External Appendix to "Overcoming Participation Constraints"; Detailed Calculations For the Example in Section 6.4.

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February 28, 2006

Abstract

This document provides step by step calculations and explanations that are omitted from the presentation of the example in Section 6.4 of the main paper.

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1 Primitives

The reader may note that there are many sets of primitives that generate the same maximized surplus function. The primitives below are picked mainly to make the calculations convenient. The reader may also take note that once the maximized surplus function has been derived, there is no need to return to the primitives, since interim expected payoffs from participation in the mechanisms depend only on the maximized surplus function. However, it is still important to specify some underlying primitives: not every surplus function $s_k : \Theta_k \to R$ and reservation payoff $r_k : \Theta_k \to R^n$ are consistent with surplus maximization, so we need to demonstrate existence of primitives generating the surplus function and reservation payoff function in the example.

1.1 Costs and Utility Functions

Let $n = 2$ and assume that, for each k and $i = 1, 2$, the single issue k type space is given by $\Theta^i = \{l, m, h\}$. Assume that the set of alternative resolutions of issue k are $D_k = \left\{d_k^0, d_k^l, d_k^m, d_k^h\right\}$. In terms of interpretation it is useful to think of d_k^0 as the "status quo" outcome, whereas d_k^l, d_k^m and d_k^h are surplus maximizing alternatives for type profiles ll, mm and hh respectively.

The cost function is given by

$$C_k(d_k) = \begin{cases} 0 & \text{if } d_k = d_k^0 \\ \frac{(k+1)^2}{k} & \text{if } d_k = d_k^l \\ \frac{k(k+1)^2}{2} & \text{if } d_k = d_k^m \\ 2(k+1)^3 & \text{if } d_k = d_k^h \end{cases},$$

Assume that the valuation functions for the three types are given by;

$$v_k^i(d_k, l) = \begin{cases} 0 & \text{if } d_k = d_k^0 \\ \frac{1}{2}\frac{(k+1)^2}{k} & \text{if } d_k = d_k^l \\ 0 & \text{if } d_k = d_k^m \\ \frac{(k+1)^2}{4}\left[k - \frac{1}{k}\right] & \text{if } d_k = d_k^h \end{cases}$$

for $\theta_k^i = l$,

$$v_k^i(d_k, m) = \begin{cases} 0 & \text{if } d_k = d_k^0 \\ \frac{3}{4}\frac{(k+1)^2}{k} + \varepsilon(k+1) & \text{if } d_k = d_k^l \\ \frac{k(k+1)^2}{2} - 2\varepsilon k(k+1)(k+2) & \text{if } d_k = d_k^m \\ \frac{k(k+1)^2}{4} - 3(k+1)^3 & \text{if } d_k = d_k^h \end{cases}$$

for $\theta_k^i = m$, and

$$v_k^i(d_k, h) = \begin{cases} 0 & \text{if } d_k = d_k^0 \\ -\varepsilon(k+1) & \text{if } d_k = d_k^l \\ -\frac{k(k+1)^2}{2} & \text{if } d_k = d_k^m \\ 3(k+1)^3 - \frac{k(k+1)^2}{4} & \text{if } d_k = d_k^h \end{cases}$$

for $\theta_k^i = h$, where $\varepsilon > 0$. Further restrictions on ε will be derived below.

1.2 Reservation Payoffs

Since $C_k\left(d_k^0\right) = 0$ we take d_k^0 as the status quo outcome. Hence, $r_k^i\left(\theta_k^i\right) = v_k^i\left(d_k^0, \theta_k^i\right) = 0$ for all θ_k^i.

2 The Surplus Maximizing Rule

The primitives in the example have been constructed to make sure that: i) d_k^l is optimal given type profiles lm and ml (which generates surplus of order k), and; ii) d_k^m is optimal given type profile mm (surplus of order k^3), and; iii) that d_k^h is optimal for profiles lh, hl and hh (surplus of order k^3, and, which is of some importance, that the surplus from hh is substantially larger than that from mm). As will be seen below, this will make it possible for us to construct sequences where there is a significant effect on interim expected payoffs when types that are realized with an arbitrarily small probability opt out from the mechanism. Below, we provide the details of the derivation of the maximized surplus function.

