ONLINE APPENDIX: OMITTED PROOFS FOR RESULTS IN "OPTIMAL PROVISION OF MULTIPLE EXCLUDABLE PUBLIC GOODS" FANG AND NORMAN (2010)

A Relevant Material from Fang and Norman (2006a) Referenced in the Paper

A.1 The Environment

There are M *excludable* and indivisible public goods, labeled by $j \in \mathcal{J} = \{1, ..., M\}$ and n agents, indexed by $i \in \mathcal{I} = \{1, ..., n\}$. The cost of providing good j, denoted $C^j(n)$, is independent of which of the other goods are provided. Notice that $C^j(n)$ depends on n, the number of *agents* in the economy, and not on the number of *users*. This assumption captures the fully non-rival nature of the public goods. Indexing the cost by n also enables us to analyze large economies without making the public goods a "free lunch" in the limit. We therefore assume that $\lim_{n\to\infty} C^j(n)/n = c^j > 0$.[1]

Agent i is described by a valuation for each good $j \in \mathcal{J}$. Her type is given by a vector $\theta_i = (\theta_i^1, ..., \theta_i^M) \in \Theta \subset R^M$, and her preferences are represented by the utility function,

$$\sum_{j\in\mathcal{J}} \mathrm{I}_i^j \theta_i^j - t_i, \tag{B1}$$

where I_i^j is a dummy variable taking value 1 when i consumes good j and 0 otherwise, and t_i is the quantity of the numeraire good transferred from i to the mechanism designer. Preferences over lotteries are of the expected utility form.

The type θ_i is private information to agent i. Unlike much of the bundling literature, valuations for different goods for any agent are allowed to be correlated. However, types are independent *across agents*. We denote by F the joint cumulative distribution over θ_i. For brevity of notation, we denote a generic *type profile* by $\theta \equiv (\theta_1, ..., \theta_n) \in \Theta^n$. In the usual fashion, we let $\theta_{-i} = (\theta_1, .., \theta_{i-1}, \theta_{i+1}, ...\theta_n)$.

A.2 Mechanisms

An outcome in our environment has three components: (1) which goods, if any, should be provided; (2) who are to be given access to the goods that are provided; and (3) how to share the costs. The set of feasible *pure* outcomes is thus

$$A = \underbrace{\{0,1\}^M}_{\substack{\text{provision/no provision}\\ \text{for each good } j}} \times \underbrace{\{0,1\}^{M\times n}}_{\substack{\text{inclusion/no inclusion}\\ \text{for each agent } i \text{ and good } j}} \times \underbrace{\mathrm{R}^n}_{\substack{\text{"taxes" for each}\\ \text{agent } i}}. \tag{B2}$$

By the revelation principle, we only consider direct mechanisms for which truth-telling is a Bayesian Nash equilibrium. A pure direct mechanism maps Θ^n onto A, and a randomized mechanism is represented in analogy with the representation of mixed strategies in Aumann (1964). That is, let $\Xi \equiv [0, 1]$ and think of $\vartheta \in \Xi$ as the outcome of a fictitious lottery where, without loss of generality, ϑ is uniformly distributed and independent of θ. A *random direct mechanism* is then a measurable mapping $\mathcal{G} : \Theta^n \times \Xi \to A$. A

[1] There is no need to give this assumption any economic interpretation. It is best viewed as a way to ensure that the public good provision problem remains "significant" when there are a large number of agents.

conceptual advantage of this representation is that it allows for a useful decomposition in the sense that we now may write $\mathcal{G} = (\{\zeta^j\}_{j\in\mathcal{J}}, \{\omega^j\}_{j\in\mathcal{J}}, \tau)$ where

$$\begin{aligned} \text{Provision Rule:} \quad & \zeta^j && : \Theta^n \times \Xi \to \{0,1\} \\ \text{Inclusion Rule:} \quad & \omega^j && : \Theta^n \times \Xi \to \{0,1\}^n \\ \text{Cost-sharing Rule:} \quad & \tau && : \Theta^n \to \mathrm{R}^n. \end{aligned} \tag{B3}$$

We refer to ζ^j as the *provision rule* for good j, and interpret $\mathrm{E}_{\Xi}\zeta^j(\theta,\vartheta)$ as the probability of provision for good j given announcements θ. The rule $\omega^j = \left(\omega_1^j, ..., \omega_n^j\right)$ is the *inclusion rule* for good j, and $\mathrm{E}_{\Xi}\omega_i^j(\theta,\vartheta)$ is interpreted as the probability that agent i gets access to good j when announcements are θ, conditional on good j being provided. Finally, $\tau = (\tau_1, ..., \tau_n)$ is the *cost-sharing rule,* where $\tau_i(\theta)$ is the transfer from agent i to the mechanism designer given announced valuation profile θ. In principle, transfers could also be random, but the pure cost-sharing rule in (B3) is without loss of generality due to risk neutrality.

A.3 The Design Problem

Let E_{-i} denote the expectation operator with respect to (θ_{-i},ϑ). A mechanism is *incentive compatible* if truth-telling is a Bayesian Nash equilibrium in the revelation game induced by $\mathcal{G}$, i.e.,$\forall i \in \mathcal{I}, \theta \in \Theta^n, \widehat{\theta}_i \in \Theta$,

$$\mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}} \zeta^j(\theta,\vartheta)\omega_i^j(\theta,\vartheta)\theta_i^j - \tau_i(\theta)\right] \geq \mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}} \zeta^j(\widehat{\theta}_i,\theta_{-i},\vartheta)\omega_i^j(\widehat{\theta}_i,\theta_{-i},\vartheta)\theta_i^j - \tau_i(\widehat{\theta}_i,\theta_{-i})\right]. \tag{IC}$$

We also require that the project be self-financing. For simplicity, this is imposed as an *ex ante balanced-budget constraint:*[2]

$$\mathrm{E}\left(\sum_{i\in\mathcal{I}} \tau_i(\theta) - \sum_{j\in\mathcal{J}} \zeta^j(\theta,\vartheta)\, C^j(n)\right) \geq 0. \tag{BB}$$

Non-rival nature of the public goods is built into (BB) because the cost depends only on the provision probabilities $\zeta^j(\theta,\vartheta)$, and not on the inclusion probabilities $\omega_i^j(\theta,\vartheta)$. Finally, we require that a voluntary participation, or *individual rationality,* condition be satisfied. Individual rationality is imposed at the interim stage as:

$$\mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}} \zeta^j(\theta,\vartheta)\omega_i^j(\theta,\vartheta)\theta_i^j - \tau_i(\theta)\right] \geq 0, \qquad \forall i \in \mathcal{I}, \theta_i \in \Theta. \tag{IR}$$

A mechanism is *incentive feasible* if it satisfies (IC), (BB) and (IR). Because utility is transferable, constrained *ex ante* Pareto efficient allocations solve a utilitarian planning problem, where a fictitious social planner seeks to maximize social surplus subject to the constraints (IC), (BB) and (IR).[3] Thus a mechanism is *constrained efficient* if it maximizes

$$\sum_{j\in\mathcal{J}} \mathrm{E}\zeta^j(\theta,\vartheta)\left[\sum_{i\in\mathcal{I}} \omega_i^j(\theta,\vartheta)\theta_i^j - C^j(n)\right], \tag{B4}$$

[2]As shown in Borgers and Norman (2008) it is without loss of generality to consider a resource constraint in *ex ante* form.

[3]The qualifier *ex ante* is crucial for the equivalence. See Ledyard and Palfrey (2007) for a characterization of *interim* efficiency.

over all incentive feasible mechanisms.[4]

Ex post efficiency requires that good j be provided if and only if $\sum_{i \in \mathcal{I}} \theta_i^j \geq C^j(n)$, and that all agents receive full access whenever a good is provided. Since the *ex post* efficient allocation requires that no agent be excluded from usage, it is implementable if and only if a *non-excludable* public good can be efficiently provided under (IC), (BB) and (IR). But we know from Mailath and Postlewaite (1990) that this is impossible except for trivial cases. Our setup is thus a second best problem.

A.4 Simple Anonymous Mechanisms

We now consider mechanisms that are less complex than general mechanisms formulated in (B3).

Definition 2 *A mechanism is called a simple mechanism if it can be expressed as a* $(2M+1)$*-tuple* $g = (\rho, \eta, t) \equiv (\{\rho^j\}_{j \in \mathcal{J}}, \{\eta^j\}_{j \in \mathcal{J}}, t)$ *such that for each good* $j \in \mathcal{J}$*,*

$$\begin{aligned} \textit{Provision Rule:} \quad & \rho^j : \Theta^n \to [0,1] \\ \textit{Inclusion Rule:} \quad & \eta^j : \Theta \to [0,1] \\ \textit{Cost-sharing Rule:} \quad & t : \Theta \to \mathrm{R}, \end{aligned} \tag{B5}$$

where ρ^j *is the provision rule for good* j*,* η^j *is the inclusion rule for good* j*, and* t *is the transfer rule, same for all agents.*

Definition 3 *A simple mechanism is called anonymous if for every* $j \in \mathcal{J}$*,* $\rho^j(\theta) = \rho^j(\theta')$ *for every* $(\theta, \theta') \in \Theta^n \times \Theta^n$ *such that* θ' *can be obtained from* θ *by permuting the indices of the agents.*

Simple mechanisms of the form (B5) are less complex than general direct revelation mechanisms of the form (B3) in several ways. First, inclusion and transfer rules are the same for all agents. Second, conditional on θ, the provision probability $\rho^j(\theta)$ is independent of all other provision probabilities as well as all inclusion probabilities. Third, the inclusion and transfer rules for any agent i are independent of θ_{-i}. Finally, all agents are treated symmetrically in terms of the transfer and inclusion rules. However, simple mechanisms as defined in (B5) do allow for asymmetric treatment of the agents in terms of the provision rule, which is why we introduce the notion of *anonymous* mechanisms in Definition 3.

Our first results show that focusing on simple anonymous mechanisms is without loss of generality:

Proposition 1 (Fang and Norman 2006a). For any incentive feasible mechanism $\mathcal{G}$ of the form (B3), there exists a simple anonymous incentive feasible mechanism g that generates the same social surplus.

The rough idea for Proposition 1 is as follows. Risk neutral agents care only about the perceived probability of consuming each good and the expected transfer. Therefore, there is nothing to gain from conditioning transfers and inclusion probabilities on θ_{-i}, or by making inclusion and provision rules conditionally dependent. Mechanisms on form (B5) are therefore sufficient. Moreover, given an incentive

[4] All these constraints are noncontroversial if the design problem is interpreted as a private bargaining agreement. If the goods are government provided, the participation constraints (IR) may seem questionable. One defense in this context is that the participation constraint is a reduced form of an environment where agents may "vote with their feet." Another defense is to view this as a reduced form for inequality aversion of the planner (see Hellwig 2005).

feasible mechanism, permuting the roles of the agents leaves the surplus unchanged and all constraints satisfied. Thus an anonymous incentive feasible mechanism that generates the same surplus as the initial mechanism can be obtained by averaging over the $n!$ permuted mechanisms.[5]

A.5 Symmetric Treatment of the Goods

Our next result, on which we rely heavily in Sections 3 and 4, identifies sufficient conditions under which it is without loss of generality to treat goods symmetrically. Given valuation profile θ and a one-to-one permutation mapping of the goods $P : \mathcal{J} \to \mathcal{J}$ with inverse P^{-1}, let $\theta_i^P = \left(\theta_i^{P^{-1}(1)}, \theta_i^{P^{-1}(2)} ..., \theta_i^{P^{-1}(M)}\right)$ denote the permutation of agent i's type by changing the role of the goods in accordance to P. The valuation profile obtained when the role of the goods is changed in accordance to P for every $i \in \mathcal{I}$ is denoted by $\theta^P \equiv \left(\theta_1^P, ..., \theta_n^P\right)$.

