Robustness of Strategic Behavior

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Preliminaries

- Fix a game $u : A \times \Theta \to \mathbb{R}^2$, and a type model (Θ, T, κ).
  - Note that A_k is not in i's space of uncertainty, and the types are not epistemic types.
- For each i and t_i, set $A_i^0(t_i) = A_i$.
- For $\ell > 0$, iteratively define $A_i^\ell(t_i)$ as $a_i \in A_i^\ell(t_i)$ if, and only if,

$$a_i \in \arg\max_{a_i'} u_i(a_i', \mathrm{marg}_{\Theta \times A_k} \mu)$$

for some $\mu \in \Delta(\Theta \times A_k \times T_k)$ satisfying

$$\mathrm{marg}_{\Theta \times T_k} \mu = \kappa_i(t_i) \text{ and } \mu(a_k \in A_k^{\ell-1}(t_k)) = 1.$$

  - Note that μ allows for correlation between $\theta \in \Theta$ and $a_k \in A_k$.

ICR

Definition (Dekel, Fudenberg, and Morris, 2007)

The set of interim correlated rationalizable (ICR) actions for i at t_i is

$$ICR_i[t_i] := \bigcap_{k=0}^{\infty} A_i^k(t_i).$$

A strategy $\sigma_i : T_i \to A_i$ is (interim correlated) rationalizable if $\sigma_i(t_i) \in ICR_i[t_i]$ for all $t_i \in T_i$.

A model (Θ, T, κ) is dominance solvable if $|ICR_i[t_i]| = 1$ for all $t_i \in T_i$ and all i.

A type $t_i^* \in T_i^*$ is dominance solvable if there is a dominance solvable model (Θ, T, κ) and a type $t_i \in T_i$ satisfying $h_i(t_i) = t_i^*$.

ICR

- ICR depends only on the hierarchy of beliefs, i.e.,

$$ICR_i[t_i] = ICR_i[h_i(t_i)].$$

- ICR is characterized by common belief in rationality.
- The set of actions rationalizable for a type t coincides with the set of actions that can be played in some Bayesian equilibrium on some type space, by some type that has the same hierarchy of beliefs as t.

Redundant Types

θ	L	R
U	$1, 0$	$0, 0$
D	$\frac{3}{5}, 0$	$\frac{3}{5}, 0$

θ'	L	R
U	$0, 0$	$1, 0$
D	$\frac{3}{5}, 0$	$\frac{3}{5}, 0$

- Both L and R are optimal for 2. If 1 assigns $\frac{1}{2}$ to θL and $\frac{1}{2}$ to $\theta' R$, then U is uniquely optimal.

Common Prior Models

θ	t_2'	t_2''
t_1'	$\frac{1}{6}$	$\frac{1}{12}$
t_1''	$\frac{1}{12}$	$\frac{1}{6}$

θ'	t_2'	t_2''
t_1'	$\frac{1}{12}$	$\frac{1}{6}$
t_1''	$\frac{1}{6}$	$\frac{1}{12}$

(a) $\widehat{T}_1 \times \widehat{T}_2 =: \widehat{T}$

θ	t_2
t_1	$\frac{1}{2}$

θ'	t_2
t_1	$\frac{1}{2}$

(b) $\widetilde{T}_1 \times \widetilde{T}_2 =: \widetilde{T}$

- In $\widetilde{T}$, common belief that both players assign equal probability to θ and θ'.
- Also true in $\widehat{T}$. The type spaces $\widehat{T}_i$ contain redundant types.
- But equilibria differ!
- $ICR_1[t_1] = \{U, D\}$ and $ICR_2[t_2] = \{L, R\}$.

Richness

Assumption

For all i and all $a_i \in A_i$, there exists $\theta^{a_i} \in \Theta$ such that

$$u_i(a_i, a_{-i}, \theta^{a_i}) > u_i(a_i', a_{-i}, \theta^{a_i}), \qquad \forall a_i' \neq a_i \text{ and all } a_{-i}.$$

A Structure Theorem

Theorem (Weinstein and Yildiz, 2007)

Under the richness assumption, for any $t_i \in T_i^{\mathrm{F}}$, and any $a_i \in ICR_i[t_i]$, there is a sequence of finite dominance solvable types $t_i^m \in T_i^{\mathrm{F}}$ such that $t_i^m \to t_i$ and $A_i^{\infty}[t_i^m] = \{a_i\}$ for each m.
The set of dominance solvable types is open and dense in the universal type space.

- In particular, for each t_i^m from the proposition, there is a neighborhood of t_i^m (in the product topology) for which all types in that neighborhood are dominance solvable with the same action as the unique rationalizable action.