2.1 Type Profile ll

It is immediate that $v_k^1\left(d_k^0, l\right) + v_k^2\left(d_k^0, l\right) - C_k\left(d_k^0\right) = 0$. If instead $d_k = d_k^l$ we have that

$$v_k^1\left(d_k^l, l\right) + v_k^2\left(d_k^l, l\right) - C_k\left(d_k^l\right) = 2\left[\frac{1}{2}\frac{(k+1)^2}{k}\right] - \frac{(k+1)^2}{k} = 0,$$

whereas, if $d_k = d_k^m$,

$$v_k^1\left(d_k^m, l\right) + v_k^2\left(d_k^m, l\right) - C_k\left(d_k^m\right) = -\frac{k\,(k+1)^2}{2} < 0$$

and, finally, if $d_k = d_k^h$

$$v_k^1\left(d_k^h, l\right) + v_k^2\left(d_k^h, l\right) - C_k\left(d_k^h\right) = 2\frac{(k+1)^2}{4}\left[k - \frac{1}{k}\right] - 2\,(k+1)^3 = (k+1)^2\left[\frac{k}{2} - \frac{1}{2k} - k - 2\right] < 0$$

We conclude that an efficient decision is $x_k^*(ll) = d_k^l$ (or d_k^0), which generates a maximized surplus of $s_k(ll) = 0$.

2.2 Type Profiles lm and ml

Again it follows trivially that $v_k^1\left(d_k^0, l\right) + v_k^2\left(d_k^0, m\right) - C_k\left(d_k^0\right) = 0$, whereas the surplus generated by $d_k = d_k^l$ is

$$v_k^1\left(d_k^l, l\right) + v_k^2\left(d_k^l, m\right) - C_k\left(d_k^l\right) = \frac{1}{2}\frac{(k+1)^2}{k} + \frac{3}{4}\frac{(k+1)^2}{k} + \varepsilon\,(k+1) - \frac{(k+1)^2}{k} = \frac{1}{4}\frac{(k+1)^2}{k} + \varepsilon\,(k+1)\,.$$

If $d_k = d_k^m$ is chosen

$$\begin{aligned} v_k^1(d_k^m, l) + v_k^2(d_k^m, m) - C_k(d_k^m) &= 0 + \frac{k(k+1)^2}{2} - 2\varepsilon k(k+1)(k+2) - \frac{k(k+1)^2}{2} \\ &= -2\varepsilon k(k+1)(k+2) < 0, \end{aligned}$$

and, for $d_k = d_k^h$

$$\begin{aligned} v_k^1\left(d_k^h, l\right) + v_k^2\left(d_k^h, m\right) - C_k\left(d_k^h\right) &= \frac{(k+1)^2}{4}\left[k - \frac{1}{k}\right] + \frac{k(k+1)^2}{4} - 3(k+1)^3 - 2(k+1)^3 \\ &= (k+1)^2\left[\frac{k}{2} - \frac{1}{4k} - 5k - 5\right] < 0 \end{aligned}$$

Using symmetry, we conclude that the efficient decision is $x_k^*(lm) = x^*(ml) = d_k^l$ and that the associated maximized surplus is

$$s_k(lm) = s_k(ml) = \frac{1}{4}\frac{(k+1)^2}{k} + \varepsilon(k+1)$$

2.3 Type Profile mm

Trivially, $v_k^1\left(d_k^0, m\right) + v_k^2\left(d_k^0, m\right) - C_k\left(d_k^0\right) = 0$. If instead $d_k = d_k^l$ we have that

$$v_k^1\left(d_k^l, m\right) + v_k^2\left(d_k^l, m\right) - C_k\left(d_k^l\right) = 2\left[\frac{3}{4}\frac{(k+1)^2}{k} + \varepsilon(k+1)\right] - \frac{(k+1)^2}{k} = \frac{1}{2}\frac{(k+1)^2}{k} + 2\varepsilon(k+1) > 0$$

and if $d_k = d_k^m$

$$\begin{aligned} v_k^1(d_k^m, m) + v_k^2(d_k^m, m) - C_k(d_k^m) &= 2\left[\frac{k(k+1)^2}{2} - 2\varepsilon k(k+1)(k+2)\right] - \frac{k(k+1)^2}{2} \\ &= \frac{k(k+1)^2}{2} - 4\varepsilon k(k+1)(k+2), \end{aligned}$$

while if $d_k = d_k^h$ we have that

$$v_k^1\left(d_k^h, m\right) + v_k^2\left(d_k^h, m\right) - C_k\left(d_k^h\right) = 2\left[\frac{k(k+1)^2}{4} - 3(k+1)^3\right] - 2(k+1)^3 = (k+1)^2\left[\frac{k}{2} - 8k - 8\right] < 0$$

We conclude that d_k^l and d_k^m are the only remaining candidates that could maximize the social surplus. Define

$$\Delta(k) = \frac{k(k+1)^2}{2} - 4\varepsilon k(k+1)(k+2) - \frac{1}{2}\frac{(k+1)^2}{k} - 2\varepsilon(k+1).$$

If $\Delta(k)$ is strictly positive d_k^m is the surplus maximizing decision, whereas if $\Delta(k)$ is strictly negative, then d_k^l is the unique maximizer.