Definition 4 *Mechanism g is symmetric if for every θ and every permutation $P : \mathcal{J} \to \mathcal{J}$,*

1. $\rho^{P^{-1}(j)}\left(\theta^P\right) = \rho^j(\theta)$ *for every* $j \in \mathcal{J}$;
2. $\eta^{P^{-1}(j)}\left(\theta_i^P\right) = \eta^j(\theta_i)$ *for every* $j \in \mathcal{J}$;
3. $t\left(\theta_i^P\right) = t(\theta_i)$.

Note that in the definition of a symmetric mechanism, the *same permutation of goods is applied for all agents.*[6]

Clearly, the environment must be symmetric if there is to be any hope for symmetric solutions. First, provision costs need to be identical across goods, i.e. $C^j(n) = C(n)$ for all j. Second, some symmetry assumption on valuations is also necessary. Symmetry of valuations is imposed by requiring that $\theta_i = \left(\theta_i^1, \cdots, \theta_i^M\right)$ is an *exchangeable* random vector, which means that $F(\theta_i) = F\left(\theta_i'\right)$ if θ_i' is a permutation of θ_i. Notice that this still allows for valuations to be correlated.

Under these restrictions it is without loss of generality to consider symmetric mechanisms:

Proposition 2 (Fang and Norman 2006a). Suppose that θ_i is an *exchangeable* random vector and $C^j(n) = C(n)$ for all $j \in \mathcal{J}$. Then, for any simple anonymous incentive feasible mechanism g, there exists a symmetric simple anonymous incentive feasible mechanism that generates the same surplus as g.

The idea is similar to that of Proposition A.4, except that it is the roles of the goods that are permuted. For intuition, consider the case with two goods, and suppose that the two goods are treated asymmetrically. Reversing the role of the goods, an alternative mechanism that generates the same surplus is obtained. Averaging over the original and the reversed mechanism creates a symmetric mechanism

[5] The proof is more complex than simply randomizing with equal probabilities over the $n!$ permutations because inclusion and provision probabilities can be correlated.

[6] As an example, suppose that there are two agents and two goods, and that the valuation for each good is either h or l. In this case $\Theta = \{(h,h), (h,l), (l,h), (l,l)\}$. Consider the type profile $\theta = (\theta_1, \theta_2) = ((h,l), (l,h)) \in \Theta^2$. Applying the only non-identity permutation of the goods, i.e., $P(1) = 2$ and $P(2) = 1$, to all agents generates a type profile $\theta^P = \left(\theta_1^P, \theta_2^P\right) = ((l,h), (h,l))$. Definition 4 requires that the allocations for type profile $((l,h), (h,l))$ is the same as the allocation for $((h,l), (l,h))$ with goods relabeled and transfers unchanged.

where surplus is unchanged.[7] Incentive feasibility of the new mechanism follows from incentive feasibility of the original mechanism. The proof of Proposition A.5 generalizes this procedure by permuting the goods ($M!$ possibilities) and creating a symmetric mechanism by averaging over these permuted mechanisms.

PROOF OF PROPOSITION 2: The proposition follows from the following Claims B1 and B2 below.

Claim B1 *For any incentive feasible mechanism $\mathcal{G}$ of the form (B3), there exist an incentive feasible mechanism*

$$G=\left(\left(\rho^j,\eta_1^j,...,\eta_n^j\right)_{j\in\mathcal{J}},(t_i)_{i\in\mathcal{I}}\right), \tag{B6}$$

that generates the same social surplus, where $\rho^j:\Theta^n\to[0,1]$ is the provision rule for good j, $\eta_i^j:\Theta\to[0,1]$ is the inclusion rule for agent i and good j,and $t_i:\Theta\to R$ is the transfer rule for agent i.

To see this, consider an incentive feasible mechanism $\mathcal{G}$. Pick any $k\in[0,1]$ and define, for each $\theta\in\Theta^n, j\in\mathcal{J}$ and $i\in\mathcal{I}$,

$$\begin{aligned}
\rho^j(\theta) &= \mathrm{E}_\Xi\zeta^j(\theta,\vartheta)=\int_0^1\zeta^j(\theta,\vartheta)\,d\vartheta \qquad\qquad (\text{B7})\\
\eta_i^j(\theta_i) &= \begin{cases}\frac{\mathrm{E}_{-i}\zeta^j(\theta,\vartheta)\omega_i^j(\theta,\vartheta)}{\mathrm{E}_{-i}\zeta^j(\theta,\vartheta)}=\frac{\int_{\Theta_{-i}^n}\int_0^1\zeta^j(\theta,\vartheta)\omega_i^j(\theta,\vartheta)d\vartheta d\mathbf{F}(\theta_{-i})}{\int_{\Theta_{-i}^n}\int_0^1\zeta^j(\theta,\vartheta)d\vartheta d\mathbf{F}(\theta_{-i})} & \text{if } \int_{\Theta_{-i}^n}\int_0^1\zeta^j(\theta,\vartheta)\,d\vartheta d\mathbf{F}(\theta_{-i})>0\\ k & \text{if } \int_{\Theta_{-i}^n}\int_0^1\zeta^j(\theta,\vartheta)\,d\vartheta d\mathbf{F}(\theta_{-i})=0\end{cases}\\
t_i(\theta_i) &= \mathrm{E}_{-i}\tau(\theta)=\int_{\Theta_{-i}^n}\tau(\theta)\,d\mathbf{F}(\theta_{-i}).
\end{aligned}$$

This is a mechanism of the form in (B6), and we will call it G. Use of the law of iterated expectations on $\rho^j(\theta)$ and $t_i(\theta_i)$ shows that (BB) is unaffected when switching from $\mathcal{G}$ to G. It remains to show that the surplus is unchanged, and that (IC) and (IR) continue to hold under G. The utility of agent i of type $\theta_i\in\Theta$ who announces $\hat\theta_i\in\Theta$ is

$$\mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}}\zeta^j\left(\hat\theta_i,\theta_{-i},\vartheta\right)\omega_i^j\left(\hat\theta_i,\theta_{-i},\vartheta\right)\theta_i-\tau\left(\hat\theta_i,\theta_{-i}\right)\right] \text{ in mechanism } \mathcal{G} \tag{B8}$$
$$\mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}}\rho^j\left(\hat\theta_i,\theta_{-i}\right)\eta_i^j\left(\hat\theta_i\right)\theta_i-t_i\left(\hat\theta_i\right)\right] \text{ in mechanism } G. \tag{B9}$$

If $\int_{\Theta_{-i}^n}\int_0^1\zeta^j\left(\hat\theta_i,\theta_{-i},\vartheta\right)d\vartheta d\mathbf{F}(\theta_{-i})=0$, we trivially have that the payoffs in (B8) and (B9) are identical, whereas if $\int_{\Theta_{-i}^n}\int_0^1\zeta^j\left(\hat\theta_i,\theta_{-i},\vartheta\right)d\vartheta d\mathbf{F}(\theta_{-i})>0$, we have that

$$\begin{aligned}
\mathrm{E}_{-i}\rho^j\left(\hat\theta_i,\theta_{-i}\right)\eta_i^j\left(\hat\theta_i\right)\theta_i &= \mathrm{E}_{-i}\zeta^j\left(\hat\theta_i,\theta_{-i},\vartheta\right)\frac{\mathrm{E}_{-i}\omega_i^j\left(\hat\theta_i,\theta_{-i},\vartheta\right)\zeta^j\left(\hat\theta_i,\theta_{-i},\vartheta\right)}{\mathrm{E}_{-i}\zeta^j\left(\hat\theta_i,\theta_{-i},\vartheta\right)} \qquad (\text{B10})\\
&= \mathrm{E}_{-i}\omega_i^j\left(\hat\theta_i,\theta_{-i},\vartheta\right)\zeta^j\left(\hat\theta_i,\theta_{-i},\vartheta\right)\theta_i.
\end{aligned}$$

Trivially, $\mathrm{E}_{-i}t_i(\theta_i)=t_i(\theta_i)=\mathrm{E}_{-i}\tau(\theta)$, which combined with (B10) implies that the payoffs in (B8) and (B9) are identical. Since the equality between (B8) and (B9) were established for any i, θ_i and $\hat\theta_i$, it

[7] Provision probabilities and taxes are given by straightforward averaging, but since inclusion and provision probabilities may be correlated the procedure is somewhat more involved for the inclusion rules.

follows that all incentive and participation constraints (IC) and (IR) hold for mechanism G given that they are satisfied in mechanism $\mathcal{G}$. Moreover, [again by (B10)]

$$\mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}}\rho^{j}\left(\theta\right)\eta_{i}^{j}\left(\theta_{i}\right)\theta_{i}\right]=\mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}}\omega_{i}^{j}\left(\theta,\vartheta\right)\zeta^{j}\left(\theta,\vartheta\right)\theta_{i}\right], \tag{B11}$$

so it follows by integration over Θ and summation over i that

$$\mathrm{E}\left[\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\rho^{j}\left(\theta\right)\eta_{i}^{j}\left(\theta_{i}\right)\theta_{i}\right]=\mathrm{E}\left[\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\zeta^{j}\left(\theta,\vartheta\right)\omega_{i}^{j}\left(\theta,\vartheta\right)\theta_{i}\right], \tag{B12}$$

By construction, we also have that $\rho^{j}\left(\theta\right)=\mathrm{E}_{\Xi}\zeta^{j}\left(\theta,\vartheta\right)$ for every θ. Thus $\mathrm{E}\left[\rho^{j}\left(\theta\right)C^{j}\left(n\right)\right]=\mathrm{E}\left[\zeta^{j}\left(\theta,\vartheta\right)C^{j}\left(n\right)\right]$, implying that

$$\sum_{j\in\mathcal{J}}\mathrm{E}\rho^{j}\left(\theta\right)\left[\sum_{i\in\mathcal{I}}\eta_{i}^{j}\left(\theta_{i}\right)\theta_{i}-C^{j}\left(n\right)\right]=\sum_{j\in\mathcal{J}}\mathrm{E}\zeta^{j}\left(\theta,\vartheta\right)\left[\sum_{i\in\mathcal{I}}\omega_{i}^{j}\left(\theta,\vartheta\right)\theta_{i}-C^{j}\left(n\right)\right]. \tag{B13}$$

Hence, $\mathcal{G}$ and G generate the same social surplus.

