Foundations for Types Spaces

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Introduction

- Fundamental space of uncertainty is Θ, a set of parameters.
- Assume Θ is finite, though can extend to metric spaces that are complete (Cauchy sequences converge), separable (countably dense subset).
- Will be concerned with beliefs over Θ, beliefs over beliefs over Θ, ...

Example

- Suppose $\Theta = \{\theta_0, \theta_1\}$, and $\mu \in \Delta(\Theta) \subset \mathbb{R}^2$, set of probability distributions.
- If $(\mu_n) \subset \Delta(\Theta)$, then $\mu_n \to \mu$ is convergence in the usual Euclidean sense: $\mu_n(\theta) \to \mu(\theta)$ for each θ.
- This is equivalent to for all $f : \Theta \to \mathbb{R}$,

$$\sum f(\theta)\mu_n(\theta) \to \sum f(\theta)\mu(\theta).$$

When are distributions close?

- Fix a metric space Z.
  - If $Z = \{\theta_0, \theta_1\}$, then give Z discrete topology: $d(\theta, \theta') = 0$ if $\theta = \theta'$, and 1 otherwise. Singletons are open.
  - If $Z = [0, 1]$, then $d(x, y) = |x - y|$. Singletons are not open.

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- The Borel σ-algebra is the σ-algebra generated by the open sets (trivial if Z is finite). A Borel measure is a measure defined over the Borel sets. The restriction to Borel sets (as events) and Borel measures is a mild one, and yields a nice mathematical structure.
- The set of Borel probability measures over the space Z is denoted $\Delta(Z)$.
  - If $Z = \{\theta_0, \theta_1\}$, then $\Delta(Z)$ is $[0, 1]$.
  - If $Z = [0, 1]$, then $\Delta(Z)$ can be described by the set of probability distribution functions on $[0, 1]$.

Topology of Weak Convergence

- Endow $\Delta(Z)$ with the topology of weak convergence: $\mu_k \to \mu$ iff for all bounded continuous functions $f : Z \to \mathbb{R}$, $\int f d\mu_k \to \int f d\mu$.
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    - If Z is finite, then all functions are contiuous, and this is equivalent to the usual convergence of probabilities.
- Suppose Z is a subset of the real line, and denote the distribution function of μ (respectively, μ_k) by F (resp., F_k). Then, μ_k converges weakly to μ if and only if for all continuity points z of F, $F_k(z) \rightarrow F(z)$.

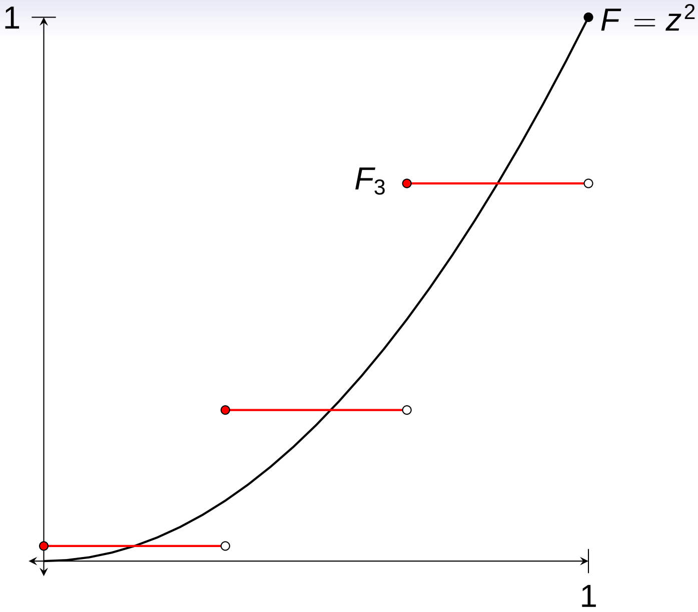
Example 1

- Suppose $Z = [0, 1]$, and $\mu \in \Delta(Z)$.
- Suppose μ_n is the simple probability measure (i.e., has finite support) given by

$$\mu_n(\{z\}) = \begin{cases} y_n^k, & z = k/n \text{ for some } k \in \{0, 1, \ldots, n-1\}, \\ 1 - \sum_k y_n^k, & z = 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Then, μ_n converges weakly to μ if