Claim 1 *Suppose that $\varepsilon \leq \frac{1}{24}$, then $\Delta(\cdot)$ is strictly increasing in k on the interval $[1, \infty)$*

Proof. Rearrange to get

$$\begin{aligned}\Delta(k) &= \frac{k(k+1)^2}{2} - 4\varepsilon k(k+1)(k+2) - \frac{1}{2}\frac{(k+1)^2}{k} - 2\varepsilon(k+1) \\ &= \frac{1}{2}(k+1)^2\left(k - \frac{1}{k}\right) - 4\varepsilon k(k+1)(k+2) - 2\varepsilon(k+1).\end{aligned}$$

Differentiation yields,

$$\Delta'(k) = (k+1)\left(k - \frac{1}{k}\right) + \frac{1}{2}(k+1)^2\left(1 + \frac{1}{k^2}\right) - 4\varepsilon\left[(k+1)(k+2) + k(k+2) + k(k+1)\right] - 2\varepsilon.$$

Simplify the bracketed expression to get

$$\begin{aligned}& (k+1)(k+2) + k(k+2) + k(k+1) = \underbrace{(k+1)^2 + (k+1)}_{=(k+1)(k+2)} + \underbrace{k(k+1) + k}_{=k(k+2)} + k(k+1) \\ = \;& (k+1)^2 + (k+1) + k(k+1) + (k+1) - 1 + k(k+1) = (k+1)^2 + 2(k+1) + 2k(k+1) - 1 \\ = \;& 3(k+1)^2 - 1.\end{aligned}$$

Hence

$$\begin{aligned}\Delta'(k) &= (k+1)\left(k - \frac{1}{k}\right) + \frac{1}{2}(k+1)^2\left(1 + \frac{1}{k^2}\right) - 4\varepsilon\left[3(k+1)^2 - 1\right] - 2\varepsilon \\ &= (k+1)\left(k - \frac{1}{k}\right) + \frac{1}{2}(k+1)^2\left(1 + \frac{1}{k^2} - 24\varepsilon\right) + 2\varepsilon.\end{aligned}$$

The claim follows since all terms are strictly positive given that $k \geq 1$ and $\varepsilon \leq \frac{1}{24}$. ■

Evaluating we have that

$$\Delta(1) = 2 - 24\varepsilon - 2 - 4\varepsilon = -28\varepsilon < 0$$

for any $\varepsilon > 0$. We conclude that (somewhat unfortunately since it is an additional complication for the example) d_1^l is the surplus maximizing decision for $k = 1$. However

$$\Delta(2) = 9 - 96\varepsilon - \frac{9}{4} - 6\varepsilon$$

Clearly, for ε small enough (the exact bound is $\varepsilon < \frac{27}{408}$, where it may be noted that $\frac{1}{24} = \frac{27}{648} < \frac{27}{408}$) we have that $\Delta(2)$ is strictly positive. Combining with the fact that $\Delta(k)$ is monotonically increasing in k under the condition that $\varepsilon \leq \frac{1}{24}$ we conclude;

Claim 2 *Suppose that $\varepsilon \leq \frac{1}{24}$. Then $x_1^*(mm) = d_1^l$ and $x_k^*(mm) = d_k^m$ for every $k \geq 2$. The associated social surplus is*

$$\begin{aligned}s_1(mm) &= \frac{1}{2}\frac{(k+1)^2}{k} + 2\varepsilon(k+1) \\ s_k(mm) &= \frac{k(k+1)^2}{2} - 4\varepsilon k(k+1)(k+2) \;\; \text{for } k \geq 2\end{aligned}$$

2.4 Type Profiles lh and hl

As for any other type profile $v_k^1\left(d_k^0, l\right)+v_k^2\left(d_k^0, h\right)-C_k\left(d_k^0\right)=0$. For the non-trivial alternatives we have that

$$v_k^1\left(d_k^l, l\right)+v_k^2\left(d_k^l, h\right)-C_k\left(d_k^l\right)=\frac{1}{2}\frac{(k+1)^2}{k}-\varepsilon(k+1)-\frac{(k+1)^2}{k}=-\frac{1}{2}\frac{(k+1)^2}{k}-\varepsilon(k+1)<0,$$

and

$$v_k^1\left(d_k^m, l\right)+v_k^2\left(d_k^m, h\right)-C_k\left(d_k^m\right)=0-\frac{k(k+1)^2}{2}-\frac{k(k+1)^2}{2}=-k(k+1)^2<0,$$

and

$$\begin{aligned} v_k^1\left(d_k^h, l\right)+v_k^2\left(d_k^h, h\right)-C_k\left(d_k^m\right) &= \frac{(k+1)^2}{4}\left[k-\frac{1}{k}\right]+3(k+1)^3-\frac{k(k+1)^2}{4}-2(k+1)^3 \\ &= \frac{(k+1)^2}{4k}+(k+1)^3=(k+1)^2\left[\frac{4k(k+1)-1}{4k}\right] \end{aligned}$$