Claim B2 *For every incentive feasible mechanism of the form (B6), there exists an anonymous simple incentive feasible mechanism g of the form (B5) that generates the same surplus.*

To see this, consider an incentive feasible simple mechanism G on form (B6). For $k\in\{1,....,n!\}$, let $P_k:\mathcal{I}\to\mathcal{I}$ denote the k-th permutation of the set of agents $\mathcal{I}$. Note that $P_k^{-1}(i)$ gives the index of the agent who takes agent i's position in permutation P_k. Moreover, for any given $\theta\in\Theta^n$, let $\theta^{P_K}=\left(\theta_{P_k^{-1}(1)},...,\theta_{P_k^{-1}(n)}\right)\in\Theta^n$ denote the corresponding k-th permutation of θ. For each $k\in\{1,...,n!\}$, let $G_k=\left(\left(\rho_k^j,\eta_{k1}^j,...,\eta_{kn}^j\right)_{j=1,2},t_{k1},...,t_{kn}\right)$ be given by

$$\begin{aligned}\rho_{k}^{j}\left(\theta\right)&=\rho^{j}\left(\theta^{P_k}\right)\quad\forall\,\theta\in\Theta^{n},\ j\in\mathcal{J},\\ \eta_{ki}^{j}\left(\theta_{i}\right)&=\eta_{P_k^{-1}(i)}^{j}(\theta_{i})\quad\forall\,\theta_{i}\in\Theta,\ j\in\mathcal{J},i\in\mathcal{I},\\ t_{ki}\left(\theta_{i}\right)&=t_{P_k^{-1}(i)}(\theta_{i})\quad\forall\,\theta_{i}\in\Theta,i\in\mathcal{I},\end{aligned} \tag{B14}$$

and let $g=\left(\left(\widetilde{\rho}^j,\widetilde{\eta}_1^j,...,\widetilde{\eta}_n^j\right)_{j=1,2},\widetilde{t}_1,...,\widetilde{t}_n\right)$ be given by

$$\begin{aligned}&\widetilde{\rho}^{j}\left(\theta\right)=\tfrac{1}{n!}\sum_{k=1}^{n!}\rho_{k}^{j}\left(\theta\right)\quad\forall\,\theta\in\Theta^{n},\ j\in\mathcal{J}\\ &\widetilde{\eta}_{i}^{j}\left(\theta_{i}\right)=\frac{\sum_{k=1}^{n!}\mathrm{E}_{-i}\left[\rho_{k}^{j}(\theta)\right]\eta_{ki}^{j}(\theta_{i})}{\sum_{k=1}^{n!}\mathrm{E}_{-i}\left[\rho_{k}^{j}(\theta)\right]}\quad\forall\,\theta_{i}\in\Theta,i\in\mathcal{I},\ j\in\mathcal{J}\\ &\widetilde{t}_{i}\left(\theta_{i}\right)=\tfrac{1}{n!}\sum_{k=1}^{n!}t_{ki}\left(\theta_{i}\right)\quad\forall\,\theta_{i}\in\Theta,i\in\mathcal{I}.\end{aligned}$$

We now note that: (1) for each $j\in\mathcal{J}$, $\widetilde{\rho}^{j}\left(\theta\right)=\widetilde{\rho}^{j}\left(\theta'\right)$ if θ' is a permutation of θ. This is immediate since the sets $\left\{\rho_{k}^{j}\left(\theta\right)\right\}_{k=1}^{n!}=\left\{\rho^{j}\left(P_{k}\left(\theta\right)\right)\right\}_{k=1}^{n!}$ and $\left\{\rho_{k}^{j}\left(\theta'\right)\right\}_{k=1}^{n!}=\left\{\rho^{j}\left(P_{k}\left(\theta'\right)\right)\right\}_{k=1}^{n!}$ are the same; (2) for $j\in\mathcal{J}$ and each pair $i,i'\in\mathcal{I},\widetilde{\eta}_{i}^{j}\left(\cdot\right)=\widetilde{\eta}_{i'}^{j}\left(\cdot\right)$. That is, the inclusion rules are the same for all agents. To see this, consider agent i and i', and suppose that $\theta_i=\theta_{i'}$. We then have that $\left\{\mathrm{E}_{-i}\left[\rho_{k}^{j}\left(\theta\right)\right]\eta_{ki}^{j}\left(\theta_{i}\right)\right\}_{k=1}^{n!}$ and $\left\{\mathrm{E}_{-i'}\left[\rho_{k}^{j}\left(\theta\right)\right]\eta_{ki'}^{j}\left(\theta_{i'}\right)\right\}_{k=1}^{n!}$ are identical and that $\mathrm{E}_{-i}\left[\widetilde{\rho}^{j}\left(\theta\right)\right]=\mathrm{E}_{-i'}\left[\widetilde{\rho}^{j}\left(\theta\right)\right]$; and (3) for each pair $i,i'\in\mathcal{I},\widetilde{t}_{i}\left(\cdot\right)=\widetilde{t}_{i'}\left(\cdot\right)$, which is obvious since the sets $\left\{t_{ki}\left(\theta_{i}\right)\right\}_{k=1}^{n!}$ and $\left\{t_{ki}\left(\theta_{i}'\right)\right\}_{k=1}^{n!}$ are identical. Together, (1), (2) and (3) establishes that g is anonymous and simple.

Now we show that g is incentive feasible and generates the same expected surplus as G. First, since G and G_k are identical except for the permutation of the agents, we have, for $k = 1, ..., n!$,

$$\sum_{j\in\mathcal{J}} \mathrm{E}\left\{\rho_k^j(\theta)\left[\sum_{i\in\mathcal{I}}\eta_{ki}^j(\theta_i)\theta_i^j - C^j(n)\right]\right\} = \sum_{j\in\mathcal{J}} \mathrm{E}\left\{\rho^j(\theta)\left[\sum_{i\in\mathcal{I}}\eta_i^j(\theta_i)\theta_i^j - C^j(n)\right]\right\}. \tag{B15}$$

Hence,

$$\begin{aligned}\sum_{j\in\mathcal{J}} \mathrm{E}\left\{\widetilde{\rho}^j(\theta)\left[\sum_{i\in\mathcal{I}}\widetilde{\eta}^j(\theta_i)\theta_i^j - C^j(n)\right]\right\} &= \sum_{j\in\mathcal{J}} \mathrm{E}\left\{\frac{1}{n!}\sum_{k=1}^{n!}\rho_k^j(\theta)\left[\sum_{i\in\mathcal{I}}\frac{\sum_{k=1}^{n!}\mathrm{E}_{-i}\rho_k^j(\theta)\eta_{ki}^j(\theta_i)}{\sum_{k=1}^{n!}\mathrm{E}_{-i}\rho_k^j(\theta)}\theta_i^j - C^j(n)\right]\right\}\\
= \sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}}\mathrm{E}_{\theta_i}\left\{\frac{1}{n!}\sum_{k=1}^{n!}\mathrm{E}_{-i}\rho_k^j(\theta)\eta_{ki}^j(\theta_i)\theta_i^j\right\} &- \mathrm{E}\left[\frac{1}{n!}\sum_{k=1}^{n!}\rho_k^j(\theta)\right]C^j(n)\\
= \frac{1}{n!}\sum_{k=1}^{n!}\sum_{j\in\mathcal{J}}\mathrm{E}\left\{\rho_k^j(\theta)\left[\sum_{i\in\mathcal{I}}\eta_{ki}^j(\theta_i)\theta_i^j - C^j(n)\right]\right\} &= \sum_{j\in\mathcal{J}}\mathrm{E}\left\{\rho^j(\theta)\left[\sum_{i\in\mathcal{I}}\eta_i^j(\theta_i)\theta_i^j - C^j(n)\right]\right\},\end{aligned}$$

where the last equality follows from (B15). Hence the surplus generated by g is identical to that by original mechanism G. To show that g is incentive feasible we first note that $\mathrm{E}\rho_k^j(\theta) = \mathrm{E}\rho^j(\theta)$ and $\mathrm{E}\sum_{i\in\mathcal{I}}t_{ki}(\theta_i) = \mathrm{E}\sum_{i\in\mathcal{I}}t_i(\theta_i)$ for all k, since the agents' valuations are drawn from identical distributions and G_k and G only differ in the index of the agents. Thus

$$\begin{aligned}\mathrm{E}\textstyle\sum_{i\in\mathcal{I}}\widetilde{t}_i(\theta_i) - \sum_{j\in\mathcal{J}}\mathrm{E}\widetilde{\rho}^j(\theta)C^j(n) &= \mathrm{E}\textstyle\sum_{i\in\mathcal{I}}\frac{1}{n!}\sum_{k=1}^{n!}t_{ki}(\theta_i) - \sum_{j\in\mathcal{J}}\mathrm{E}\frac{1}{n!}\sum_{k=1}^{n!}\rho_k^j(\theta)C^j(n)\\
&= \mathrm{E}\textstyle\sum_{i\in\mathcal{I}}t_i(\theta_i) - \sum_{j\in\mathcal{J}}\mathrm{E}\rho^j(\theta)C^j(n)\ ,\end{aligned}$$

so g satisfies (BB) if G does. Second, (IC) holds for any permuted mechanism, that is,

$$\mathrm{E}_{-i}\textstyle\sum_{j\in\mathcal{J}}\rho_k^j(\theta)\eta_{ki}^j(\theta_i)\theta_i^j - t_{ki}(\theta_i) \geq \mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\rho_k^j(\widehat{\theta}_i,\theta_{-i})\eta_{ki}^j(\widehat{\theta}_i,\theta_{-i})\theta_i^j - t_{ki}(\widehat{\theta}_i,\theta_{-i}) \tag{B16}$$

for all $i \in \mathcal{I}$, and $\theta_i, \widehat{\theta}_i \in \Theta$. Hence,

$$\begin{aligned}\mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\widetilde{\rho}^j(\theta)\widetilde{\eta}^j(\theta_i)\theta_i^j - \widetilde{t}(\theta_i) &= \mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\left[\frac{1}{n!}\sum_{k=1}^{n!}\rho_k^j(\theta)\right]\frac{\sum_{k=1}^{n!}\mathrm{E}_{-i}\left[\rho_k^j(\theta)\right]\eta_{ki}^j(\theta_i)}{\sum_{k=1}^{n!}\mathrm{E}_{-i}\left[\rho_k^j(\theta)\right]}\theta_i^j - \frac{1}{n!}\sum_{k=1}^{n!}t_{ki}(\theta_i)\\
= \frac{1}{n!}\sum_{k=1}^{n!}\left[\mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\rho_k^j(\theta)\eta_{ki}^j(\theta_i)\theta_i^j - t_{ki}(\theta_i)\right] &\geq \frac{1}{n!}\sum_{k=1}^{n!}\left[\mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\rho_k^j(\widehat{\theta}_i,\theta_{-i})\eta_{ki}^j(\widehat{\theta}_i,\theta_{-i})\theta_i^j - t_{ki}(\widehat{\theta}_i,\theta_{-i})\right]\\
= \mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\frac{1}{n!}\sum_{k=1}^{n!}\rho_k^j(\widehat{\theta}_i,\theta_{-i})\eta_{ki}^j(\widehat{\theta}_i,\theta_{-i})\theta_i^j - \frac{1}{n!}\sum_{k=1}^{n!}t_{ki}(\widehat{\theta}_i,\theta_{-i}) &= \sum_{j\in\mathcal{J}}\mathrm{E}_{-i}\widetilde{\rho}^j(\widehat{\theta}_i,\theta_{-i})\widetilde{\eta}_i^j\left(\widehat{\theta}_i\right)\theta_i^j - \widetilde{t}(\widehat{\theta}_i)\end{aligned} \tag{B17}$$

where the inequality follows from (B16). Hence g is satisfies (IC). Finally, g also satisfies the (IR) because (see the second line in (B17)) all the permuted mechanisms satisfy participation constraints. Proposition A.4 follows by combining Claims B1 and B2. ■

PROOF OF PROPOSITION A.5: This proof requires us to be explicit about the coordinates of the vector θ when permuting $\mathcal{J}$. We therefore need some extra notation for this proof (only).