$$y_n^k = \mu((2k-1)/2n, (2k+1)/2n].$$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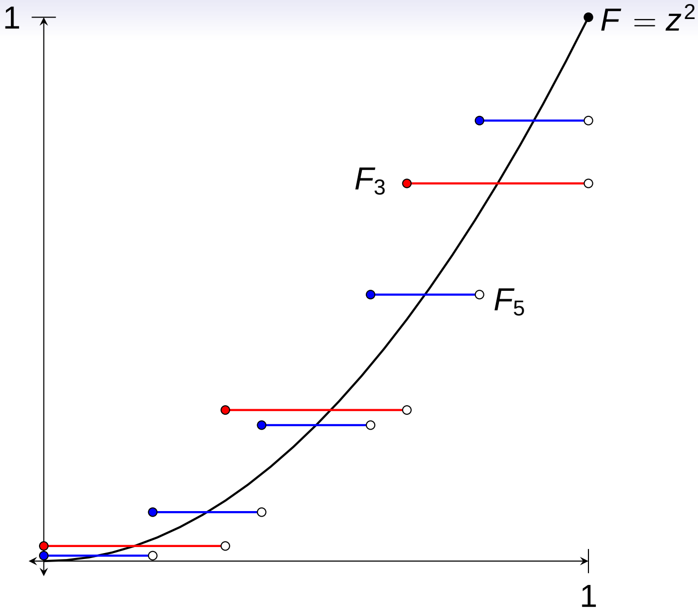
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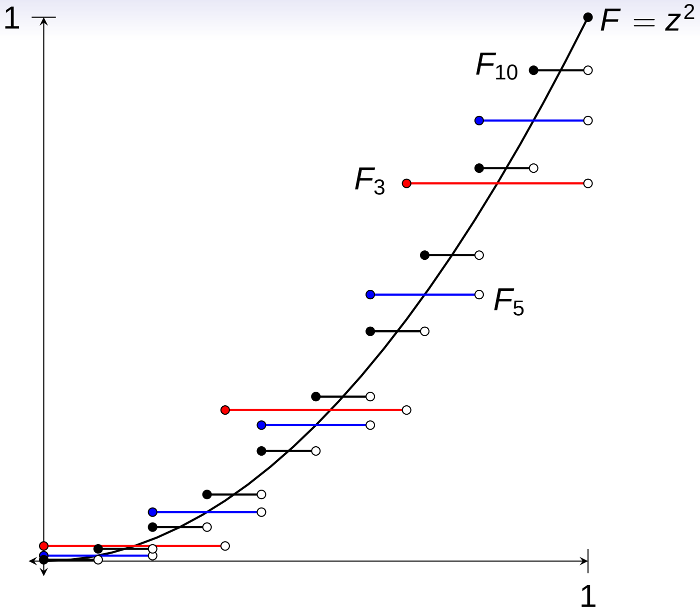
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F_{10}
F_3
F_5
1

Example 1

- If
$$y_n^k = \mu[(3k-1)/3n, (3k+1)/3n].$$
then distributions do not converge.
- While both definitions of y_n^k assign probabilities to k/n using an interval containing k/n, the difficulty with the second one is that the intervals exclude too much of the state: for n large, almost one third of the interval $[0,1]$ is excluded. So, if μ is uniform for example, the limit of μ_n has an atom of size $1/3$ at $z=1$.

Example 2

- Suppose $Z = [0,1]$, and $\mu \in \Delta(Z)$ is the simple probability measure

$$\mu(\{z\}) = \begin{cases} \frac{1}{2}, & z = \frac{1}{3}, \frac{2}{3}, \\ 0, & \text{otherwise.} \end{cases}$$

- Let μ_n be the probability measure with density

$$f_n(z) = \begin{cases} \frac{1}{n}, & z \in \left[0, \frac{n-3}{3n}\right], \\ \frac{1}{4}(n-2), & z \in \left[\frac{n-3}{3n}, \frac{n+3}{3n}\right], \\ \frac{1}{n}, & z \in \left[\frac{n+3}{3n}, \frac{2n-3}{3n}\right], \\ \frac{1}{4}(n-2), & z \in \left[\frac{2n-3}{3n}, \frac{2n+3}{3n}\right], \\ a(n), & z \in \left[\frac{2n+3}{3n}, 1\right]. \end{cases}$$