Hence, again using symmetry, $x_k^*(lh)=x_k^*(hl)=d_k^h$ and the resulting maximized surplus is

$$s_k(lh)=s_k(hl)=(k+1)^2\left[\frac{4k(k+1)-1}{4k}\right]$$

2.5 Type Profiles mh and hm

Obviously, $v_k^1\left(d_k^0, m\right)+v_k^2\left(d_k^0, h\right)-C_k\left(d_k^0\right)=0$. If instead $d_k=d_k^l$

$$v_k^1\left(d_k^l, m\right)+v_k^2\left(d_k^l, h\right)-C_k\left(d_k^l\right)=\frac{3}{4}\frac{(k+1)^2}{k}+\varepsilon(k+1)-\varepsilon(k+1)-\frac{(k+1)^2}{k}=-\frac{1}{4}\frac{(k+1)^2}{k}<0,$$

and if $d_k=d_k^m$,

$$\begin{aligned} v_k^1\left(d_k^m, m\right)+v_k^2\left(d_k^m, h\right)-C_k\left(d_k^m\right) &= \frac{k(k+1)^2}{2}-2\varepsilon k(k+1)(k+2)-\frac{k(k+1)^2}{2}-\frac{k(k+1)^2}{2} \\ &= -2\varepsilon k(k+1)(k+2)-\frac{k(k+1)^2}{2}<0, \end{aligned}$$

whereas if $d_k=d_k^h$, we get that

$$\begin{aligned} v_k^1\left(d_k^h, m\right)+v_k^2\left(d_k^h, h\right)-C_k\left(d_k^h\right) &= \frac{k(k+1)^2}{4}-3(k+1)^3+3(k+1)^3-\frac{k(k+1)^2}{4}-2(k+1)^3 \\ &= -2(k+1)^2<0 \end{aligned}$$

Hence, $x_k^*(mh)=x_k^*(hm)=d_k^0$, and the maximized surplus is $s_k(mh)=s_k(hm)=0$

2.6 Type Profile hh

Again, $v_k^1\left(d_k^0, h\right)+v_k^2\left(d_k^0, h\right)-C_k\left(d_k^0\right)=0$, and

$$v_k^1\left(d_k^l, h\right)+v_k^2\left(d_k^l, h\right)-C_k\left(d_k^l\right)=-2\varepsilon(k+1)-\frac{(k+1)^2}{k}<0$$

and

$$v_k^1\left(d_k^m,h\right)+v_k^2\left(d_k^m,h\right)-C_k\left(d_k^m\right)=-2\left[\frac{k\left(k+1\right)^2}{2}\right]-\frac{k\left(k+1\right)^2}{2}=-\frac{3k\left(k+1\right)^2}{2}<0$$

and

$$v_k^1\left(d_k^h,h\right)+v_k^2\left(d_k^h,h\right)-C_k\left(d_k^h\right)=2\left[3\left(k+1\right)^3-\frac{k\left(k+1\right)^2}{4}\right]-2\left(k+1\right)^3=4\left(k+1\right)^3-\frac{k\left(k+1\right)^2}{2}>0$$

Hence, $x_k^*\left(hh\right)=d_k^h$ and

$$s_k\left(hh\right)=4\left(k+1\right)^3-\frac{\left(k+1\right)^2}{2k}=\left[4\left(k+1\right)-\frac{k}{2}\right]\left(k+1\right)^2$$

2.7 Summary: An Optimal Decision Rule and the Maximized Surplus

Combining all the cases above and ignoring $k=1$ we have that an optimal social decision rule is[1][2]

$$x_k^*\left(\theta_k\right)=\begin{cases} d_k^0 & \text{if } \theta_k\in\{ll,mh,hm\} \\ d_k^l & \text{if } \theta_k\in\{lm,ml\} \\ d_k^m & \text{if } \theta_k=mm \\ d_k^h & \text{if } \theta_k\in\{lh,hl,hh\} \end{cases}, \tag{1}$$

and the maximized surplus (given $k\geq 2$) is

$$s_k\left(\theta_k\right)=\begin{cases} 0 & \text{if } \theta_k\in\{ll,mh,hm\} \\ \frac{1}{4}\frac{(k+1)^2}{k}+\varepsilon\left(k+1\right) & \theta_k\in\{lm,ml\} \\ \frac{k(k+1)^2}{2}-4\varepsilon k\left(k+1\right)\left(k+2\right) & \text{if } \theta_k=mm \\ \left(k+1\right)^2\left[\frac{4k(k+1)-1}{4k}\right] & \text{if } \theta_k\in\{lh,hl\} \\ \left[4\left(k+1\right)-\frac{k}{2}\right]\left(k+1\right)^2 & \text{if } \theta_k=hh \end{cases}. \tag{2}$$