Implications

- Any refinement of rationalizability depends upon arbitrarily high order beliefs: Consider a complete information game with multiple Nash equilibria $\hat{a}$ and $\tilde{a}$ (they can even be strict), with $\hat{a}_i \neq \tilde{a}_i$. Suppose at t_i, player i believes it is common belief that the game is the complete information game. Then, trivially, $t_i \in T_i^F$ and $\hat{a}_i, \tilde{a}_i \in ICR_i[t_i]$. Moreover, there are two sequences of dominance solvable types, $(\hat{t}_i^m)_m$ and $(\tilde{t}_i^m)_m$, such that
  1. $\hat{a}_i$ is the unique rationalizable for each $\hat{t}_i^m$,
  2. $\tilde{a}_i$ is the unique rationalizable for each $\tilde{t}_i^m$, and
  3. both $\hat{t}_i^m$ and $\tilde{t}_i^m$ converge to t_i in the product topology, and so for any K and $\varepsilon > 0$, there exists M such that for all $m > M$ and all $k \leq K$, the level k belief under $\hat{t}_i^m$ and $\tilde{t}_i^m$ are ε-close.

If the modeler does not observe the entire hierarchy of beliefs precisely, for any rationalizable action, the modeler cannot rule out an open set of types for whom that action is uniquely rationalizable and so survives any sensible refinement of rationalizability.

Robust Multiplicity?

- Weinstein and Yildiz (2007) result relies on the assumption that all players are unlimited in their reasoning ability.
- If the set of possible payoff functions is sufficiently rich, then robust multiplicity is consistent with an infinite depth of reasoning and almost-common belief in an infinite depth. That is, given a set A' of actions with $|A'| > 1$, there exist types h^m, $m = 1, 2, \ldots$, with an infinite depth of reasoning and mth-order mutual belief in an infinite depth for whom A' is robustly rationalizable (Heifetz and Kets, 2018)

The Strategic Topology

- Consider games $G : A \times \Theta \to [-M, M]^n$.
- Till this point, we have been fixing the game and varying type spaces. Here, we will be fixing the type space, T, on a finite space of parameters Θ and varying the games G.
- Set of ε-interim correlated rationalizable actions $ICR_i[t_i, G, \varepsilon]$ is just ICR_i at t_i in the game G, with BR_i replaced by ε-BR_i.
- Like ICR, its ε version only depends on the hierarchy of beliefs.

The Topology

Definition (Dekel, Fudenberg, and Morris, 2006)

A sequence of types $(t_i^n)_n$ converges strategically to t_i if for every game G, and every action a_i,

$$a_i \in ICR[t_i, G, 0] \iff \forall \varepsilon > 0,\ \exists N_{\varepsilon,G},\ \forall n \geq N_{\varepsilon,G},\ a_i \in ICR[t_i^n, G, \varepsilon].$$

The strategic topology is the topology of strategic convergence in T_i^*.

If $\iff$ is weakened to $\Longleftarrow$ (a form of upper hemicontinuity), the resulting topology is the product topology.

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  4. But, the set of critical types is open and dense in the strategic topology (Chen, Di Tillio, Faingold, and Xiong, 2017).
- The set of non-common prior types contains a set of types that is open and dense in T_i^* under the strategic topology.

Full Surplus Extraction

- Parameter space Θ_i, $|\Theta_i| < \infty$.
- Social alternatives X.
- Quasi linear preferences, $v_i(x, \theta_i) + \tau_i$.
- Suppose there is a common prior μ on $\Theta := \prod_i \Theta_i$.

Definition

The prior μ satisfies the Crémer-McLean condition if there does not exist i, θ_i and $\lambda_i : \Theta_i \setminus \{\theta_i\} \to \mathcal{R}_+$ for which

$$\mu(\theta_{-i} \mid \theta_i) = \sum_{\theta_i' \neq \theta_i} \lambda_i(\theta_i')\mu(\theta_{-i} \mid \theta_i') \qquad \forall \theta_{-i} \in \Theta_{-i}.$$

Theorem (Crémer and McLean, 1988)

Suppose μ satisfies the Crémer-McLean condition. To any direct mechanism (ξ_X, τ), there is an equivalent Bayesian incentive compatible direct mechanism (ξ_X, τ'); that is,

1. *the two mechanisms have the same allocation rule, ξ_X, and*
2. *the two mechanisms have the same interim expected payments, i.e.,*

$$\sum_{\theta_{-i}} \tau_i(\theta_i, \theta_{-i})\mu(\theta_{-i} \mid \theta_i) = \sum_{\theta_{-i}} \tau_i'(\theta_i, \theta_{-i})\mu(\theta_{-i} \mid \theta_i)$$

for all i and θ_i.

- Independent private information violates the Crémer-McLean condition. If it is satisfied, mechanism designer can extract all the surplus (hence full surplus extraction).
- The space of distribution functions on Θ is a finite dimensional Euclidean space, and so the set of distributions that satisfy the Crémer-McLean condition is generic.
- The key property is that under the the Crémer-McLean condition, different θ_i's have different beliefs over θ_{-i} (i.e., “beliefs determine preferences,” which can be used to elicit the beliefs using “side bets”.

