An Introduction to Epistemic Game Theory

George J. Mailath

University of Pennsylvania
and
Australian National University

February 2019

Rationality

- What are the implications of rationality?

Rationality

- What are the implications of rationality?
- Fix a normal form game $\{(A_i, u_i)\}_{i \in \{1,\dots,n\}}$.

Definition

An action a_i' *strictly dominates* another action a_i'' if for all $a_{-i} \in A_{-i}$,

$$u_i(a_i', a_{-i}) > u_i(a_i'', a_{-i}).$$

A strategy a_i is a *strictly dominant strategy* if it strictly dominates every strategy $a_i'' \neq a_i, a_i'' \in A_i$.

Rationality

- What are the implications of rationality?
- Fix a normal form game $\{(A_i, u_i)\}_{i\in\{1,\ldots,n\}}$.

Definition

An action a_i' *strictly dominates* another action a_i'' if for all $a_{-i} \in A_{-i}$,

$$u_i(a_i', a_{-i}) > u_i(a_i'', a_{-i}).$$

A strategy a_i is a *strictly dominant strategy* if it strictly dominates every strategy $a_i'' \neq a_i, a_i'' \in A_i$.

- A rational player i will not play a strictly dominated action.

Rationality and Best Replies

	L	R
T	3	0
M	2	2
B	0	3

	L	R
T	5	0
M	2	2
B	0	5

- On the left, M is not dominated, and is the unique best reply to $\frac{1}{2} \circ L + \frac{1}{2} \circ R$.

Rationality and Best Replies

	L	R
T	3	0
M	2	2
B	0	3

	L	R
T	5	0
M	2	2
B	0	5
$\frac{1}{2} \circ T + \frac{1}{2} \circ B$	$2\frac{1}{2}$	$2\frac{1}{2}$

- On the left, M is not dominated, and is the unique best reply to $\frac{1}{2} \circ L + \frac{1}{2} \circ R$.
- On the right, M is not dominated by L or by R, is never a best reply, and is dominated by $\frac{1}{2} \circ T + \frac{1}{2} \circ B$.

Rationality and Best Replies

Definition

The strategy $a_i' \in A_i$ is strictly dominated by the mixed strategy $\alpha_i \in \Delta(A_i)$ if

$$U_i(\alpha_i, a_{-i}) > U_i(a_i', a_{-i}) \qquad \forall a_{-i} \in A_{-i}.$$

Rationality and Best Replies

Definition

The strategy $a_i' \in A_i$ is strictly dominated by the mixed strategy $\alpha_i \in \Delta(A_i)$ if

$$U_i(\alpha_i, a_{-i}) > U_i(a_i', a_{-i}) \qquad \forall a_{-i} \in A_{-i}.$$

Lemma

Suppose $n = 2$. The action $a_1' \in A_1$ is not strictly dominated by a pure or mixed strategy if, and only if, $a_1' \in \arg\max u_1(a_1, \alpha_2)$ for some $\alpha_2 \in \Delta(A_2)$.

Iterated Deletion of Dominated Strategies

Common Belief in Rationaity

	L	C	R
T	$1, 0$	$1, 2$	$0, 1$
B	$0, 3$	$0, 1$	$2, 0$

Iterated Deletion of Dominated Strategies

Common Belief in Rationaity

	L	C	~~R~~
T	$1, 0$	$1, 2$	~~$0, 1$~~
B	$0, 3$	$0, 1$	~~$2, 0$~~

Iterated Deletion of Dominated Strategies

Common Belief in Rationaity

	L	C	$\cancel{R}$
T	1, 0	1, 2	0, 1
$\cancel{B}$	0, 3	0, 1	2, 0

Iterated Deletion of Dominated Strategies

Common Belief in Rationaity

	~~*L*~~	*C*	~~*R*~~
T	~~1, 0~~	1, 2	~~0, 1~~
~~*B*~~	~~0, 3~~	~~0, 1~~	~~2, 0~~

Iterated Deletion of Dominated Strategies

Common Belief in Rationaity

	~~*L*~~	*C*	~~*R*~~
T	~~1, 0~~	1, 2	~~0, 1~~
~~*B*~~	~~0, 3~~	~~0, 1~~	~~2, 0~~

- Result of these iterated deletions is *TC*.