2.8 Interim Expected Payoffs from Participation in the Groves Mechanism

For $k=2,3...$ we assume that the probability distribution over Θ_k^i be given by

$$\left(\Pr\left[\theta^i=l\right],\Pr\left[\theta^i=m\right],\Pr\left[\theta^i=h\right]\right)=\left(\frac{k\left(k+2\right)}{\left(k+1\right)^2},\frac{1}{2\left(k+1\right)^2},\frac{1}{2\left(k+1\right)^2}\right). \tag{3}$$

---

[1] There is multiplicity for type profile ll. We could easily get rid of this without affecting the maximized social surplus by adjusting the costs and preferences slightly. This would add some extra terms for the calculations, and, ultimately, only the maximized surplus is relevant, so we have opted to go with the simpler primitives.

[2] Hence, we will start our sequence at $k=2$. Obviously, we could replace every k with $k+1$ and start the sequence at $k=1$, but the formulas get somewhat less transparent, which is why we stick with the current formulation.

With probability distribution (3), the interim expected value of the maximized issue k surplus is

$$
\begin{aligned}
\mathrm{E}\left[s_k\left(\theta_k\right)|l\right] &= \Pr\left[\theta^i = l\right] s_k\left(ll\right) + \Pr\left[\theta^i = m\right] s_k\left(lm\right) + \Pr\left[\theta^i = h\right] s_k\left(lh\right) \qquad (4)\\
&= \frac{1}{2\left(k+1\right)^2}\left[\frac{1}{4}\frac{\left(k+1\right)^2}{k} + \varepsilon\left(k+1\right)\right] + \frac{1}{2\left(k+1\right)^2}\left[\left(k+1\right)^2\left[\frac{4k\left(k+1\right)-1}{4k}\right]\right]\\
&= \frac{1}{2\left(k+1\right)^2}\left[\frac{1}{4}\frac{\left(k+1\right)^2}{k} + \varepsilon\left(k+1\right) + \left(k+1\right)^2\left[\frac{4k\left(k+1\right)-1}{4k}\right]\right]\\
&= \frac{1}{2}\left[\frac{1}{4k} + \frac{\varepsilon}{\left(k+1\right)} + \frac{4k\left(k+1\right)-1}{4k}\right]\\
&= \frac{1}{2}\left[\frac{1}{4k} + \frac{\varepsilon}{\left(k+1\right)} + \left(k+1\right) - \frac{1}{4k}\right] = \frac{1}{2}\left[\frac{\varepsilon}{\left(k+1\right)} + \left(k+1\right)\right] = \frac{k+1}{2} + \frac{\varepsilon}{2\left(k+1\right)}
\end{aligned}
$$

for $\theta_i = l$. For $\theta_i = m$ we have that

$$
\begin{aligned}
\mathrm{E}\left[s_k\left(\theta_k\right)|m\right] &= \Pr\left[\theta^i = l\right] s_k\left(ml\right) + \Pr\left[\theta^i = m\right] s_k\left(mm\right) + \Pr\left[\theta^i = h\right] s_k\left(mh\right) \qquad (5)\\
&= \frac{k\left(k+2\right)}{\left(k+1\right)^2}\left[\frac{1}{4}\frac{\left(k+1\right)^2}{k} + \varepsilon\left(k+1\right)\right] + \frac{1}{2\left(k+1\right)^2}\left[\frac{k\left(k+1\right)^2}{2} - 4\varepsilon k\left(k+1\right)\left(k+2\right)\right]\\
&= \frac{k+2}{4} + \frac{\varepsilon k\left(k+2\right)}{\left(k+1\right)} + \frac{k}{4} - \frac{2\varepsilon k\left(k+2\right)}{k+1} = \frac{k+1}{2} - \frac{\varepsilon k\left(k+2\right)}{\left(k+1\right)}\\
&= \frac{k+1}{2} - \frac{\varepsilon\left[k\left(k+1\right)+k\right]}{\left(k+1\right)} = \frac{k+1}{2} - \varepsilon k - \frac{\varepsilon k}{k+1}
\end{aligned}
$$