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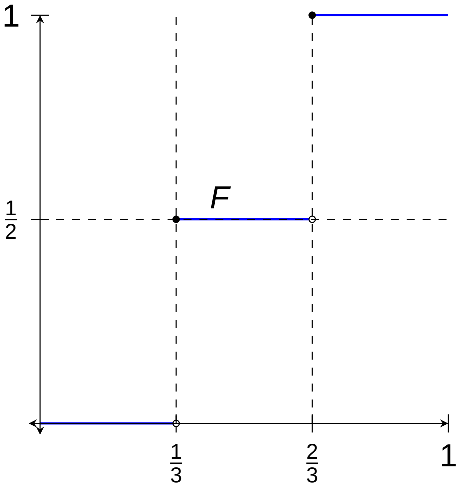

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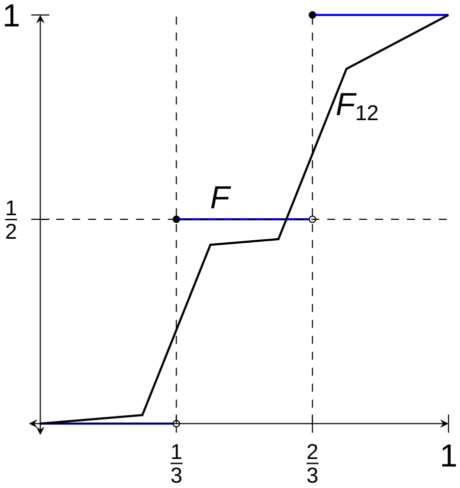

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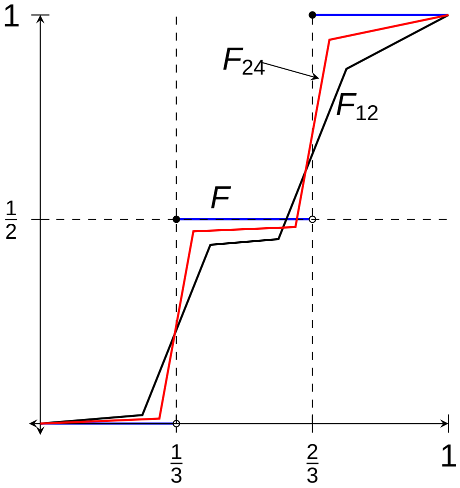

Prohorov metric

- If Z is complete, separable, metric (with metric d), then so is $\Delta(Z)$, that is, $\Delta(Z)$ is also complete, separable, metric with the topology of weak convergence.
- The standard metric used to metrize the space of probability measures is the Prohorov metric: For any Borel set $B \subset Z$, define

$$B^{\varepsilon} := \{x \mid d(x, B) < \varepsilon\} = \{x \mid \inf_{y \in B} d(x, y) < \varepsilon\}.$$

For any $\mu, \lambda \in \Delta(Z)$, the Prohorov distance between μ and λ is given by

$$d^{P}(\mu, \lambda) := \inf\{\varepsilon > 0 : \mu(B) \leq \lambda(B^{\varepsilon}) + \varepsilon, \lambda(B) \leq \mu(B^{\varepsilon}) + \varepsilon, \ \forall B \text{ Borel}\}.$$

Hierarchies of Beliefs

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- Similarly, k's first order belief is δ_k^1:

$$\delta_k^1 \in \Delta(X_0^k).$$

Hierarchies of Beliefs

- But i does not know k's beliefs, i.e., i does not know $X_0^i \times \Delta(X_0^k)$ and so has second order beliefs:

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- Note that the second order beliefs allow for i's beliefs over Θ to be correlated with the beliefs over k's beliefs.
- Moreover, for sensible beliefs (i.e., coherent, defined soon), the second order beliefs subsume the first order beliefs since the second order marginal on X_0^i should equal the first order belief.

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- $\Delta(X_0^k \times \Delta(X_0^i))$, player k's beliefs over X_1^k, i.e., jointly over Θ and i's beliefs over Θ.