Notation B1 *We will, with some abuse of notation, write* $\mathbf{F}(\theta) \equiv \Pi_{i\in\mathcal{I}}F(\theta_i)$ *and* $\mathbf{F}(\theta_{-i}) \equiv \Pi_{k\in\mathcal{I}\setminus i}F(\theta_k)$ *as the joint distribution of* θ *and* θ_{-i} *respectively. We write* $\theta_i^{-j} = \left(\theta_i^1, ..., \theta_i^{j-1}, \theta_i^{j+1}, ..., \theta_i^M\right)$ *for a type* *vector where good j has been removed. Analogously, $\theta^{-j} = \left(\theta_1^{-j}, ..., \theta_n^{-j}\right)$ stands for the type profile with good j coordinate removed for all agents and $\theta^j = \left(\theta_1^j, ..., \theta_n^j\right)$ is the vector collecting the valuations for good j for all agents. Furthermore, $\theta_{-i}^{-j} = \left(\theta_1^{-j}, ..., \theta_{i-1}^{-j}, \theta_{i+1}^{-j}, ..., \theta_n^{-j}\right)$ and $\theta_{-i}^j = \left(\theta_1^j, ..., \theta_{i-1}^j, \theta_{i+1}^j ..., \theta_n^j\right)$ are used for the vectors obtained respectively from θ^{-j} and θ^j by removing agent i. These conventions are used also on the distributions, so, for example, $\mathbf{F}_{-i}^{-j}$ denotes the cumulative distribution of θ_{-i}^{-j}. Conditional distributions are denoted in the natural way: for example $\mathbf{F}_{-i}^{-j}\left(\cdot\middle|\,\theta_i^j\right)$ denotes the joint distribution of θ_{-i}^{-j} conditional on θ_i^j. Since no integrals are taken over subsets of the range of integration, we also conserve space and write $\int_\theta h(\theta)\, d\mathbf{F}(\theta)$ rather than $\int_{\theta\in\Theta^n} h(\theta)\, d\mathbf{F}(\theta)$ when integrating a function h over θ and similarly for integrals over various components of θ.*

Consider a simple anonymous incentive feasible mechanism g. For $k \in \{1, ..., M!\}$ let $P_k : \mathcal{J} \to \mathcal{J}$ be the k-th permutation of $\mathcal{J}$ and $\theta_i^{P_k} = \left(\theta_i^{P_k^{-1}(1)}, ..., \theta_i^{P_k^{-1}(M)}\right) \in \Theta$ be the permutation of θ_i when the goods are permuted according to P_k. Let $\theta^{P_k} = \left(\theta_1^{P_k}, ..., \theta_n^{P_k}\right) \in \Theta^n$ denote the corresponding permutation of θ.[8] For each $k \in \{1, ..., M!\}$ define mechanism $g_k = (\left\{\rho_k^j\right\}_{j\in\mathcal{J}}, \left\{\eta_k^j\right\}_{j\in\mathcal{J}}, t_k)$, where for every $\theta \in \Theta^n$;

1. $\rho_k^j(\theta) = \rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)$ for every $j \in \mathcal{J}$;[9]
2. $\eta_k^j(\theta_i) = \eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)$ for every $j \in \mathcal{J}$;[10]
3. $t_k(\theta_i) = t\left(\theta_i^{P_k}\right)$.

By construction, each g_k is simple. Each g_k is also anonymous by the anonymity of g. Using the definition of g_k and manipulating the result by observing that the labeling of the variables is irrelevant, we get:[11]

$$\begin{aligned}
\mathrm{E}\rho_k^j(\theta)\,\eta_k^j(\theta_i)\theta_i^j &= \int_\theta \rho_k^j(\theta)\,\eta_k^j(\theta_i)\theta_i^j d\mathbf{F}(\theta) \,/\text{def of } g_k/ = \int_{\theta\in\Theta^n} \rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)\eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)\theta_i^j d\mathbf{F}(\theta) \\
&= \int_{\theta^j}\left[\int_{\theta^{-j}} \rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)\eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)\theta_i^j d\mathbf{F}^{-j}\left(\theta^{-j}\middle|\,\theta^j\right)\right] d\mathbf{F}^j\left(\theta^j\right) \qquad \text{(B18)} \\
/\text{relabel}/ &= \int_{\theta^{P_k^{-1}(j)}}\left[\int_{\left(\theta^{-j}\right)^{P_k}} \rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\,\theta_i^{P_k^{-1}(j)} d\mathbf{F}^{-j}\left(\left(\theta^{-j}\right)^{P_k}\middle|\, \underbrace{\theta^{P_k^{-1}(j)}}_{j\text{-th argument}}\right)\right] d\mathbf{F}^j\left(\theta^{P_k^{-1}(j)}\right)
\end{aligned}$$

where we recall,

$$\left(\theta^{-j}\right)^{P_k} \equiv \left(\theta^{P_k^{-1}(1)}, ..., \theta^{P_k^{-1}(j-1)}, \theta^{P_k^{-1}(j+1)}, ..., \theta^{P_k^{-1}(n)}\right). \qquad \text{(B19)}$$

[8] To illustrate, suppose $n = 2, M = 3$, and $\theta = (\theta_1, \theta_2) = ((1, 2, 0), (3, 2, 1))$. Consider, for example, purmutation k given by $P_k(1) = 2, P_k(2) = 1, P_k(3) = 3$. Then $P_k^{-1}(1) = 2, P_k^{-1}(2) = 1, P_k^{-1}(3) = 3$ and $\theta_1^{P_k} = \left(\theta_1^{P_k^{-1}(1)}, \theta_1^{P_k^{-1}(2)}, \theta_1^{P_k^{-1}(3)}\right) = (2, 1, 0), \theta_2^{P_k} = \left(\theta_2^{P_k^{-1}(1)}, \theta_2^{P_k^{-1}(2)}, \theta_2^{P_k^{-1}(3)}\right) = (2, 3, 1), \theta^{P_k} = \left(\theta_1^{P_k}, \theta_2^{P_k}\right) = ((2, 1, 0), (2, 3, 1))$.

[9] This implies that $\rho_k^{P_k^{-1}(j)}\left(\theta^{P_k}\right) = \rho^j(\theta)$ for every $j \in \mathcal{J}$.

[10] This implies that $\eta_k^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right) = \eta^j(\theta_i)$ for every $j \in \mathcal{J}$.

[11] It is important to point out that, in reaching the fourth equality in (B18), we can relabel the integrating varibles (since they are dummies) but not the integrating functions.

By exchangeability, we have

$$d\mathbf{F}^{-j}\left(\left(\theta^{-j}\right)^{P_k}\Big|\underbrace{\theta^{P_k^{-1}(j)}}_{j\text{-th (vector) argument}}\right) \tag{B20}$$
$$= d\mathbf{F}^{-j}\left(\theta^{P_k^{-1}(1)},...,\theta^{P_k^{-1}(j-1)},\theta^{P_k^{-1}(j+1)},...,\theta^{P_k^{-1}(n)}|j\text{-th (vector) argument} = \theta^{P_k^{-1}(j)}\right)$$
$$= d\mathbf{F}^{-j}\left(\theta^{-j}|j\text{-th (vector) argument} = \theta^{P_k^{-1}(j)}\right)$$
$$= d\mathbf{F}^{-P_k^{-1}(j)}\left(\theta^{-P_k^{-1}(j)}|P_k^{-1}(j)\text{-th (vector) argument } = \theta^{P_k^{-1}(j)}\right);$$

and

$$d\mathbf{F}^{j}\left(\theta^{P_k^{-1}(j)}\right) = d\mathbf{F}^{\theta^{P_k^{-1}(j)}}\left(\theta^{P_k^{-1}(j)}\right). \tag{B21}$$

Using (B18), (B20) and (B21), we have that

$$\mathrm{E}\rho_k^j(\theta)\,\eta_k^j(\theta_i)\theta_i^j \tag{B22}$$
$$= \int_{\theta^{P_k^{-1}(j)}}\left[\int_{\left(\theta^{-j}\right)^{P_k}}\rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\,\theta_i^{P_k^{-1}(j)}d\mathbf{F}^{-j}\left(\left(\theta^{-j}\right)^{P_k}\Big|\underbrace{\theta^{P_k^{-1}(j)}}_{j\text{-th argument}}\right)\right]d\mathbf{F}^{j}\left(\theta^{P_k^{-1}(j)}\right)$$
$$= \int_{\theta^{P_k^{-1}(j)}}\left[\int_{\left(\theta^{-j}\right)^{P_k}}\rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\,\theta_i^{P_k^{-1}(j)}d\mathbf{F}^{-P_k^{-1}(j)}\left(\theta^{-P_k^{-1}(j)}|\underbrace{\theta^{P_k^{-1}(j)}}_{P_k^{-1}(j)\text{-th argument}}\right)\right]d\mathbf{F}^{P_k^{-1}(j)}\left(\theta^{P_k^{-1}(j)}\right)$$
$$= \int_{\theta}\rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\,\theta_i^{P_k^{-1}(j)}d\mathbf{F}(\theta) = \mathrm{E}\rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\theta_i^{P_k^{-1}(j)}.$$

Moreover, exchangeability implies that $\mathrm{E}t_k(\theta_i) = \mathrm{E}t\left(\theta_i^{P_k}\right) = \mathrm{E}t(\theta_i)$. The ex ante utility,

$$\mathrm{E}\left[\sum_{j=1}^M \rho_k^j(\theta)\,\eta_k^j(\theta_i)\theta_i^j - t_k(\theta_i)\right] = \left[\sum_{j=1}^M \mathrm{E}\rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\theta_i^{P_k^{-1}(j)}\right] - \mathrm{E}t(\theta_i)$$
$$= \left[\sum_{j=1}^M \mathrm{E}\rho^j(\theta)\,\eta^j(\theta_i)\theta_i^j\right] - \mathrm{E}t(\theta_i), \tag{B23}$$

is thus unchanged when changing from g to g_k. The same steps as in (B18) through (B22) (only somewhat simpler) establishes that $\mathrm{E}\rho_k^j(\theta) = \mathrm{E}\rho^{P_k^{-1}(j)}$ for every j, implying that

$$\mathrm{E}\left[\sum_{j=1}^M \rho_k^j(\theta)\,C^j(n) - \sum_i t_k(\theta_i)\right] = \left[C(n)\,\mathrm{E}\sum_{j=1}^M \rho_k^j(\theta) - \sum_i \mathrm{E}t_k(\theta_i)\right] \tag{B24}$$
$$= \left[C(n)\,\mathrm{E}\sum_{j=1}^M \rho^j(\theta) - \sum_i \mathrm{E}t(\theta_i)\right] = \mathrm{E}\left[\sum_{j=1}^M \rho^j(\theta)\,C(n) - \sum_i t(\theta_i)\right],$$

so the feasibility constraint is unaffected when changing from g to g_k. Next, write Write $U(\theta_i,\theta_i';g)$ and $U(\theta_i,\theta_i';g_k)$ for the expected utility from announcing θ_i' when the true type is θ_i in mechanisms g and g_k respectively. Next, by a calculation in the same spirit as (B18) through (B22):

$$
\begin{aligned}
\mathrm{E}_{-i}\rho_k^j\left(\theta_{-i},\theta_i'\right) &= \int_{\theta_{-i}}\rho_k^j\left(\theta_{-i},\theta_i'\right)d\mathbf{F}_{-i}\left(\theta_{-i}\right)/\text{def of } g_k/ = \int_{\theta_{-i}}\rho^{P_k^{-1}(j)}\left(\left(\theta_{-i},\theta_i'\right)^{P_k}\right)d\mathbf{F}_{-i}\left(\theta_{-i}\right) \\
&= \int_{\theta_{-i}^j}\left[\int_{\theta_{-i}^{-j}}\rho^{P_k^{-1}(j)}\left(\left(\theta_{-i},\theta_i'\right)^{P_k}\right)d\mathbf{F}_{-i}^{-j}\left(\theta_{-i}^{-j}\middle|\theta_{-i}^j\right)\right]d\mathbf{F}_{-i}^j\left(\theta_{-i}^j\right) \qquad \text{(B25)}\\
/\text{relabel}/ &= \int_{\theta_{-i}^{P_k^{-1}(j)}}\left[\int_{\theta_{-i}^{-P_k^{-1}(j)}}\rho^{P_k^{-1}(j)}\left(\theta_{-i},\theta_i'^{P_k}\right)d\mathbf{F}_{-i}^{-j}\left(\left(\theta_{-i}^{-j}\right)^{P_k}\middle|\theta_{-i}^{P_k^{-1}(j)}\right)\right]d\mathbf{F}_{-i}^j\left(\theta_{-i}^{P_k^{-1}(j)}\right) \\
/\text{exchangeability}/ &= \int_{\theta_{-i}^{P_k^{-1}(j)}}\left[\int_{\theta_{-i}^{-P_k^{-1}(j)}}\rho^{P_k^{-1}(j)}\left(\theta_{-i},\theta_i'^{P_k}\right)d\mathbf{F}_{-i}^{-P_k^{-1}(j)}\left(\theta_{-i}^{-P_k^{-1}(j)}\middle|\theta_{-i}^{P_k^{-1}(j)}\right)\right]d\mathbf{F}_{-i}^j\left(\theta_{-i}^{P_k^{-1}(j)}\right) \\
&= \int_{\theta_{-i}}\rho^{P_k^{-1}(j)}\left(\theta_{-i},\theta_i'^{P_k}\right)d\mathbf{F}_{-i}\left(\theta_{-i}\right) = \mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta_{-i},\theta_i'^{P_k}\right)
\end{aligned}
$$