Finally, for $\theta_i = h$, we have that

$$
\begin{aligned}
\mathrm{E}\left[s_k\left(\theta_k\right)|h\right] &= \Pr\left[\theta^i = l\right] s_k\left(hl\right) + \Pr\left[\theta^i = m\right] s_k\left(hm\right) + \Pr\left[\theta^i = h\right] s_k\left(hh\right) \qquad (6)\\
&= \frac{k\left(k+2\right)}{\left(k+1\right)^2}\left[\left(k+1\right)^2\left[\frac{4k\left(k+1\right)-1}{4k}\right]\right] + \frac{1}{2\left(k+1\right)^2}\left[4\left(k+1\right) - \frac{k}{2}\right]\left(k+1\right)^2\\
&= k\left(k+2\right)\left[\frac{4k\left(k+1\right)-1}{4k}\right] + \left[2\left(k+1\right) - \frac{k}{4}\right]\\
&= k\left(k+1\right)\left(k+2\right) + 2\left(k+1\right) - \frac{k+2}{4} - \frac{k}{4} = k\left(k+1\right)\left(k+2\right) + 2\left(k+1\right) - \frac{k+1}{2}\\
&= \left(k+1\right)\left[k^2+2k+1+1\right] - \frac{k+1}{2} = \left(k+1\right)^3 + \left(k+1\right) - \frac{k+1}{2} = \left(k+1\right)^3 + \frac{\left(k+1\right)}{2}
\end{aligned}
$$

The ex ante expected surplus is thus

$$
\begin{aligned}
\mathrm{E}\left[s_k\left(\theta_k\right)\right] &= \frac{k\left(k+2\right)}{\left(k+1\right)^2}\left[\frac{k+1}{2} + \frac{\varepsilon}{2\left(k+1\right)}\right] + \frac{1}{2\left(k+1\right)^2}\left[\frac{k+1}{2} - \frac{\varepsilon k\left(k+2\right)}{\left(k+1\right)}\right] \qquad (7)\\
&\quad + \frac{1}{2\left(k+1\right)^2}\left[\left(k+1\right)^3 + \frac{\left(k+1\right)}{2}\right]\\
&= \frac{k\left(k+2\right)}{2\left(k+1\right)} + \varepsilon\left[\frac{k\left(k+2\right)}{2\left(k+1\right)^3}\right] + \frac{1}{4\left(k+1\right)} - \varepsilon\left[\frac{\varepsilon k\left(k+2\right)}{2\left(k+1\right)^3}\right] + \frac{k+1}{2} + \frac{1}{4\left(k+1\right)}\\
&= \frac{k\left(k+2\right)}{2\left(k+1\right)} + \frac{1}{2\left(k+1\right)} + \frac{k+1}{2} = \frac{k^2+2k+1}{2\left(k+1\right)} + \frac{k+1}{2} = \frac{\left(k+1\right)^2}{2\left(k+1\right)} + \frac{k+1}{2} = k+1
\end{aligned}
$$

2.9 Summary of the Expected Surplus Calculations

We have shown that

$$\begin{aligned} \mathrm{E}\left[\left. s_k(\theta_k)\right| l\right] &= \frac{k+1}{2}+\frac{\varepsilon}{2(k+1)} && (8)\\ \mathrm{E}\left[\left. s_k(\theta_k)\right| m\right] &= \frac{k+1}{2}-\varepsilon k-\frac{\varepsilon k}{(k+1)} && (9)\\ \mathrm{E}\left[\left. s_k(\theta_k)\right| h\right] &= (k+1)^3+\frac{(k+1)}{2} && (10)\\ \mathrm{E}\left[s_k(\theta_k)\right] &= k+1 && (11) \end{aligned}$$

3 The Probability that a Participation Constraint is Violated Converges to zero as $K\to\infty$.

Since $R_K^i\left(\theta^i\right)=0$ for all θ^i, the interim expected payoff from participation in the Groves mechanism under consideration simplifies to

$$U_K^i\left(\theta^i\right)=\mathrm{E}_{-i}\left[S_K(\theta)\right]-\frac{1}{2}\mathrm{E}\left[S_K(\theta)\right]=\sum_{k=2}^{K}\mathrm{E}_{-i}\left[s_k(\theta_k)\right]-\frac{1}{2}\sum_{k=2}^{K}\mathrm{E}\left[s_k(\theta_k)\right] \qquad (12)$$