Rationalizability

Bernheim (1984); Pearce (1984)

- Suppose we have a two player game, $u : A \rightarrow \mathbb{R}^2$.
- Finite action spaces, $|A_i| < \infty$. Define, for all $\mu \in \Delta(A_k)$,

$$BR_i(\mu_k) := \arg\max_{a_i \in A_i} u_i(a_i, \mu).$$

Rationalizability

Bernheim (1984); Pearce (1984)

- Suppose we have a two player game, $u : A \to \mathbb{R}^2$.
- Finite action spaces, $|A_i| < \infty$. Define, for all $\mu \in \Delta(A_k)$,

$$BR_i(\mu_k) := \arg\max_{a_i \in A_i} u_i(a_i, \mu).$$

Definition

Let $A_i^0 = A_i$ for $i = 1, 2$. Set

$$A_i^{\ell+1} = \bigcup_{\mu_k \in \Delta(A_k^\ell)} BR_i(\mu_k).$$

The set of rationalizable strategies for i is the set $\mathcal{R}_i \subset A_i$ given by

$$\mathcal{R}_i := \cap_\ell A_i^\ell.$$

Best Reply Set

Definition

A set $B = B_1 \times B_2 \subset A$ is a best reply set if for all i, every $a_i \in B_i$ is a best reply to some $\mu \in \Delta(B_k)$.

The set B is a full best reply set if for all $a_i \in B_i$, there exists $\mu \in \Delta(B_k)$ such that

$$a_i \in BR_i(\mu) \subset B_i.$$

Best Reply Set

Definition

A set $B = B_1 \times B_2 \subset A$ is a best reply set if for all i, every $a_i \in B_i$ is a best reply to some $\mu \in \Delta(B_k)$.

The set B is a full best reply set if for all $a_i \in B_i$, there exists $\mu \in \Delta(B_k)$ such that

$$a_i \in BR_i(\mu) \subset B_i.$$

- A strategy profile is a best reply set (considered as a singleton) if, and only if, it is a Nash equilibrium.

Best Reply Set

Definition

A set $B = B_1 \times B_2 \subset A$ is a best reply set if for all i, every $a_i \in B_i$ is a best reply to some $\mu \in \Delta(B_k)$.

The set B is a full best reply set if for all $a_i \in B_i$, there exists $\mu \in \Delta(B_k)$ such that

$$a_i \in BR_i(\mu) \subset B_i.$$

- A strategy profile is a best reply set (considered as a singleton) if, and only if, it is a Nash equilibrium.
- A strategy profile is a full best reply set (considered as a singleton) if, and only if, it is a strict Nash equilibrium.

Epistemic Models

Dekel and Siniscalchi (2014)

Definition

An epistemic model for the complete information normal form game u is a collection $(A_i, T_i, \kappa_i)_i$, where T_i is a compact metric space, and κ_i is a continuous mapping with $\kappa_i : T_i \longrightarrow \Delta(A_k \times T_k)$.

A type in T_i is called an epistemic type.

Epistemic Models

Dekel and Siniscalchi (2014)

Definition

An epistemic model for the complete information normal form game u is a collection $(A_i, T_i, \kappa_i)_i$, where T_i is a compact metric space, and κ_i is a continuous mapping with $\kappa_i : T_i \longrightarrow \Delta(A_k \times T_k)$.

A type in T_i is called an epistemic type.

- Each epistemic type $t_i \in T_i$ induces a hierarchy of beliefs, $(\delta^1_{t_i}, \delta^2_{t_i}, \ldots)$, where $\delta^1_{t_i} \in \Delta(A_k)$, $\delta^2_{t_i} \in \Delta(A_k \times \Delta(A_i)), \ldots$.