Robust Mechanism Design

- Parameter space Θ_i, $|\Theta_i| < \infty$.
- Outcomes Z, general preferences $u_i : Z \times \Theta \to \mathbb{R}$.
- Type space $\mathcal{T} = \big((T_i, p_i, b_i)_i\big)$, where

$$p_i : T_i \to \Theta_i,$$

so that $p(t_i)$ is i's payoff type at $t_i \in T_i$, and

$$b_i : T_i \to \Delta(T_{-i}),$$

so that $b_i(t_i)$ is i's belief type at t_i (Bergemann and Morris, 2005).

Typically, applications restrict attention to payoff type spaces:

Definition

A type space $\mathcal{T} = \big((T_i, p_i, b_i)_i\big)$ is a payoff type space if $T_i = \Theta_i$ and p_i is the identity map, for all i.

Belief Types

Theorem

Suppose the type space is finite with a full support common prior μ. Then, for all $\beta \in \prod_i b_i(T_i) = \prod_i \Delta(T_{-i})$,

$$\mu(\theta \mid \beta) = \prod_i \mu(\theta_i \mid \beta).$$

Proof

- We first verify that

$$\mu(\theta_{-1} \mid \theta_1, \beta) = \mu(\theta_{-1} \mid \theta_1', \beta) \qquad \forall \theta_1, \theta_1' \in \Theta_1, \beta \in \prod_i b_i(T_i).$$

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- If not, then there exists $t, t' \in T$ such that

$$\mu(\theta_{-1} \mid t) \neq \mu(\theta_{-1} \mid t') \quad \text{and} \quad b(t) = b(t') = \beta, p_1(t_1) = \theta_1, p_1(t_1') = \theta_1'.$$

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- But since t and t' have the same belief types, the beliefs over θ_{-1} must be the same, contradiction.

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- But since t and t' have the same belief types, the beliefs over θ_{-1} must be the same, contradiction.
- But then,

$$\mu(\theta \mid \beta) = \mu(\theta_1 \mid \beta)\mu(\theta_{-1} \mid \beta).$$

$\square$

Solution Concepts

- A direct mechanism is $f : T \to Z$.

Definition

A direct mechanism f is interim incentive compatible if, for all for all i and $t_i \in T_i$,

$$\int_{t_{-i}} u_i(f(t_i, t_{-i}), p(t_i, t_{-i})) db_i(t_i) \geq \int_{t_{-i}} u_i(f(\hat{t}_i, t_{-i}), p(t_i, t_{-i})) db_i(t_i) \quad \forall \hat{t}_i \in T_i.$$

- Equivalently, f is interim incentive compatible if the incomplete information game induced by f has truthtelling as a Bayesian equilibrium.

- A social choice correspondence (ssc) is $F : \Theta \rightrightarrows Z$.

Definition

A scc F is interim implementable on $\mathcal{T}$ if there exists an interim incentive compatible direct mechanism $f : T \to Z$ such that $f(t) \in F(p(t))$ for all $t \in T$.

- A social choice function (scf) is $\xi : \Theta \to Z$.
- An scf can also be viewed as reduced direct mechanism.

Definition

A reduced direct mechanism $\xi : \Theta \to Z$ is ex post incentive compatible (EPIC) if, for all i and all $\theta \in \Theta$,

$$u_i(\xi(\theta), \theta) \geq u_i(\xi(\theta_i', \theta_{-i}), \theta) \qquad \forall \theta_i' \in \Theta_i.$$

A scc F is ex post implementable if a selection is EPIC.

Theorem

If F is ex post implementable, then F is interim implementable on all type spaces.

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Definition

A reduced direct mechanism $\xi : \Theta \to Z$ is dominant strategy incentive compatible if, for all i and all $\theta \in \Theta$,

$$u_i(\xi(\theta_i, \theta'_{-i}), \theta) \geq u_i(\xi(\hat{\theta}_i, \theta'_{-i}), \theta) \qquad \forall \theta'_{-i} \in \Theta_{-i}, \hat{\theta}_i \in \Theta_i.$$

In private value settings ($u_i(z, (\theta_i, \theta_{-i})) = u_i(z, (\theta_i, \theta'_{-i}))$ for all z, θ_i, θ_{-i}, and θ'_{-i}), ex post and dominant strategy incentive compatibility are equivalent.

Interdependent Value Example

- Single good allocation problem with values ($\gamma \leq 1$):

$$v_i(\theta) = \theta_i + \gamma \sum_{k \neq i} \theta_k .$$

- Generalized VCG: players announce their θ_i's, the highest announcement receives the object, and pays

$$\max_{k \neq i} \hat{\theta}_k + \gamma \sum_{k \neq i} \hat{\theta}_k$$

where i is the identity of the winning bidder. The winning bidder's utility is

$$\theta_i + \gamma \sum_{k \neq i} \theta_k - \left\{ \max_{k \neq i} \hat{\theta}_k + \gamma \sum_{k \neq i} \hat{\theta}_k \right\} .$$

- If the other bidders tell the truth, the utility is

$$\theta_i - \max_{k \neq i} \theta_k ,$$

and reporting truthfully is optimal.

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