Consider a type on the form $(m,....,m,l,....,l)$. Specifically, assume that $\theta_k^i=m$ for $k=2,...,K^*$ and $\theta_k^i=l$ for $k=K^*+1,...,K$ and denote this type by $\left(\mathbf{m}_{K^*},\mathbf{l}_{K^*-K}\right)$. Substituting (8), (9) and (11) into (12) we have that type $\left(\mathbf{m}_{K^*},\mathbf{l}_{K^*-K}\right)$ earns an interim expected payoff of

$$\begin{aligned} U_K^i\left(\mathbf{m}_{K^*},\mathbf{l}_{K^*-K}\right) &= \sum_{k=2}^{K^*}\mathrm{E}\left[\left. s_k(\theta_k)\right| m\right]+\sum_{k=K^*+1}^{K}\mathrm{E}\left[\left. s_k(\theta_k)\right| l\right]-\frac{1}{2}\sum_{k=2}^{K}\mathrm{E}\left[s_k(\theta_k)\right] \qquad (13)\\ &= \sum_{k=2}^{K^*}\left[\frac{k+1}{2}-\varepsilon k-\frac{\varepsilon k}{(k+1)}\right]+\sum_{k=K^*+1}^{K}\frac{k+1}{2}+\frac{\varepsilon}{2(k+1)}-\frac{1}{2}\sum_{k=2}^{K}(k+1)\\ &= \varepsilon\left[\sum_{k=2}^{K^*}\left[-k-\frac{k}{k+1}\right]+\frac{1}{2}\sum_{k=K^*+1}^{K}\left[\frac{1}{k+1}\right]\right] \end{aligned}$$

Define

$$H\left(K^*,K\right)=\sum_{k=2}^{K^*}\left[-k-\frac{k}{k+1}\right]+\frac{1}{2}\sum_{k=K^*+1}^{K}\frac{1}{k+1} \qquad (14)$$

We note that;

1. $H\left(K^*,K\right)$ is strictly decreasing in K^*
2. $H(1,K)=\frac{\varepsilon}{2}\sum_{k=2}^{K}\frac{1}{(k+1)}>0$
3. $H(K,K)=-\varepsilon\sum_{k=2}^{K}\left[k+\frac{k}{(k+1)}\right]<0$

These three properties imply that for every $K \geq 2$ there exists a unique integer $K^*(K) \in \{1, ..., K\}$ such that

$$\begin{aligned} H(K^*(K), K) &> 0 \\ H(K^*(K)+1, K) &< 0 \end{aligned} \tag{15}$$

Moreover, $K^*(K)$ is monotonically increasing and goes (slowly) to infinity as K goes to infinity. To see this, we first observe that, for K^* fixed, the positive term in (14) is divergent. That is,

$$\begin{aligned} \sum_{k=K^*+1}^{K} \frac{1}{k+1} &= \sum_{k=K^*+1}^{K} \left[\int_{k+1}^{k+2} \frac{1}{k+1} dz \right] > \sum_{k=K^*+1}^{K} \left[\int_{k+1}^{k+2} \frac{1}{z} dz \right] \\ &= \int_{K^*+2}^{K+2} \frac{1}{z} dz = \ln(K+2) - \ln(K^*+2). \end{aligned} \tag{16}$$

For contradiction, assume that there exists some $\overline{K}$ such that $K^*(K) < \overline{K} - 1$ for all K. Then, for any K we have that

$$\begin{aligned} H(K^*(K)+1, K) &> H\left(\overline{K}, K\right) = -\sum_{k=2}^{\overline{K}} \left[k + \frac{k}{k+1} \right] + \frac{1}{2} \sum_{k=\overline{K}+1}^{K} \left[\frac{1}{k+1} \right] \\ / \text{ using (16)}/ &> \sum_{k=2}^{\overline{K}} \left[-k - \frac{k}{k+1} \right] + \frac{\ln(K+2) - \ln\left(\overline{K}+2\right)}{2}. \end{aligned} \tag{17}$$

Since $\ln(K+2) \to \infty$ as $K \to \infty$ and the other two terms are finite we conclude that $H(K^*(K)+1, K) > 0$ for K sufficiently large, which contradicts the definition of $K^*(K)$.

Next, consider a type on form $\left(\theta_2^i, ..., \theta_{K^*(K)}^i, l, ..., l\right)$, where the signals for problems $2, ..., K^*(K)$ are arbitrary, and $\theta_k^i = l$ for $k = K^*(K)+1, .., K$. We denote such a type $\left(\boldsymbol{\theta}_{K^*(K)}^i, \mathbf{l}_{K^*-K(K)}\right)$. Since $\mathrm{E}\left[s_k(\theta_k) | m \right] \leq \mathrm{E}\left[s_k(\theta_k) | \theta_k^i \right]$ for all $\theta_k^i \in \{l, m, h\}$ and every k it follows that