Epistemic Models

Dekel and Siniscalchi (2014)

Definition

An epistemic model for the complete information normal form game u is a collection $(A_i, T_i, \kappa_i)_i$, where T_i is a compact metric space, and κ_i is a continuous mapping with $\kappa_i : T_i \to \Delta(A_k \times T_k)$.

A type in T_i is called an epistemic type.

- Each epistemic type $t_i \in T_i$ induces a hierarchy of beliefs, $(\delta^1_{t_i}, \delta^2_{t_i}, \ldots)$, where $\delta^1_{t_i} \in \Delta(A_k)$, $\delta^2_{t_i} \in \Delta(A_k \times \Delta(A_i)), \ldots$.
- Moreover, any coherent hierarchy of beliefs induces a universal type.

- Player i believes an event $E_k \subset A_k \times T_k$ if

$$\kappa_i(t_i)(E_k) = 1.$$

- Define

$$B_i(E_k) = \{(a_i, t_i) \in A_i \times T_i : \kappa_i(t_i)(E_k) = 1\}.$$

Note that $B_i(E_k)$ places no restrictions on A_i and so has a cross product structure. So, will sometimes treat $B_i(E_j)$ as a subset of T_i.

Rationality and Belief in Rationality

Definition

Action a_i is rational for t_i if

$$a_i \in BR_i(\mathrm{marg}_{A_k} \kappa_i(t_i)).$$

- $R_i := \{(a_i, t_i) \in A_i \times T_i : a_i \text{ is rational for } t_i\}$.
- Player i believes k is rational at t_i if $t_i \in B_i(R_k)$.
- Player i is rational and believes k is rational if

$$(a_i, t_i) \in R_i \cap B_i(R_k).$$

- Both players are rational and believe the other player is rational if

$$\begin{aligned}((a_1, t_1), (a_2, t_2)) \in & R_1 \cap B_1(R_2) \times R_2 \cap B_2(R_1) \\ &= (R_1 \times R_2) \cap (B_1(R_2) \times B_2(R_1)) =: R \cap B(R).\end{aligned}$$

An Example

	L	R
T	$1, 1$	$2, 0$
B	$0, 0$	$0, 1$

	Lt_2'	Rt_2'	Lt_2''	Rt_2''
$\kappa_1(t_1')$	1	0	0	0
$\kappa_1(t_1'')$	0	0	0	1

	Tt_1'	Bt_1'	Tt_1''	Bt_1''
$\kappa_2(t_2')$	1	0	0	0
$\kappa_2(t_2'')$	0	0	0	1

An Example

	L	R
T	$1, 1$	$2, 0$
B	$0, 0$	$0, 1$

	Lt_2'	Rt_2'	Lt_2''	Rt_2''
$\kappa_1(t_1')$	1	0	0	0
$\kappa_1(t_1'')$	0	0	0	1

	Tt_1'	Bt_1'	Tt_1''	Bt_1''
$\kappa_2(t_2')$	1	0	0	0
$\kappa_2(t_2'')$	0	0	0	1

- $R_1 = \{Tt_1', Tt_1''\}$; $R_2 = \{Lt_2', Rt_2''\}$.

An Example

	L	R
T	$1, 1$	$2, 0$
B	$0, 0$	$0, 1$

	Lt_2'	Rt_2'	Lt_2''	Rt_2''
$\kappa_1(t_1')$	1	0	0	0
$\kappa_1(t_1'')$	0	0	0	1

	Tt_1'	Bt_1'	Tt_1''	Bt_1''
$\kappa_2(t_2')$	1	0	0	0
$\kappa_2(t_2'')$	0	0	0	1

- $R_1 = \{Tt_1', Tt_1''\}$; $R_2 = \{Lt_2', Rt_2''\}$.
- $B_1(R_2) = A_1 \times T_1$; $B_2(R_1) = A_2 \times \{t_2'\}$.