Fourth Order Beliefs

- Define

$$X_1^i := X_0^i \times \Delta(X_0^k),$$

and then recursively

$$X_n^i := X_{n-1}^i \times \Delta(X_{n-1}^k) = X_0^i \times \prod_{\ell=0}^{n-1} \Delta(X_\ell^k).$$

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- So, i's fourth level of uncertainty is over

$$X_3^i = \underbrace{X_0^i}_{=\Theta} \times \underbrace{\Delta(X_0^k)}_{k\text{'s beliefs over }\Theta} \times \underbrace{\Delta(X_1^k)}_{k\text{'s belief over }X_1^k} \times \underbrace{\Delta(X_2^k)}_{k\text{'s belief over }X_2^k}.$$

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- Player i's type, $t_i := (\delta_i^1, \delta_i^2, \ldots) \in \prod_{\ell=0}^{\infty} \Delta(X_\ell^i) =: T_i^0$.

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- Similarly, $t_k \in T_k^0$.

A Detour and Some Examples

- We are thus naturally led to dealing with sequences (infinite hierarchies) of beliefs. Need to think about convergence of sequences of sequences of beliefs.

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- Suppose $Z_n = \{0, 1\}$ for all $n \in \{1, 2, \dots\}$, and consider the sequence in $\prod_n Z_n = Z_1 \times Z_2 \times \cdots$ whose mth term is given by $z^m := (z_n^m)$, where

$$z_n^m = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

A Detour and Some Examples

- Then,

$$\begin{aligned} z^1 &= (1, 0, 0, 0, \dots), \\ z^2 &= (0, 1, 0, 0, \dots), \\ z^3 &= (0, 0, 1, 0, \dots), \\ z^4 &= (0, 0, 0, 1, \dots), \\ &\vdots \end{aligned}$$

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- Then z^m converges pointwise to the zero sequence, $z^\infty := (0, 0, 0, 0 \dots)$: For all n, $z_n^m = 0$ for $m > n$.
- The sequence does not converge uniformly: For all m (no matter how large), there exists n such that $z_n^m \neq 0$ (in particular, $n = m$).

Another Example

- Suppose $Z_n = \mathbb{R}$ for all $n \in \{1, 2, \dots\}$, and consider the sequence in $\prod_n Z_n = Z_1 \times Z_2 \times \cdots$ whose mth term is given by $z^m := (z_n^m)$, where

$$z_n^m = n/m.$$

Then,

$$\begin{aligned} z^1 &= (1, 2, 3, \dots), \\ z^2 &= (\tfrac{1}{2}, \tfrac{2}{2}, \tfrac{3}{2}, \dots), \\ z^3 &= (\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, \dots), \end{aligned}$$

and so on.

- Then z^m converges pointwise to the zero sequence, $z^\infty := (0, 0, \dots)$: For all n, $z_n^m < \varepsilon$ for $m > n/\varepsilon$.
- The sequence does not converge uniformly: for all m, $z_n^m \to \infty$ as $n \to \infty$.

Pointwise Convergence

- Given a collection $\{Z_n\}_{n\geq 1}$, the product topology on $Z = \prod_{n\geq 1} Z_n$ is the weakest topology making the projections continuous: Let $\pi_n : Z \longrightarrow Z_n$ be the nth coordinate projection ($\pi_n(z_1, z_2, \ldots,) = z_n$). Then $z^m \longrightarrow z^0$ if, for all n, $\pi_n(z^m) \longrightarrow z_n^0$ (that is, this is the topology of pointwise convergence).

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- A set $G \subset Z$ is open in the product topology if, and only if, $\pi_n(G) = Z_n$ for all but finitely many n. This implies that if G is open, then there exists n' such that for all $n > n'$, $\pi_n(G) = Z_n$.

Pointwise Convergence

- If Z_n is a metric space, with metric d_n, the product topology is metrizable. Fix $\rho \in (0,1)$, Define

$$d_\rho((z_1, z_2, \ldots,), (z_1', z_2', \ldots,)) := \sum_n \rho^n \max\{d_n(z_n, z_n'), 1\}.$$

  Then d_ρ is a metric for the product topology (any ρ induces the same topology).