That is, the perceived probability of getting j when announcing θ_i' in mechanism g_k is the same as the perceived probability of getting good $P_k^{-1}(j)$ when announcing $\left(\theta_i'\right)^{P_k}$, so that

$$
\begin{aligned}
U(\theta_i,\theta_i';g_k) &= \mathrm{E}_{-i}\sum_{j=1}^M\rho_k^j\left(\theta_{-i},\theta_i'\right)\eta_k^j(\theta_i')\theta_i^j - t_k\left(\theta_i'\right) \qquad \text{(B26)}\\
&= \sum_{j=1}^M\eta_k^{P_k^{-1}(j)}((\theta_i')^{P_k})\theta_i^j\mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta_{-i},\theta_i'^{P_k}\right) - t\left(\left(\theta_i'\right)^{P_k}\right),
\end{aligned}
$$

whereas

$$
\begin{aligned}
U(\theta_i,\theta_i';g) &= \sum_{j=1}^M\eta_k^j(\theta_i')\theta_i^j\mathrm{E}_{-i}\rho_k^j\left(\theta_{-i},\theta_i'\right) - t\left(\theta_i'\right) \Rightarrow \qquad \text{(B27)}\\
&\quad U(\theta_i,\theta_i';g)\Big|_{\substack{\theta_i=\theta_i^{P_k}\\ \theta_i'=\theta_i'^{P_k}}} = \sum_{j=1}^M\eta_k^j((\theta_i')^{P_k})\theta_i^{P_k^{-1}(j)}\mathrm{E}_{-i}\rho_k^j\left(\theta_{-i},\theta_i'^{P_k}\right) - t\left(\left(\theta_i'\right)^{P_k}\right) \\
&\quad = \sum_{j=1}^M\eta_k^{P_k^{-1}(j)}((\theta_i')^{P_k})\theta_i^j\mathrm{E}_{-i}\rho_k^{P_k^{-1}(j)}\left(\theta_{-i},\theta_i'^{P_k}\right) - t\left(\left(\theta_i'\right)^{P_k}\right) = U(\theta_i,\theta_i';g_k),
\end{aligned}
$$

which establishes that type θ_i who announces θ_i' in mechanism g_k gets the same utility as type $\theta_i^{P_k}$ who announces $\left(\theta_i'\right)^{P_k}$ in mechanism g. Hence incentive compatibility and individual rationality of g_k follows from incentive compatibility and individual rationality of g. Now, construct a new mechanism $\widetilde{g} = (\{\widetilde{\rho}^j\}_{j\in\mathcal{J}}, \{\widetilde{\eta}^j\}_{j\in\mathcal{J}}, \widetilde{t})$ by letting

$$
\begin{aligned}
\widetilde{\rho}^j\left(\theta\right) &= \tfrac{1}{M!}\sum_{k=1}^{M!}\rho_k^j\left(\theta\right) = \tfrac{1}{M!}\sum_{k=1}^{M!}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right) \qquad \text{(B28)}\\
\widetilde{\eta}^j\left(\theta_i\right) &= \frac{\sum_{k=1}^{M!}\eta_k^j\left(\theta_i\right)\mathrm{E}_{-i}\rho_k^j\left(\theta\right)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^j\left(\theta\right)} = \frac{\sum_{k=1}^{M!}\eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)\mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)} \\
\widetilde{t}\left(\theta_i\right) &= \tfrac{1}{M!}t_k\left(\theta_i\right) = \tfrac{1}{M!}t\left(\theta_i^{P_k}\right)
\end{aligned}
$$

let $P:\mathcal{J}\to\mathcal{J}$ be an arbitrary perturbation of the set of goods. Then,

$$
\widetilde{\rho}^{P^{-1}(j)}\left(\theta^P\right) = \tfrac{1}{M!}\sum_{k=1}^{M!}\rho^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta^P\right)^{P_k}\right) = \tfrac{1}{M!}\sum_{k=1}^{M!}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right) = \widetilde{\rho}^j\left(\theta\right), \qquad \text{(B29)}
$$

since the sets $\left\{\rho^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta^P\right)^{P_k}\right)\right\}_{k=1}^{M!}$ and $\left\{\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)\right\}_{k=1}^{M!}$ are identical. Furthermore

$$
\begin{aligned}
\widetilde{\eta}^{P^{-1}(j)}\left(\theta_i^P\right) &= \frac{\sum_{k=1}^{M!}\eta_k^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta_i^P\right)^{P_k}\right)\mathrm{E}_{-i}\rho_k^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta^P\right)^{P_k}\right)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta^P\right)^{P_k}\right)} \qquad \text{(B30)}\\
&= \frac{\sum_{k=1}^{M!}\eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)\mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)} = \widetilde{\eta}^j\left(\theta_i\right)
\end{aligned}
$$

for the same reason. It is obvious that $\widetilde{t}\left(\theta_i^P\right) = \widetilde{t}\left(\theta_i\right)$, which together with (B29) and (B30) establishes that $\widetilde{g}$ is symmetric. To complete the proof we need to show that $\widetilde{g}$ is incentive feasible and generates the same surplus as g. We note that

$$\begin{aligned}
&\mathrm{E}\widetilde{\rho}^j\left(\theta\right)\widetilde{\eta}^j\left(\theta_i\right)\theta_i^j = \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\mathrm{E}\rho_k^j\left(\theta\right)\frac{\sum_{k=1}^{M!}\eta_k^j(\theta_i)\mathrm{E}_{-i}\rho_k^j(\theta)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^j(\theta)}\theta_i^j \\
&= \tfrac{1}{M!}\mathrm{E}_{\theta_i}\textstyle\sum_{k=1}^{M!}\left[\mathrm{E}_{-i}\rho_k^j\left(\theta\right)\frac{\sum_{k=1}^{M!}\eta_k^j(\theta_i)\mathrm{E}_{-i}\rho_k^j(\theta)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^j(\theta)}\theta_i^j\right] = \tfrac{1}{M!}\mathrm{E}\left[\sum_{k=1}^{M!}\eta_k^j\left(\theta_i\right)\rho_k^j\left(\theta\right)\theta_i^j\right] \\
&\Rightarrow \mathrm{E}\textstyle\sum_{j=1}^{M}\left[\widetilde{\rho}^j\left(\theta\right)\widetilde{\eta}^j\left(\theta_i\right)\theta_i^j - \widetilde{t}\left(\theta_i\right)\right] = \tfrac{1}{M!}\sum_{k=1}^{M!}\mathrm{E}\left[\sum_{j=1}^{M}\eta_k^j\left(\theta_i\right)\rho_k^j\left(\theta\right)\theta_i^j - t_k\left(\theta_i\right)\right] \\
&= \mathrm{E}\left[\textstyle\sum_{j=1}^{M}\eta^j\left(\theta_i\right)\rho^j\left(\theta\right)\theta_i^j - t\left(\theta_i\right)\right],
\end{aligned}$$

where the last equality follows from (B22) and (B23), which establishes that the ex ante utility from $\widetilde{g}$ and g are the same for all agents. Moreover,

$$\begin{aligned}
&\mathrm{E}\left[\textstyle\sum_{j=1}^{M}\widetilde{\rho}^j\left(\theta\right)C^j\left(n\right) - \sum_{i=1}^{n}\widetilde{t}\left(\theta_i\right)\right] = \mathrm{E}\left[C\left(n\right)\sum_{j=1}^{M}\frac{1}{M!}\sum_{k=1}^{M!}\rho_k^j\left(\theta\right) - \sum_{i=1}^{n}\sum_{k=1}^{M!}\frac{1}{M!}t_k\left(\theta_i\right)\right] \\
&= \textstyle\sum_{k=1}^{M!}\mathrm{E}\left[C\left(n\right)\sum_{j=1}^{M}\rho_k^j\left(\theta\right) - \sum_{i=1}^{n}t_k\left(\theta_i\right)\right] \\
&= \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\mathrm{E}\left[\sum_{j=1}^{M}\rho^j\left(\theta\right)C\left(n\right) - \sum_{i=1}^{n}t\left(\theta_i\right)\right] = \mathrm{E}\left[\sum_{j=1}^{M}\rho^j\left(\theta\right)C^j\left(n\right) - \sum_{i=1}^{n}t\left(\theta_i\right)\right],
\end{aligned}$$

where the third equality follows from (B24). So the budget balance constraint is unaffected. All incentive compatibility constraints hold since,

$$\begin{aligned}
U(\theta_i,\theta_i';\widetilde{g}) &= \sum_{j=1}^{M}\widetilde{\eta}^j(\theta_i')\theta_i^j\mathrm{E}_{-i}\widetilde{\rho}^j\left(\theta_{-i},\theta_i'\right) - \widetilde{t}\left(\theta_i'\right) \\
&= \frac{\sum_{k=1}^{M!}\eta_k^j(\theta_i')\mathrm{E}_{-i}\rho_k^j(\theta_{-i},\theta_i')}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^j(\theta_{-i},\theta_i')}\mathrm{E}_{-i}\left[\tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\rho_k^j\left(\theta_{-i},\theta_i'\right)\right] - \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}t_k\left(\theta_i'\right) \\
&= \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\left[\eta_k^j\left(\theta_i'\right)\mathrm{E}_{-i}\rho_k^j\left(\theta_{-i},\theta_i'\right) - t_k\left(\theta_i'\right)\right] \\
&= \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}U(\theta_i,\theta_i';g_k) \;\leq \frac{1}{M!}\sum_{k=1}^{M!}U\left(\theta;g_k\right) = U(\theta;\widetilde{g}).
\end{aligned}$$

where the third equality follows from (B26). By the same calculation, $U(\theta;\widetilde{g}) = \frac{1}{M!}\sum_{k=1}^{M!}U(\theta;g_k) \geq 0$, since all participation constraints hold for each k. This completes the proof. ■

B Other Omitted Proofs

PROOF OF THE EXISTENCE OF SOLUTION TO PROBLEM (9) IN SECTION 3.1: The only variables that are not automatically in a compact set are the transfers. However, $t_i\left(\mathbf{l}\right) \leq Ml < Mh$ and $t_i\left(\theta_i\right) - t_i\left(\theta_i|l_k\right) \leq m\left(\theta_i\right)h + \left[M - m\left(\theta_i\right)\right]l < Mh$. Recursive application of (10) therefore implies that we may bound $t_i\left(\theta_i\right)$ from above by M^2h. Since this is also an upper bound for the difference between $t_i\left(\theta_i\right)$ and $t_i\left(\theta_i|l_k\right)$, it follows from (5) that we may bound $t_i\left(\theta_i\right)$ from below by $-M^2h$. Hence, existence of a solution to (9) follows from Weierstrass maximum theorem. ■

PROOF OF LEMMA 1:: Let $m\left(\theta_i\right) = m\left(\widehat{\theta}_i\right) = m$, let $\lambda_i\left(\theta_i,\theta_i'\right)$ denote the multiplier associated with one of the $m-1$ downwards adjacent constraints for type θ_i and let $\lambda_i\left(\widehat{\theta}_i,\widehat{\theta}_i'\right)$ denote the multiplier associated with one of the $m-1$ downwards adjacent constraints for type $\widehat{\theta}_i$. Proposition A.5 ensures that provision and inclusion rules are symmetric and by use of strong duality in linear programming we find that it is without loss to assume that $\lambda_i\left(\theta_i,\theta_i'\right)=\lambda_i\left(\widehat{\theta}_i,\widehat{\theta}_i'\right)$. ■