An Example

	L	R
T	$1,1$	$2,0$
B	$0,0$	$0,1$

	Lt_2'	Rt_2'	Lt_2''	Rt_2''
$\kappa_1(t_1')$	1	0	0	0
$\kappa_1(t_1'')$	0	0	0	1

	Tt_1'	Bt_1'	Tt_1''	Bt_1''
$\kappa_2(t_2')$	1	0	0	0
$\kappa_2(t_2'')$	0	0	0	1

- $R_1 = \{Tt_1', Tt_1''\}$; $R_2 = \{Lt_2', Rt_2''\}$.
- $B_1(R_2) = A_1 \times T_1$; $B_2(R_1) = A_2 \times \{t_2'\}$.
- $B_1B_2(R_1) = A_1 \times \{t_1'\}$, $R_1 \cap B_1(R_2) \cap B_1B_2(R_1) = \{Tt_1'\}$.

An Example

	L	R
T	$1, 1$	$2, 0$
B	$0, 0$	$0, 1$

	Lt'_2	Rt'_2	Lt''_2	Rt''_2
$\kappa_1(t'_1)$	1	0	0	0
$\kappa_1(t''_1)$	0	0	0	1

	Tt'_1	Bt'_1	Tt''_1	Bt''_1
$\kappa_2(t'_2)$	1	0	0	0
$\kappa_2(t''_2)$	0	0	0	1

- $R_1 = \{Tt'_1, Tt''_1\}$; $R_2 = \{Lt'_2, Rt''_2\}$.
- $B_1(R_2) = A_1 \times T_1$; $B_2(R_1) = A_2 \times \{t'_2\}$.
- $B_1B_2(R_1) = A_1 \times \{t'_1\}$, $R_1 \cap B_1(R_2) \cap B_1B_2(R_1) = \{Tt'_1\}$.
- $B_2B_1(R_2) = A_2 \times T_2$, $R_2 \cap B_2(R_1) \cap B_2B_1(R_2) = \{Lt'_2\}$.

Another Example

	L	R
T	$3, 1$	$0, 0$
B	$0, 0$	$1, 3$

	Lt_2'	Rt_2'	Lt_2''	Rt_2''
$\kappa_1(t_1')$	$\frac{1}{4}$	0	0	$\frac{3}{4}$
$\kappa_1(t_1'')$	0	1	0	0

	Tt_1'	Bt_1'	Tt_1''	Bt_1''
$\kappa_2(t_2')$	1	0	0	0
$\kappa_2(t_2'')$	0	1	0	0

Another Example

	L	R
T	$3,1$	$0,0$
B	$0,0$	$1,3$

	Lt_2'	Rt_2'	Lt_2''	Rt_2''
$\kappa_1(t_1')$	$\frac{1}{4}$	0	0	$\frac{3}{4}$
$\kappa_1(t_1'')$	0	1	0	0

	Tt_1'	Bt_1'	Tt_1''	Bt_1''
$\kappa_2(t_2')$	1	0	0	0
$\kappa_2(t_2'')$	0	1	0	0

- $R_1 = \{Tt_1', Bt_1', Bt_1''\}$; $R_2 = \{Lt_2', Rt_2''\}$.

Another Example

	L	R
T	$3, 1$	$0, 0$
B	$0, 0$	$1, 3$

	Lt_2'	Rt_2'	Lt_2''	Rt_2''
$\kappa_1(t_1')$	$\frac{1}{4}$	0	0	$\frac{3}{4}$
$\kappa_1(t_1'')$	0	1	0	0

	Tt_1'	Bt_1'	Tt_1''	Bt_1''
$\kappa_2(t_2')$	1	0	0	0
$\kappa_2(t_2'')$	0	1	0	0

- $R_1 = \{Tt_1', Bt_1', Bt_1''\}$; $R_2 = \{Lt_2', Rt_2''\}$.
- $B_1(R_2) = A_1 \times \{t_1'\}$; $B_2(R_1) = A_2 \times T_2$.