- Endowing $Z_n = \{0, 1\}$ with the discrete metric, $d(z, z') = 0$ if $z = z'$ and 1 otherwise, we have

$$d_\rho(z^m, 0) = \rho^{-m} \to 0 \text{ as } m \to \infty.$$

Uniform Convergence

- The uniform (or box) topology, is metrized by the sup metric:

$$d_S((z_1, z_2, \ldots,), (z_1', z_2', \ldots,)) := \sup_n \max\{d_n(z_n, z_n'), 1\}.$$

- Endowing $Z_n = \{0, 1\}$ with the discrete metric, $d(z, z') = 0$ if $z = z'$ and 1 otherwise, we have

$$d_S(z^m, 0) = 1 \not\to 0 \text{ as } m \to \infty.$$

Now back to our story

- We have
$$X_1^i := X_0^i \times \Delta(X_0^j),$$
and
$$X_n^i := X_{n-1}^i \times \Delta(X_{n-1}^k) = X_0^i \times \prod_{\ell=0}^{n-1} \Delta(X_\ell^k).$$
- Player i's nth order belief is $\delta_i^n \in \Delta(X_{n-1}^i)$.
- i's type is the infinite hierarchy, $t_i := (\delta_i^1, \delta_i^2, \dots) \in \prod_{n=0}^{\infty} \Delta(X_n^i) =: T_i^0$.
- With the Prohorov metric, each space X_n^i is a nice metric space, and $t_i^{(m)} = (\delta_{i,(m)}^1, \delta_{i,(m)}^2, \dots) \to t_i = (\delta_i^1, \delta_i^2, \dots)$ in the product topology if
$$\delta_{i,(m)}^n \to \delta_i^n \qquad \forall n.$$

The email game

- Recall $\Theta = \{\theta_0, \theta_1\}$, with prob p on θ_1.
- Let $t_i^{(m)}$ denote player i's type after sending m messages.
- At the type $t_2^{(0)}$, player 2 has belief $\bar{\delta}_2^1$: it assigns prob $p\varepsilon/[(1-p)+p\varepsilon] =: p'$ to θ_1 and $1-p'$ to θ_0.
  Player 2 assigns prob $1-p'$ to player 1 assigning prob 0 to θ_1, and prob p' to 1 assigning prob 1 to θ_1. That is,

  $$\delta_2^2 = (1-p') \circ (\theta_0, \bar{\delta}_1^1) + p' \circ (\theta_1, \tilde{\delta}_1^1)$$

  where $\bar{\delta}_1^1 = 1 \circ \theta_0 + 0 \circ \theta_1$ and $\tilde{\delta}_1^1 = 0 \circ \theta_0 + 1 \circ \theta_1$.
- At the type $t_1^{(0)}$, player 1 assigns prob 0 to θ_1 (this is $\bar{\delta}_1^1$). Player 1 assigns probability 1 to player 2 being of type t_2^0 and so having first order belief

  $$\bar{\delta}_2^1 = (1-p') \circ \theta_0 + p' \circ \theta_1, \text{ i.e., } \delta_1^2 = 1 \circ (\theta_0, \bar{\delta}_2^1).$$

- At the type $t_1^{(1)}$, player 1 assigns prob 1 to θ_1 (this is $\tilde{\delta}_1^1$).
  Player 1 assigns probability $\varepsilon/[\varepsilon + (1-\varepsilon)\varepsilon] =: p''$ to player 2 being type $t_2^{(0)}$, and so assigning prob p' to θ_1, and prob $1 - p''$ to 2 being type $t_2^{(1)}$ and so assigning prob 1 to θ_1. Denote this second order belief by

$$\tilde{\delta}_1^2 := p'' \circ (\theta_1, (1-p') \circ \theta_0 + p' \circ \theta_1) + (1-p'') \circ (\theta_1, 0 \circ \theta_0 + 1 \circ \theta_1).$$

- Player 2's third order belief at $t_2^{(0)}$ is given by

$$\delta_2^3 = (1-p') \circ (\theta_0, \bar{\delta}_1^1, 1 \circ \bar{\delta}_1^2) + p' \circ (\theta_1, \tilde{\delta}_1^1, \tilde{\delta}_1^2),$$

and so on.

i's beliefs over t_k

Lemma

Suppose $\{Z_n\}_{n\geq 0}$ *is a collection of Polish spaces, and define*

$$D := \{(\delta^1, \delta^2, \ldots) \mid \delta^n \in \Delta(Z_0 \times \cdots \times Z_{n-1}), \forall n \geq 1, \\ \mathrm{marg}_{Z_0 \times \cdots \times Z_{n-2}} \delta^n = \delta^{n-1}, \forall n \geq 2\}.$$

