return to PD

- Maximal total PPE payoff: $\lambda = (1, 1)$.
- If only two signals, $\underline{y}$ and $\overline{y}$, EE cannot be orthogonally enforced in the direction $(1, 1)$, and so all PPE inefficient:
  Need $x_i(\underline{y}) < x_i(\overline{y})$, $\forall i$, and so $\lambda \cdot x(\underline{y}) < \lambda \cdot x(\overline{y}) = 0$.
- With two signals, there are three equations in two unknowns, and so typically cannot satisfy all constraints.
- With three signals, now have three unknowns, and so can solve (provided the three equations are independent).

Statistically Informative Monitoring

Rank Conditions

Definition

The profile α has individual full rank for player i if the $|A_i| \times |Y|$-matrix $R_i(\alpha_{-i})$, with

$$[R_i(\alpha_{-i})]_{a_i y} := \rho(y \mid a_i \alpha_{-i}),$$

has full row rank.

Statistically Informative Monitoring

Rank Conditions

Definition

The profile α has individual full rank for player i if the $|A_i| \times |Y|$-matrix $R_i(\alpha_{-i})$, with

$$[R_i(\alpha_{-i})]_{a_i y} := \rho(y \mid a_i \alpha_{-i}),$$

has full row rank.

The profile α has pairwise full rank for players i and j if the $(|A_i| + |A_j|) \times |Y|$-matrix

$$R_{ij}(\alpha) := \begin{bmatrix} R_i(\alpha_{-i}) \\ R_j(\alpha_{-j}) \end{bmatrix}$$

has rank $|A_i| + |A_j| - 1$.

Another Folk Theorem

The Public Monitoring Folk Theorem (Fudenberg, Levine, and Maskin 1994)

Suppose the set of feasible and individually rational payoffs has nonempty interior, and that all action profiles satisfy pairwise full rank for all players. Every strictly individually rational and feasible payoff is a perfect public equilibrium payoff, provided players are patient enough.

Another Folk Theorem

The Public Monitoring Folk Theorem (Fudenberg, Levine, and Maskin 1994)

Suppose the set of feasible and individually rational payoffs has nonempty interior, and that all action profiles satisfy pairwise full rank for all players. Every strictly individually rational and feasible payoff is a perfect public equilibrium payoff, provided players are patient enough.

- Pairwise full rank fails for our prisoners' dilemma example (can be satisfied if there are three signals).
- Also fails for Green Porter noisy oligopoly example, since distribution of the market clearing price only depends on total market quantity.
- Folk theorem holds under weaker assumptions.

Role of Patience

- The monitoring can be arbitrarily noisy, as long as it remains statistically informative.
- But, the noisier the monitoring the more patient the players must be.

Role of Patience

- The monitoring can be arbitrarily noisy, as long as it remains statistically informative.
- But, the noisier the monitoring the more patient the players must be.
- Suppose time is continuous, and decisions are taken at points Δ, 2Δ, $3\Delta, \ldots$.
- If r is continuous rate of time discounting, then $\delta = e^{-r\Delta}$.
- As $\Delta \to 0$, $\delta \to 1$.
  - For games of perfect monitoring, high δ can be interpreted as Δ.
  - But, this is problematic for games of imperfect monitoring: As $\Delta \to 0$, the monitoring becomes increasingly precise over a fixed time interval.