PROOF OF LEMMA 2: Fix m and let $\theta_i\in\Theta$ with $m\left(\theta_i\right)=m\left\{1,...,M-1\right\}$. Consider the incentive constraints that involves θ_i :

$$0\leq\underbrace{\sum_{\theta_{-i}}\beta_{-i}\left(\theta_{-i}\right)\sum_{j=1}^{M}\rho^j\left(\theta\right)\eta_i^j\left(\theta_i\right)\theta_i^j-t_i\left(\theta_i\right)}_{\text{Term }A}-\underbrace{\sum_{\theta_{-i}}\beta_{-i}\left(\theta_{-i}\right)\sum_{j=1}^{M}\rho^j\left(\theta_{-i},\theta_i|l_k\right)\eta_i^j\left(\theta_i|l_k\right)\theta_i^j-t_i\left(\theta_i|l_k\right)}_{\text{Term }B}\tag{B31}$$

There are m different adjacent downward deviations from θ_i (i.e. to replace a single high-value coordinate in θ_i with a low-value); thus $t_i\left(\theta_i\right)$ enters in Term A for m conditions. From Lemma 1, the multiplier associated with each of these conditions is $\lambda\left(m\right)$. Moreover, there are $M-m$ types of θ_i' with $m\left(\theta_i'\right)=m+1$ such that an adjacent downward deviation from θ_i' can "turn into" type θ_i; thus $t_i\left(\theta_i\right)$ enters in Term B for $M-m$ downwards incentive constraints for types with $m+1$ high valuations. Again from Lemma 1, the multiplier associated with each of these conditions is $\lambda\left(m+1\right)$. Finally, $t_i\left(\theta_i\right)$ appears in the balanced budget constraint (5). The first order condition with respect to $t_i\left(\theta_i\right)$ can thus be written as

$$\begin{aligned}&-m\lambda\left(m\right)+\left(M-m\right)\lambda\left(m+1\right)+\Lambda\beta_i\left(\theta_i\right)\\=\;&-m\lambda\left(m\right)+\left(M-m\right)\lambda\left(m+1\right)+\Lambda\frac{m!\left(M-m\right)!}{M!}\beta_m=0,\end{aligned}\tag{B32}$$

where we used (12) for the first equality.

When $m\left(\theta_i\right)=0$, i.e. when $\theta_i=\mathbf{l}=\left(\mathbf{l},...,\mathbf{l}\right)$, there is no adjacent downward deviation from θ_i; instead, $t_i\left(\mathbf{1}\right)$ appears in the individual rationality constraint (4) for type-$\mathbf{l}$. Thus the optimality condition with respect to $t_i\left(\mathbf{l}\right)$ is:

$$-\lambda\left(0\right)+\lambda\left(1\right)M+\Lambda\beta_i\left(\mathbf{l}\right)=0.$$

Similarly, the optimality condition with respect to $t_i\left(\mathbf{h}\right)$ where $\mathbf{h}=\left(h,....,h\right)$ is

$$-M\lambda\left(M\right)+\Lambda\beta_i\left(\mathbf{h}\right)=0.$$

For $m=M$, condition (B32) reads $M\lambda\left(M\right)=\Lambda\beta_M$, implying that Lemma 2 is also true for $m=M$. Now suppose that $\lambda\left(m\right)$ is given by the expression in (13) for some $m\leq M$. The optimality condition (B32) with respect to $t_i\left(\theta_i'\right)$ where $m\left(\theta_i'\right)=m-1$ then reads:

$$\begin{aligned}0&=-\left(m-1\right)\lambda\left(m-1\right)+\left(M-m+1\right)\lambda\left(m\right)+\frac{\left(m-1\right)!\left(M-m+1\right)!}{M!}\beta_{m-1}\\&=-\left(m-1\right)\lambda\left(m-1\right)+\frac{\left(M-m+1\right)}{m}m\lambda\left(m\right)+\frac{\left(m-1\right)!\left(M-m+1\right)!}{M!}\beta_{m-1}\\&=-\left(m-1\right)\lambda\left(m-1\right)+\frac{\left(M-m+1\right)}{m}\frac{m!\left(M-m\right)!}{M!}\Lambda\sum_{j=m}^{M}\beta_j+\frac{\left(m-1\right)!\left(M-m+1\right)!}{M!}\beta_{m-1}\\&=-\left(m-1\right)\lambda\left(m-1\right)+\frac{\left(m-1\right)!\left(M-m+1\right)!}{M!}\Lambda\sum_{j=m-1}^{M}\beta_j,\end{aligned}$$

where the third equality follows from induction hypothesis. Thus, (13) holds also for $m-1$. The result follows from induction. ■

PROOF OF PROPOSITION 1: This follows from the following two lemmas: Lemma B1 and Lemma B2.

Lemma B1 ***(Relations Among Various Incentive Constraints)***

1. *If the downward adjacent constraints in (9)* ***bind*** *at (ρ, η, t), then all upward adjacent incentive constraints in (6) are satisfied.*

2. *If (ρ, η, t) is monotonic and that all downward (upward, respectively) adjacent incentive constraints hold, then all downward (upward, respectively) constraints in (6) are satisfied.*

3. *If (ρ, η, t) is monotonic and that all downward and upward incentive constraints hold, then all incentive constraints in (6) are satisfied.*

PROOF: **[Part 1]** Consider types θ_i and θ_i' with $m(\theta_i) = m$ and $m(\theta_i') = m+1$. For ease of notation, define $U(m,m)$ and $U(m+1,m+1)$ as the payoff of truth-telling for type m and $m+1$; and denote the payoff from a type with $m+1$ high valuations to announce a type with m high valuations as $U(m+1,m)$, and the payoff from a type with m high valuations to announce $m+1$ high valuations as $U(m,m+1)$. $U(m+1,m)$ and $U(m,m+1)$ are respectively given as:

$$\begin{aligned} U(m+1,m) &= \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \Big\{P^h(\theta_{-i},m)\,mh + P^l(\theta_{-i},m)\,\eta(m)\left[(M-m-1)\,l+h\right]\Big\} - t(m), \\ U(m,m+1) &= \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \Big[P^h(\theta_{-i},m+1)(mh+l) + P^l(\theta_{-i},m+1)\,\eta(m+1)(M-m-1)\,l\Big] - t(m+1) \end{aligned}$$

We then have that

$$\begin{aligned} & U(m,m) - U(m,m+1) \\ = & \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\, P^h(\theta_{-i},m)\,mh + P^l(\theta_{-i},m)\,\eta(m)(M-m)\,l - t(m) \\ & - \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},m+1)[mh+l] + P^l(\theta_{-i},m+1)\,\eta(m+1)(M-m-1)\,l\right] - t(m+1) \\ = & \underbrace{\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\, P^h(\theta_{-i},m)\,mh + P^l(\theta_{-i},m)\,\eta(m)\{(M-m-1)\,l+h\} - t(m)}_{=U(m+1,m)} \\ & + \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\, P^l(\theta_{-i},m)\,\eta(m)(l-h) \\ & \underbrace{- \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},m+1)(m+1)\,h + P^l(\theta_{-i},m+1)\,\eta(m+1)(M-m-1)\,l\right] - t(m+1)}_{=U(m+1,m+1)} \\ & + \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},m+1)[h-l]\right] \\ = & \underbrace{U(m+1,m) - U(m+1,m+1)}_{=0 \text{ by hypothesis}} + \underbrace{\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left\{P^h(\theta_{-i},m+1) - P^l(\theta_{-i},m)\,\eta(m)\right\}[h-l]}_{\geq 0 \text{ as } P^h(\theta_{-i},m+1)-P^l(\theta_{-i},m)\eta(m)\geq 0} \\ \geq & \; 0. \end{aligned}$$

[Part 2] [Downward Incentive Constraints] The proof is by induction. Pick an arbitrary m. Assume that there is some $K < m$ such that a type with m high valuations has no incentives to pretend to be of any type with $k \in \{m-1, ..., K\}$ high valuations. It follows that

$$\begin{aligned} U(m,m) &= \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},m)\, mh + P^l(\theta_{-i},m)\,\eta(m)(M-m)\,l\right] - t(m) \\ &\geq \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},k)\, Kh + P^l(\theta_{-i},K)\,\eta(K)(M-m)\,l + (m-K)\,h\right] - t(K) \\ &= U(m,K) \end{aligned} \tag{B33}$$

is satisfied by hypothesis. By assumption the downwards adjacent incentive constraint for type K holds, implying that

$$\begin{aligned} U(K,K) &= \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},K)\, Kh + P^l(\theta_{-i},K)\,\eta(K)(M-K)\,l\right] - t(K) \\ &\geq \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},K-1)(K-1)\,h + P^l(\theta_{-i},K-1)\,\eta(K-1)(M-K)\,l + h\right] - t(K-1) \\ &= U(K,K-1) \end{aligned} \tag{B34}$$

But, the payoff of announcing type $K-1$ for type m is

$$\begin{aligned} &\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},K-1)(K-1)\,h + P^l(\theta_{-i},K)\,\eta(K-1)(M-m)\,l + (m-K+1)\,h\right] - t(K-1) \\ &= \underbrace{\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},K-1)(K-1)\,h + P^l(\theta_{-i},K)\,\eta(K-1)(M-K)\,l + h\right] - t(K-1)}_{\text{RHS in (B34)}} \\ &\quad + \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\, P^l(\theta_{-i},K-1)\,\eta(K-1)(m-K)(h-l) \\ /(\text{B34})/ &\leq \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},K)\, Kh + P^l(\theta_{-i},K)\,\eta(K)(M-K)\,l\right] - t(K) \\ &\quad + \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\, P^l(\theta_{-i},K-1)\,\eta(K-1)(m-K)(h-l) \\ &= \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},K)\, Kh + P^l(\theta_{-i},K)\,\eta(K)(M-m)\,l + (m-K)\,h\right] - t(K) \\ &\quad - \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\, P^l(\theta_{-i},K)\,\eta(K)(m-K)(h-l) \\ &\quad + \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\, P^l(\theta_{-i},K-1)\,\eta(K-1)(m-K)(h-l) \\ \Big/ \begin{matrix} (\text{B33}) \text{ and} \\ P^l(\theta_{-i},K-1)\eta(K-1) \\ \leq P^l(\theta_{-i},K)\eta(K) \end{matrix} \Big/ &\leq \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},m)\, mh + P^l(\theta_{-i},m)\,\eta(m)(M-m)\,l\right] - t(m), \end{aligned}$$

implying that m has no incentive to mimic $K-1$. By induction it follows that all downwards constraints are satisfied. ■

[Upward Incentive Constraints] The proof is by induction. Let $K > m$ and assume that

$$\begin{aligned}
U(m,m) &= \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left\{P^h(\theta_{-i},m)\, mh + P^l(\theta_{-i},m)\,\eta(m)(M-m)\, l\right\} - t(m) \\
&\geq \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left\{P^h(\theta_{-i},k)\,[mh+(k-m)\, l] + P^l(\theta_{-i},k)\,\eta(k)(M-k)\, l\right\} - t(k) \\
&= U(m,k)
\end{aligned}$$

for all $k \in \{m+1,...,K\}$. If $K = M$, all upwards constraint hold by assumption. If $K < M$, the upwards adjacent constraint for type K implies that

$$\begin{aligned}
U(K,K) &= \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\, P^h(\theta_{-i},K)\, Kh + P^l(\theta_{-i},K)\,\eta(K)(M-K)\, l - t(K) \\
&\geq \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\, P^h(\theta_{-i},K+1)\,[Kh+l] + P^l(\theta_{-i},K+1)\,\eta(K+1)(M-K-1)\, l - t(K+1) \\
&= U(K,K+1)\,.
\end{aligned} \tag{B35}$$