There exists a homeomorphism (i.e., a one-to-one and onto continuous function with a continuous inverse)

$$f : D \to \Delta\left(\prod_n Z_n\right)$$

satisfying

$$\mathrm{marg}_{Z_0 \times \cdots \times Z_{n-1}} f(\delta^1, \delta^2, \ldots) = \delta^n.$$

Proof

- Kolmogorov's extension (existence) theorem implies that for all $(\delta^1, \delta^2, \ldots) \in D$, there exists unique measure $f(\delta^1, \delta^2, \ldots) := \delta \in \Delta(\prod_n Z_n)$ satisfying
$$\mathrm{marg}_{Z_0 \times \cdots \times Z_{n-1}} \delta = \delta^n.$$
- It remains to verify that f and f^{-1} are both continuous.
- Since $f^{-1}(\delta) = (\mathrm{marg}_{Z_0} \delta, \mathrm{marg}_{Z_0 \times Z_1} \delta, \ldots)$, and if (δ_k) converges to δ, then so do the marginals of δ_k, f^{-1} is trivially continuous.
  - Note the role of the product topology here. This does not prove that f^{-1} is continuous under a stronger topology, such as the box topology (which implies uniform, not pointwise, convergence) on D.

Proof (concl.)

- We now prove the continuity of f.
- Suppose $\left((\delta_k^1, \delta_k^2, \dots)\right)_k$ is a sequence in D converging to $(\delta^1, \delta^2, \dots)$.
- Then, for each n, $\delta_k^n \to \delta^n$. We need to show $f(\delta_k^1, \delta_k^2, \dots) =: \delta_k$ weakly converges to $f(\delta^1, \delta^2, \dots) =: \delta$.
- A cylinder set is a set C with property that there exists a finite set J and $(z'_n)_{n \in J}$ such that $z \in C$ if $z_n = z'_n$ for all $n \in J$.
- The collection of all cylinder sets is a convergence-determining class for weak convergence. Consequently, we need only show convergence on every cylinder set.
- For any cylinder C, there is an $\bar{n}$ such that δ_k^n agrees with δ_k on C for all $n \geq \bar{n}$, and so $\delta_k(C) \to \delta(C)$. □

Coherency

Definition

A type $t_i \in T_i^0$ is coherent if for all $n \geq 2$,

$$\mathrm{marg}_{X_i^{n-2}} \delta_i^n = \delta_i^{n-1}.$$

The set of coherent types is denoted T_i^1.

Theorem

There is a homeomorphism $f : T_i^1 \to \Delta(\Theta \times T_k^0)$ *satisfying*

$$\mathrm{marg}_{X_{n-1}^i} f(\delta^1, \delta^2, \ldots) = \delta^n.$$

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and

$$T_i^* := \cap_{\ell=1}^\infty T_i^\ell.$$

The Universal Type Space

- Define

$$T_i^\ell := \{t_i \in T_i^1 \mid f(t_i)(\Theta \times T_k^{\ell-1}) = 1\},$$

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Definition

The universal type space for player i is the set T_i^*.

- The set $T_1^* \times T_2^*$ is the set of pairs of types for which it is common belief that players' types are coherent.

The infinite regress does end

Theorem

There is a homeomorphism $g : T_i^* \to \Delta(\Theta \times T_\ell^*)$ *satisfying*

$$\mathrm{marg}_{X_{n-1}^i}\, g(\delta^1, \delta^2, \ldots) = \delta^n.$$

Belief-Closed Subsets

Definition

A set $T_1 \times T_2$ is belief-closed subset of the universal type space $T_1^* \times T_2^*$ if for all $t_i \in T_i$,

$$g(t_i)(\Theta \times T_j) = 1.$$

- In the email game, let $t_i^{(\infty)}$ denote the hierarchy of beliefs that player i believes $\theta = \theta_1$ and believes it is common belief that the game is $\theta = \theta_1$. Then $\{t_1^{(\infty)}\} \times \{t_2^{(\infty)}\}$ is belief closed. (Moreover, $t_i^{(m)} \to t_i^{(\infty)}$.)