Another Example

	L	R
T	$3, 1$	$0, 0$
B	$0, 0$	$1, 3$

	Lt_2'	Rt_2'	Lt_2''	Rt_2''
$\kappa_1(t_1')$	$\frac{1}{4}$	0	0	$\frac{3}{4}$
$\kappa_1(t_1'')$	0	1	0	0

	Tt_1'	Bt_1'	Tt_1''	Bt_1''
$\kappa_2(t_2')$	1	0	0	0
$\kappa_2(t_2'')$	0	1	0	0

- $R_1 = \{Tt_1', Bt_1', Bt_1''\}$; $R_2 = \{Lt_2', Rt_2''\}$.
- $B_1(R_2) = A_1 \times \{t_1'\}$; $B_2(R_1) = A_2 \times T_2$.
- $B_1B_2(R_1) = A_1 \times T_1$, $R_1 \cap B_1(R_2) \cap B_1B_2(R_1) = A_1 \times \{t_1'\}$.

Another Example

	L	R
T	$3,1$	$0,0$
B	$0,0$	$1,3$

	Lt_2'	Rt_2'	Lt_2''	Rt_2''
$\kappa_1(t_1')$	$\frac{1}{4}$	0	0	$\frac{3}{4}$
$\kappa_1(t_1'')$	0	1	0	0

	Tt_1'	Bt_1'	Tt_1''	Bt_1''
$\kappa_2(t_2')$	1	0	0	0
$\kappa_2(t_2'')$	0	1	0	0

- $R_1 = \{Tt_1', Bt_1', Bt_1''\}$; $R_2 = \{Lt_2', Rt_2''\}$.
- $B_1(R_2) = A_1 \times \{t_1'\}$; $B_2(R_1) = A_2 \times T_2$.
- $B_1B_2(R_1) = A_1 \times T_1$, $R_1 \cap B_1(R_2) \cap B_1B_2(R_1) = A_1 \times \{t_1'\}$.
- $B_2B_1(R_2) = A_2 \times T_2$, $R_2 \cap B_2(R_1) \cap B_2B_1(R_2) = \{Lt_2', Rt_2''\}$.

Common Belief in Rationality

- Define $R := R_1 \times R_2 \subset A_1 \times T_1 \times A_2 \times T_2$ and $B(R) := B_1(R_2) \times B_2(R_1) \subset A_1 \times T_1 \times A_2 \times T_2$.

Definition

The event that there is rationality and common belief in rationality is

$$RCBR := R \cap B(R) \cap B^2(R) \cdots = R \cap CB(R).$$

- In simple iterated deletion example, $RCBR = \{(Tt_1', Lt_2')\}$.
- In battle of the sexes, $RCBR = A_1 \times \{(t_1', LT_2'), (t_1', RT_2'')\}$.

Theorem

Fix a game.

1. *In any model, the projection of the event RCBR on A is a full best response set.*
2. *In any complete model (i.e., the κ_i are onto), the projection of the event RCBR on A is the set of rationalizable strategy profiles.*
3. *For every full best response set B, there is a finite model in which the projection of the event RCBR on A is B.*

What about Iterated Admissibility?

	L	M	R
U	$1,1$	$1,1$	$0,0$
D	$1,1$	$0,0$	$1,1$

- Iterated admissibility leads to UL, DL.
- But order matters!

BERNHEIM, B. D. (1984): “Rationalizable Strategic Behavior,” Econometrica, 52(4), 1007–1028.

DEKEL, E., AND M. SINISCALCHI (2014): “Epistemic game theory,” in Handbook of Game Theory, volume 4, ed. by H. P. Young, and S. Zamir. North Holland.

PEARCE, D. (1984): “Rationalizable Strategic Behavior and the Problem of Perfection,” Econometrica, 52(4), 1029–50.