We then note that

$$\begin{aligned}
& U(m,m) - U(m,K+1) \;\geq\; U(m,K) - U(m,K+1) \\
=\; & \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left\{P^h(\theta_{-i},K)\,[mh+(K-m)\, l] + P^l(\theta_{-i},K)\,\eta(K)(M-K)\, l\right\} - t(K) \\
& - \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\, P^h(\theta_{-i},K+1)\,[mh+(K+1-m)\, l] + P^l(\theta_{-i},K+1)\,\eta(K+1)(M-K-1)\, l - t(K+1) \\
=\; & \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left\{P^h(\theta_{-i},K)\, Kh + P^l(\theta_{-i},K)\,\eta(K)(M-K)\, l\right\} - t(K) \\
& + \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left\{P^h(\theta_{-i},K)(K-m)(l-h)\right\} \\
& - \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\, P^h(\theta_{-i},K+1)\,[Kh+l] + P^l(\theta_{-i},K+1)\,\eta(K+1)(M-K-1)\, l - t(K+1) \\
& + \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\, P^h(\theta_{-i},K+1)(K-m)(h-l) \\
=\; & \underbrace{U(K,K) - U(K,K+1)}_{\geq 0 \text{ by (B35)}} + \underbrace{\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},K+1) - P^h(\theta_{-i},K)\right](K-m)(h-l)}_{\geq 0 \text{ by monotonicity of } P^h},
\end{aligned}$$

implying that type m has no incentive to mis-report as type $K+1$. ■

[Part 3] The solution to (9) violates an incentive constraint in (6) if there exists θ_i, θ_i' such that

$$\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^{M} \rho^j(\theta)\,\eta_i^j(\theta_i)\,\theta_i^j - t_i(\theta_i) < \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^{M} \rho^j\left(\theta_{-i},\theta_i'\right)\eta_i^j\left(\theta_i'\right)\theta_i^j - t_i\left(\theta_i'\right). \tag{B36}$$

Since θ_i is exchangeable and costs are identical for all goods, we can apply Proposition A.5 (in conjunction with Lemma 3) to conclude that it is without loss of generality to assume that there exists $\left\{\eta(m), P^h(\theta_{-i},m), P^l(\theta_{-i},m), t(m)\right\}_{m=0}^{M}$, such that:

- $\eta_i^j(\theta_i) = \eta(m)$ for every (θ_i, j) such that $\theta_i^j = l$ for good j and $m(\theta_i) = m$;
- $\rho^j\left(\theta_{-i},\theta_i'\right) = P^h(\theta_{-i},m)$ for every (θ_i, j) such that $\theta_i^j = h$ for good j and $m(\theta_i) = k$;

- $\rho^j(\theta_{-i},\theta_i') = P^l(\theta_{-i},m)$ for every (θ_i,j) such that $\theta_i^j = l$ for good j and $m(\theta_i) = k$;
- $t_i(\theta_i) = t(m)$ for every θ_i such that $\theta_i^k = h$ and $m(\theta_i) = k$.

Consider an arbitrary announcement θ_i' with $m(\theta_i') = m'$. Let $r \leq \min\{m', m\}$ be the number of coordinates such that $\theta_i^j = \theta_i'^j = h$; and let $s \leq \min\{M - m', M - m\}$ be the number of coordinates such that $\theta_i^j = \theta_i'^j = l$. We can then express the failure of an incentive constraint for the full problem in (B36) as

$$\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},m)\, mh + P^l(\theta_{-i},m)\,\eta(m)(M-m)\,l\right] - t(m) \tag{B37}$$
$$< \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left\{P^h(\theta_{-i},m')\left[rh + (m'-r)\,l\right] + P^l(\theta_{-i},m')\,\eta(m')\left[sl + (M-m'-s)\,h\right]\right\} - t(m').$$

We note that if $r < m'$ and $s < M - m'$, then it is possible to announce a type θ_i'' (that differs from θ_i') with $m(\theta_i'') = m(\theta_i') = m'$, but there are $r+1$ coordinates with $\theta_i'^j = \theta_i''^j = h$ and $s+1$ coordinates with $\theta_i'^j = \theta_i''^j = l$. The utility for agent i with type θ_i' from announcing θ_i'' is, using the mechanism described above,

$$\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left\{P^h(\theta_{-i},m')\left[(r+1)\,h + (m'-r-1)\,l\right] + P^l(\theta_{-i},m')\,\eta(m')\left[(s+1)\,l + (M-m'-s-1)\,h\right]\right\} - t(m')$$
$$= \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left\{P^h(\theta_{-i},m')\left[rh + (m'-r)\,l\right] + P^l(\theta_{-i},m')\,\eta(m')\left[sl + (M-m'-s)\,h\right]\right\} - t(m')$$
$$+ \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},m') - P^l(\theta_{-i},m')\,\eta(m')\right](h-l)$$
$$\geq \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left\{P^h(\theta_{-i},m')\left[rh + (m'-r)\,l\right] + P^l(\theta_{-i},m')\,\eta(m')\left[sl + (M-m'-s)\,h\right]\right\} - t(m')$$

where the inequality follows from the assumed monotonicity [i.e. $P^h(\theta_{-i},m') \geq P^l(\theta_{-i},m')$] and $\eta(m') \leq 1$. This is a violation of the upwards incentive constraints, a contradiction to the postulate of the Lemma that all upwards incentive constraints hold. Thus, we conclude that a failure of an incentive constraint implies that either $r = m < m'$ and $s = M - m'$, in which case an upwards incentive constraint fails or $r = m' < m$ and $s = M - m$ in which case a downwards incentive constraint fails. ■

Lemma B2 *Let (ρ,η,t) be an optimal solution to (9). If (ρ,η,t) is **not*** ex post *efficient, then every downward adjacent incentive constraint and the participation constraint for type-$\mathbf{l} = (l,...,l)$ bind.*

PROOF: Since by assumption (ρ,η,t) is not *ex post* efficient, it must be the case that $(n-1)\,h > nc$. Otherwise, the *ex post* optimal mechanism is either to provide the public goods if and only if $\theta_i^j = h$ for all i when $(n-1)\,h < nc < nh$, or to never provide any of the goods when $nh \leq nc$; in either case, the *ex post* optimal mechanism is trivially implementable under the constraints in (9). Thus we assume that $(n-1)h > nc$ in the remainder of the proof.

Suppose that a downward adjacent incentive constraint does not bind, then there exists some θ_i such that

$$\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^{M} \rho^j(\theta)\,\eta_i^j(\theta_i)\,\theta_i^j - t_i(\theta_i) > \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^{M} \rho^j(\theta_{-i},\theta_i|l_k)\,\eta_i^j(\theta_i|l_k)\,\theta_i^j - t_i(\theta_i|l_k).$$

If $\eta_i^j(\theta_i|l_k) < 1$, we can increase $\eta_i^j(\theta_i|l_k)$ slightly without upsetting the constraint. Since $(n-1)h > nc$, it follows that $\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\rho^j(\theta_{-i},\theta_i|l_k) > 0$ for every j. Hence, an increase in $\eta_i^j(\theta_i|l_k)$ increases the value of the objective function, contradicting the postulated optimality of (ρ,η,t). Suppose instead that $\eta_i^j(\theta_i|l_k) = 1$. Then, we can find some $\varepsilon > 0$ and raise the tax to $\tilde{t}_i(\theta_i) = t_i(\theta_i) + \varepsilon$ for θ_i and every θ_i' with $m(\theta_i') = m(\theta_i)$. This raises some extra revenue. We can then proceed inductively as follows. First consider types θ_i^A with $m(\theta_i^A) = m(\theta_i) - 2$. If $\eta_i^j(\theta_i^A) < 1$ we can increase the surplus by increasing $\eta_i^j(\theta_i^A)$ for θ_i^A with $m(\theta_i^A) = m(\theta_i) - 2$ and simultaneously decrease $t_i(\theta_i^B)$ for types with $m(\theta_i^B) = m(\theta_i) - 1$ so as to keep the downward incentive constraint for types with θ_i^B. If $\eta_i^j(\theta_i^A) = 1$ and $m(\theta_i) - 2 \geq 1$ repeat the argument for types θ_i^C with $m(\theta_i^C) = m(\theta_i) - 3$. The resulting increase in access probabilities is always desirable from the social welfare viewpoint. Thus by induction, we conclude that either there exists a feasible allocation with a higher surplus or that $\eta_i^j(\theta_i) = 1$ for all θ_i and j.

However, if $\eta_i^j(\theta_i) = 1$ for all θ_i and j, i.e., if there are no active use of exclusions in the solution, then the extra revenue from $\tilde{t}_i(\theta_i)$ could be used to improve the surplus by by increasing $\rho^j(\theta)$ for some θ, since the postulated mechanism (ρ,η,t) is not *ex post* efficient.

The result that the participation constraint for type-**l** must bind can be proved analogously. ■

PROOF OF PROPOSITION 2: First note that (18) is irrelevant when $m = M$. Moreover, it is immediate from (21) that the probability of provision given $(\mathbf{h},\theta_{-i})$ is always weakly higher than under (θ_i,θ_{-i}) for any $\theta_{-i}\in\Theta_{-i}$. Finally, from (18) and (21), we know that a sufficient condition for monotonicity is that $\frac{1}{\beta_m(M-m)}\max\{0, G_m(\Phi)\}$ is increasing in m on $\{0,....M-1\}$. But,

$$\frac{G_m(\Phi)}{\beta_m(M-m)} = l - \Phi\left[\frac{(h-l)}{\beta_m(M-m)}\sum_{j=m+1}^{M}\beta_j\right]. \quad ■ \tag{B38}$$

PROOF OF REMARK 1: If valuations are independent across goods, say the probability that an agent has a high valuation for any good j is α, then the probability that an agent has k high valuations is given by $\beta_k = \frac{M!}{k!(M-k)!}\alpha^k(1-\alpha)^{M-k}$. Thus,

$$\beta_{k+1} = \frac{M!}{(k+1)!(M-k-1)!}\alpha^{k+1}(1-\alpha)^{M-k-1} = \frac{(M-k)\alpha}{(k+1)(1-\alpha)}\beta_k. \tag{B39}$$

Using (B39), we have that

$$\frac{1}{\beta_{m+1}(M-m-1)}\sum_{j=m+2}^{M}\beta_j = \frac{1}{\beta_{m+1}(M-m-1)}\sum_{j=m+1}^{M-1}\beta_{j+1}$$

$$= \frac{(m+1)(1-\alpha)}{(M-m)\alpha(M-m-1)\beta_m}\sum_{j=m+1}^{M-1}\frac{(M-j)\alpha}{(j+1)(1-\alpha)}\beta_j$$

$$= \frac{1}{(M-m)\beta_m}\sum_{j=m+1}^{M-1}\frac{(m+1)(M-j)}{(j+1)(M-m-1)}\beta_j < \frac{1}{(M-m)\beta_m}\sum_{j=m+1}^{M}\beta_j.$$

where the inequality follows from the fact that $j \geq m+1$ and $\beta_M > 0$. ■

PROOF of COROLLARY 1: We need to verify that $G_{m+1}(\Phi) > 0$ is implied by $G_m(\Phi) \geq 0$, and that $G_{m-1}(\Phi) < 0$ is implied by $G_m(\Phi) \leq 0$. Both are immediate from (A12). ■

PROOF OF LEMMA 6: To prove this lemma, it turns out to be useful to consider a related auxiliary problem which aims to maximize the average probability of provision (instead of social welfare) under the same constraints as the relaxed problem (9):

$$\max_{\{\rho,\eta,t\}} \sum_{\theta\in\Theta^n} \beta(\theta) \sum_{j=1}^{M} \rho^j(\theta) \quad \text{(B40)}$$
$$\text{s.t. } (10),\ (4),\ (5).$$

We first show in Claim B3 below that the characterization of the solution to (B40) is qualitatively similar to the constrained welfare problem (9):

Claim B3 *Let (ρ, η, t) be an optimal solution to (B40) and assume that $\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^j(\theta_i, \theta_{-i}) > 0$. Then,*

1. *conditional on provision, all consumers get access to all their high valuation goods and the inclusion rule for low valuation goods is given by:*

$$\eta_i^j(\theta_i) = \eta(m) \equiv \begin{cases} 0 & \text{if } G_m(1) < 0 \\ z\in[0,1] & \text{if } G_m(1) = 0 \\ 1 & \text{if } G_m(1) > 0; \end{cases} \quad \text{(B41)}$$

2. *there exists some $\Lambda \geq 0$ such that the provision rule for good j satisfies*

$$\rho^j(\theta) = \begin{cases} 0 & \text{if } 1+\Lambda\sum_{m=0}^{M}\left[H^j(\theta,m)h + L^j(\theta,m)\frac{\max\{0,G_m(1)\}}{\beta_m(M-m)} - cn\right] < 0 \\ z\in[0,1] & \text{if } 1+\Lambda\sum_{m=0}^{M}\left[H^j(\theta,m)h + L^j(\theta,m)\frac{\max\{0,G_m(1)\}}{\beta_m(M-m)} - cn\right] = 0 \\ 1 & \text{if } 1+\Lambda\sum_{m=0}^{M}\left[H^j(\theta,m)h + L^j(\theta,m)\frac{\max\{0,G_m(1)\}}{\beta_m(M-m)} - cn\right] > 0 \end{cases} \quad \text{(B42)}$$

Proof of Claim B3: The derivation of the inclusion rules follow the analysis of the constrained welfare problem step by step and is omitted.