Models

Definition

A model or type structure is the collection (Θ, T, κ), where $T = T_1 \times T_2$ is a type space, and $\kappa = (\kappa_1, \kappa_2)$ is a pair of mappings with $\kappa_i : T_i \longrightarrow \Delta(\Theta \times T_j)$. The model is complete if each κ_i is onto.

Model for email game 1

	$\theta_0 t_2^{(0)}$	$\theta_1 t_2^{(0)}$	$\theta_1 t_2^{(1)}$	$\cdots$	$\theta_1 t_2^{(m-1)}$	$\theta_1 t_2^{(m)}$	$\cdots$
$\kappa_1(t_1^{(0)})$	1	0	0	$\ldots$	0	0	$\cdots$
$\kappa_1(t_1^{(1)})$	0	p''	$1-p''$	$\cdots$	0	0	$\cdots$
$\kappa_1(t_1^{(2)})$	0	0	p''	$\cdots$	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	
$\kappa_1(t_1^{(m)})$	0	0	0	$\cdots$	p''	$1-p''$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	

Model for email game 2

	$\theta_0 t_1^{(0)}$	$\theta_1 t_1^{(1)}$	$\theta_1 t_1^{(2)}$	$\cdots$	$\theta_1 t_1^{(m)}$	$\theta_1 t_1^{(m+1)}$	$\cdots$
$\kappa_2(t_2^{(0)})$	p'	$1-p'$	0	$\ldots$	0	0	$\cdots$
$\kappa_2(t_2^{(1)})$	0	p''	$1-p''$	$\cdots$	0	0	$\cdots$
$\kappa_2(t_2^{(2)})$	0	0	p''	$\cdots$	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	
$\kappa_2(t_2^{(m)})$	0	0	0	$\cdots$	p''	$1-p''$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	

- The mapping $\kappa = (\kappa_1, \kappa_2)$, $\kappa_i : T_i \to \Delta(\Theta \times T_k)$, induces a hierarchy of beliefs for each player. For example,

$$\delta_i^1 = \operatorname*{marg}_{\Theta} \kappa_i(t_i),$$

for all Borel $B \subset \Theta \times \Delta(\Theta)$,

$$\delta_i^2(B) = \kappa_i(t_i)(\{(\theta, t_k) \mid (\theta, \operatorname{marg}_\Theta \kappa_k(t_k)) \in B\}),$$

and for all Borel $B \subset \Theta \times \Delta(\Theta) \times \Delta(\Theta \times \Delta(\Theta))$,

$$\delta_i^3(B) = \kappa_i(t_i)(\{(\theta, t_k) \mid (\theta, \operatorname{marg}_\Theta \kappa_k(t_k), \operatorname{marg}_{\Theta \times \Delta(\Theta)} \kappa_k(t_k)) \in B\}).$$

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- Let $h_i : T_i \to T_i^*$ be the mapping describing for each type t_i, player i's hierarchy of beliefs $h_i(t_i) \in T_i^*$. Clearly, $h_1(T_1) \times h_2(T_2)$ is belief closed.

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- Let $h_i : T_i \to T_i^*$ be the mapping describing for each type t_i, player i's hierarchy of beliefs $h_i(t_i) \in T_i^*$. Clearly, $h_1(T_1) \times h_2(T_2)$ is belief closed.
- Suppose (Θ, T, κ) is a model with Θ and T_i Polish spaces, and κ continuous. Then,

$$t_i^m \to t_i^\infty \Longrightarrow h_i(t_i^m) \to h_i(t_i^\infty).$$

- Define $\tilde{h}_j : \Delta(\Theta \times T_j) \to \Delta(\Theta \times T_j^*)$ by

$$\tilde{h}_j(\lambda)(B) = \lambda\{(\theta, t_j) : (\theta, h_j(t_j)) \in B\} \quad \forall \text{ Borel } B \subset \Theta \times T_j^*.$$

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- Then we have the following commutative diagram:

$$\begin{array}{ccc} T_i & \xrightarrow{\kappa_i} & \Delta(\Theta \times T_j) \\ h_i \big\downarrow & & \tilde{h}_j \big\downarrow \\ T_i^* & \xrightarrow{g} & \Delta(\Theta \times T_j^*) \end{array}$$

- so that for all $t_i \in T_i$,

$$g(h_i(t_i)) = \tilde{h}_j(\kappa_i(t_i)).$$

Two Special Models

Definition

A model (Θ, T, κ) is finite if $|\Theta \times T| < \infty$.

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A model (Θ, T, κ) satisfies the common prior assumption (CPA) if there exists a probability measure $\mu \in \Delta(\Theta \times T)$ such that for all i and Borel subsets B of $\Theta \times T_k$, and for all $t_i \in T_i$,

$$\kappa_i(t_i)(B) = \mu(B \mid \{t_i\}).$$

How Restrictive is CPA?

Definition

Let T_i^{F} be the set of all belief hierarchies for i corresponding to a finite model, i.e., $\tilde{t}_i \in T_i^{\mathrm{F}}$ if $\tilde{t}_i \in h_i(T_i)$ for some finite model $(\Theta \times T, \kappa)$; the set T_i^{F} is the set of finite types for i.

Define

$$T_i^{\mathrm{CPA}} := \{h_i(t_i) \mid t_i \in T_i \text{ for some finite model } (\Theta \times T, \kappa) \text{ that satisfies the CPA}\}.$$

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Theorem (Mertens and Zamir, 1985, Lipman, 2003)

Suppose Θ is finite. Both T_i^{F} and T_i^{CPA} are dense subsets of the universal type space T_i.

Common Knowledge

- In partition model, structure of players' information is "common knowledge," but only in an informal sense (since common knowledge is defined given the information partitions or σ-algebras).

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$$\Omega = \Theta \times T_1^* \times T_2^*.$$

- Let $\mathcal{F}$ denote the Borel σ-algebra of $\Theta \times T_1^* \times T_2^*$. Then 1's information is described by the sub σ-algebra $\mathcal{F}^1 := \{\Theta \times B \times T_2^* \mid B \text{ a Borel subset of } T_1^*\}$, and similarly for 2.

Defining Common Belief

- Given $A \subset \Theta \times T_1^* \times T_2^*$ (and $A \in \mathcal{F}$), at state $\omega = (\theta, t_1, t_2)$ player 1 assigns probability $g(t_1)(A_{t_1})$ to A, where $A_{t_1} := \{(\theta, t_2) \mid (\theta, t_1, t_2) \in A\}$.

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- Fix $E \subset \Theta$. Then i believes E at t_i if

$$t_i \in V_i^1(E) := \{\hat{t}_i \in T_i^* \mid g(\hat{t}_i)(E \times T_k^*) = 1\},$$

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- Proceeding recursively, define for $\ell \geq 2$,

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- Player i believes E is common belief at t_i if $t_i \in V_i(E)$, where $V_i(E) := \cap_\ell V_i^\ell(E)$. Note that $V_1(E) \times V_2(E)$ is a belief closed set.

Common Knowledge and Common Belief

- In partition interpretation, 1 knows $A \in \mathcal{F}$ at ω if

$$\omega \in K^1(A) := \{(\theta, t_1, t_2) \mid g(t_1)(A_{t_1}) = 1\},$$

where $A_{t_1} := \{(\theta, t_2) \mid (\theta, t_1, t_2) \in A\}$.

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- Define $K(A) := K^1(A) \cap K^2(A)$.

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Theorem (Common Belief=Common Knowledge)

For all $E \subset \Theta$,

$$\Theta \times V_1(E) \times V_2(E) = K_\infty(E \times T_1^* \times T_2^*).$$

Proof that CB=CK

- $$\begin{aligned} K^1(E \times T_1^* \times T_2^*) &= \{(\theta, t_1, t_2) \mid g(t_1)(E \times T_2^*) = 1\} \\ &= \Theta \times V_1^1(E) \times T_2^*, \\ \Longrightarrow\ K(E \times T_1^* \times T_2^*) &= \Theta \times V_1^1(E) \times V_2^1(E). \end{aligned}$$

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- $$\begin{aligned} K^1K(E \times T_1^* \times T_2^*) &= K^1(\Theta \times V_1^1(E) \times V_2^1(E)) \\ &= \{(\theta, t_1, t_2) \mid g(t_1)(\Theta \times V_2^1(E)) = 1\} \\ &= \Theta \times V_1^2(E) \times T_2^*, \\ \implies KK(E \times T_1^* \times T_2^*) &= \Theta \times V_1^2(E) \times V_2^2(E). \end{aligned}$$

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- Continue to iterate and take intersections. □

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