To derive the provision rule, note that the optimality conditions for $\rho^j(\theta)$ associated with the problem (B40) may be written as

$$\begin{aligned} &\beta(\theta) + \sum_{m=0}^{M} \lambda(m)\, m\left[H^j(\theta,m)\beta_{-i}(\theta_{-i})h + L^j(\theta,m)\eta(m)l\right] \\ &-\sum_{m=0}^{M-1}\lambda(m+1)\left\{H^j(\theta,m)\beta_{-i}(\theta_{-i})(M-m)h + L^j(\theta,m)\beta_{-i}(\theta_{-i})\eta(m)\left[(M-m)l+(h-l)\right]\right\} \\ &-\Lambda\beta(\theta)cn + \gamma^j(\theta) - \phi^j(\theta) \;=\; 0, \end{aligned}$$

together with the complementary slackness conditions. By noting that $\frac{\beta_{-i}(\theta_{-i})}{\beta(\theta)} = \frac{1}{\beta_i(\theta_i)}$, we can write the condition as

$$\begin{aligned} &1 + \sum_{m=0}^{M}\lambda(m)\left[H^j(\theta,m)\frac{m}{\beta_i(\theta_i)}h + L^j(\theta,m)\frac{m}{\beta_i(\theta_i)}\eta(m)l\right] \\ &-\sum_{m=0}^{M-1}\lambda(m+1)\left\{H^j(\theta,m)\frac{(M-m)}{\beta_i(\theta_i)}h + L^j(\theta,m)\frac{1}{\beta_i(\theta_i)}\eta(m)\left[(M-m)l+(h-l)\right]\right\} \\ &-\Lambda cn + \frac{\gamma^j(\theta)-\phi^j(\theta)}{\beta(\theta)} \;=\; 0. \end{aligned}$$

Collecting terms, we get

$$\begin{aligned}1+ & \sum_{m=0}^{M} H^{j}(\theta, m)\left\{\frac{h}{\beta_{i}(\theta_{i})}[\lambda(m) m-\lambda(m+1)(M-m)]\right\} \\ & +\sum_{m=0}^{M} L^{j}(\theta, m) \frac{\eta(m)}{\beta_{i}(\theta_{i})}\{\lambda(m) m l-\lambda(m+1)[(M-m) l+(h-l)]\} \\ & -\Lambda c n+\frac{\gamma^{j}(\theta)-\phi^{j}(\theta)}{\beta(\theta)} = 0.\end{aligned}$$

Using the difference equation for the multipliers in (14), we can simplify this further to:

$$1+\sum_{m=0}^{M} H^{j}(\theta, m) h \Lambda+\sum_{m=0}^{M} L^{j}(\theta, m)\left\{\Lambda \eta(m) l-\frac{\eta(m)}{\beta_{i}(\theta_{i})} \lambda(m+1)(h-l)\right\}-\Lambda c n+\frac{\gamma^{j}(\theta)-\phi^{j}(\theta)}{\beta(\theta)} = 0.$$

Using (A11), we can eliminate $\lambda(m+1)$ and get

$$1+\sum_{m=0}^{M} H^{j}(\theta, m) h \Lambda+\sum_{m=0}^{M} L^{j}(\theta, m)\left\{\Lambda \eta(m) l-\frac{\eta(m)(h-l)}{\beta_{m}(M-m)} \Lambda \sum_{j=m+1}^{M} \beta_{j}\right\}-\Lambda c n+\frac{\gamma^{j}(\theta)-\phi^{j}(\theta)}{\beta(\theta)} = 0.$$

Furthermore,

$$\begin{aligned}\Lambda \eta(m) l-\frac{\eta(m)(h-l)}{\beta_{m}(M-m)} \Lambda \sum_{j=m+1}^{M} \beta_{j} & = \frac{\Lambda \eta(m)}{\beta_{m}(M-m)}\left[\beta_{m}(M-m) l-(h-l) \sum_{j=m+1}^{M} \beta_{j}\right] \\ & = \frac{\Lambda}{\beta_{m}(M-m)} \max\{0, G_{m}(1)\},\end{aligned}$$

where $G_m(\cdot)$ is defined in (17). Substituting this back into (A10) gives

$$1+\Lambda \sum_{m=0}^{M}\left[H^{j}(\theta, m) h+L^{j}(\theta, m) \frac{\max\{0, G_{m}(1)\}}{\beta_{m}(M-m)}-c n\right]+\frac{\gamma^{j}(\theta)-\phi^{j}(\theta)}{\beta(\theta)}=0.$$

which, by combining with the complementary slackness conditions, gives the result. ■

Before getting into the details of the proof, it is also useful to observe that (B42) can be rewritten as

$$\rho^{j}(\theta)=\begin{cases}0 & \text{if } 1+\Lambda\left[\sum_{i=1}^{n} Z^{j}(\theta_{i})-c n\right]<0 \\ z \in[0,1] & \text{if } 1+\Lambda\left[\sum_{i=1}^{n} Z^{j}(\theta_{i})-c n\right]=0 \\ 1 & \text{if } 1+\Lambda\left[\sum_{i=1}^{n} Z^{j}(\theta_{i})-c n\right]>0,\end{cases}$$

where

$$Z^{j}(\theta^{i})=\begin{cases}h & \text{if } \theta_{i}^{j}=h \\ \frac{\max\{0, G_{m}(1)\}}{\beta_{m}(M-m)} & \text{if } \theta_{i}^{j}=l \text{ and } \theta_{i}^{k}=h \text{ for exactly } m \text{ goods } k \in \mathcal{J}.\end{cases}$$

The point with this formulation is that $\{Z^j(\theta^i)\}_{i=1}^n$ is a sequence of i.i.d. random variables. Observe that $Z^j(\theta^i)$ is bounded below by 0 and above by h, so the variance is bounded. This allows us to use the central limit theorem to establish a "generalized paradox of voting", which simply states the intuitively obvious fact that the influence of a single agent approaches zero as the number of agents goes out of bounds. To state this, we will now need to be careful about the fact that the solution depends on the number of agents in the economy, and a mechanism with n agents will therefore now be denoted by (ρ_n, η_n, t_n) and the multiplier on the resource constraint will be denoted by Λ_n.

(Proof of Lemma 6, continued): Consider a mechanism that solves (B40) first. As $Z^j\left(\theta^i\right) \in [0,h]$ for every $\theta^i \in \Theta$, it follows that for any pair $\left(\theta_i', \theta_i''\right)$ we have

$$\begin{aligned}
\mathrm{E}\left[\rho_n^j(\theta)\,|\theta_i'\right] - \mathrm{E}\left[\rho_n^j(\theta)\,|\theta_i''\right] &\leq \Pr\left[1+\Lambda_n\left(h+\sum_{k\neq i} Z^j(\theta_k)\right) - cn \geq 0\right] - \Pr\left[1+\Lambda_n\left(0+\sum_{k\neq i} Z^j(\theta_k)\right) - cn > \right. \\
&= \Pr\left[\frac{1-cn}{\Lambda_n} - h \leq \sum_{k\neq i} Z^j(\theta_k) \leq \frac{1-cn}{\Lambda_n}\right].
\end{aligned}$$

Let $\sigma = \mathrm{VAR}\left(Z^j\left(\theta^i\right)\right)$. We can then rewrite the probability statement as

$$\Pr\left[k_n \leq \frac{\sum_{k\neq i} Z^j(\theta_k) - \mathrm{E}Z^j(\theta_i)}{\sigma\sqrt{n-1}} \leq k_n + \frac{h}{\sigma\sqrt{n-1}}\right],$$

where $k_n \equiv \frac{1-cn}{\Lambda_n \sigma\sqrt{n-1}} - \frac{h}{\sigma\sqrt{n-1}} - \frac{\mathrm{E}Z^j(\theta_i)}{\sigma\sqrt{n-1}}$. As σ is finite, $\mathrm{E}\left\{Z^j(\theta_k) - \mathrm{E}Z^j(\theta_i)\right\} = 0$, and $\left\{Z^j\left(\theta^i\right)\right\}_{i=1}^n$ is an i.i.d. sequence, we know that the central limit theorem is applicable, thus $\frac{\sum_{k\neq i} Z^j(\theta_k) - \mathrm{E}Z^j(\theta_i)}{\sigma\sqrt{n-1}}$ is asymptotically distributed as a standard normal distribution. Moreover, $\frac{h}{\sigma\sqrt{n-1}} \to 0$, which together with the convergence in distribution implies that for every real number k and any $\varepsilon > 0$, there exists $N < \infty$ such that

$$\Pr\left[k \leq \frac{\sum_{k\neq i} Z^j(\theta_k) - \mathrm{E}Z^j(\theta_i)}{\sigma\sqrt{n-1}} \leq k + \frac{h}{\sigma\sqrt{n-1}}\right] \leq \frac{1}{\sqrt{2\pi}} \int_k^{k+\varepsilon} \exp\left(\frac{y^2}{2}\right) dy + \varepsilon$$

for $n \geq N$. But, the standard normal is symmetric and single-peaked, so for $n \geq N$,

$$\begin{aligned}
\Pr\left[k_n \leq \frac{\sum_{k\neq i} Z^j(\theta_k) - \mathrm{E}Z^j(\theta_i)}{\sigma\sqrt{n-1}} \leq k_n + \frac{h}{\sigma\sqrt{n-1}}\right] &\leq \frac{1}{\sqrt{2\pi}} \int_{k_n}^{k_n+\varepsilon} \exp\left(\frac{y^2}{2}\right) dy + \varepsilon \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \exp\left(\frac{y^2}{2}\right) dy + \varepsilon \to 0
\end{aligned}$$

as $\varepsilon \to 0$. The result follows.

For the case in which $\{\rho_n, \eta_n, t_n\}_{n=1}^\infty$ is a sequence from the solution to (9), we replace $Z^j(\theta_i)$ above with

$$Z_n^j(\theta_i) = \begin{cases} h & \text{if } \theta_i^j = h \\ \frac{\max\{0, G_m(\Phi_n)\}}{\beta_m(M-m)} & \text{if } \theta_i^j = l \text{ and if } \theta_i^k = h \text{ and } m(\theta_i) = m. \end{cases}$$

By choice of subsequences such that $G_m(\Phi_n) \to G_m^*$, we can approximate $\frac{\max\{0, G_m(\Phi_n)\}}{\beta_m(M-m)}$ with $\frac{\max\{0, G_m^*\}}{\beta_m(M-m)}$. The rest of the argument follows the one above step by step. ■