$$\begin{aligned} U_K^i\left(\boldsymbol{\theta}_{K^*(K)}^i, \mathbf{l}_{K^*-K(K)}\right) &= \sum_{k=2}^{K^*(K)} \mathrm{E}\left[s_k(\theta_k) | \theta_k^i \right] - \sum_{k=K(K)^*+1}^{K} \mathrm{E}\left[s_k(\theta_k) | l \right] - \frac{1}{2} \sum_{k=2}^{K} \mathrm{E}\left[s_k(\theta_k) \right] \\ &\geq \sum_{k=2}^{K^*(K)} \mathrm{E}\left[s_k(\theta_k) | m \right] - \sum_{k=K(K)^*+1}^{K} \mathrm{E}\left[s_k(\theta_k) | l \right] - \frac{1}{2} \sum_{k=2}^{K} \mathrm{E}\left[s_k(\theta_k) \right] \\ &= \varepsilon H(K^*(K), K) > 0. \end{aligned} \tag{18}$$

We conclude that the participation constraint holds for any such type. Since,

$$\Pr\left[\left(\theta_{K^*(K)+1}^i, ..., \theta_K^i\right) = (l, ..., l) \right] = \prod_{k=K^*(K)+1}^{K} \frac{k(k+2)}{(k+1)^2} = \frac{(K^*(K)+1)(K+2)}{(K^*(K)+2)(K+1)} \to 1 \tag{19}$$

as $K \to \infty$ it follows that the probability that θ^i is on the form $\left(\boldsymbol{\theta}_{K^*(K)}^i, \mathbf{l}_{K^*-K(K)}\right)$ tends to unity as K goes out of bounds. We conclude;

Claim 3 *The probability that all participation constraints hold converges to 1 as K tends to infinity.*

Hence, the Groves mechanism is almost incentive feasible in this example.

4 Unraveling when the Veto Game is Introduced

Note that for type $\mathbf{l} = (l, ..., l)$

$$U_K^i(\mathbf{l}) = \sum_{k=2}^{K} \mathrm{E}\left[s_k(\theta_k) | \, l \right] - \frac{1}{2}\sum_{k=2}^{K} \mathrm{E}\left[s_k(\theta_k) \right] = \sum_{k=2}^{K} \left[\frac{k+1}{2} + \frac{\varepsilon}{2(k+1)} - \frac{k+1}{2} \right] = \frac{\varepsilon}{2}\sum_{k=2}^{K} \frac{1}{(k+1)} \tag{20}$$

where

$$\sum_{k=2}^{K} \frac{1}{k+1} = \sum_{k=2}^{K} \left[\int_k^{k+1} \frac{1}{k+1} dz \right] < \sum_{k=2}^{K} \left[\int_k^{k+1} \frac{1}{z} dz \right] = \int_2^{K+1} \frac{1}{z} dz = \ln\left(\frac{K+1}{2}\right). \tag{21}$$

Hence,

$$U_K^i(\mathbf{l}) < \frac{\varepsilon}{2} \ln\left(\frac{K+1}{2}\right). \tag{22}$$

However, consider type θ^i where all coordinates are l except for $\theta_{k^*}^i = m$. Denote this type $\theta^i = \mathbf{l}|\theta_{k^*}^i = m$ and note that

$$\begin{aligned}
U\left(\mathbf{l}|\theta_{k^*}^i = m\right) &= \sum_{k \neq k^*}^{K} \mathrm{E}\left[s_k(\theta_k) | \, l \right] + \mathrm{E}\left[s_{k^*}(\theta_{k^*}) | \, m \right] - \frac{1}{2}\sum_{k=2}^{K} \mathrm{E}\left[s_k(\theta_k) \right] \qquad (23) \\
&= \underbrace{\sum_{k=2}^{K} \mathrm{E}\left[s_k(\theta_k) | \, l \right] - \frac{1}{2}\sum_{k=2}^{K} \mathrm{E}\left[s_k(\theta_k) \right]}_{=U^i(\mathbf{l})} + \mathrm{E}\left[s_{k^*}(\theta_{k^*}) | \, m \right] - \mathrm{E}\left[s_{k^*}(\theta_{k^*}) | \, l \right] \\
\text{/using (22)/} \quad &\leq \frac{\varepsilon}{2} \ln\left(\frac{K+1}{2}\right) + \frac{k^*+1}{2} - \varepsilon k^* - \frac{\varepsilon k^*}{(k^*+1)} - \left[\frac{k^*+1}{2} + \frac{\varepsilon}{2(k^*+1)} \right] \\
&= \varepsilon \left[\frac{1}{2} \ln\left(\frac{K+1}{2}\right) - k^* - \frac{k^*}{(k^*+1)} - \frac{1}{2(k^*+1)} \right] < \varepsilon \left[\frac{1}{2} \ln\left(\frac{K+1}{2}\right) - k^* \right]
\end{aligned}$$

Define $\widetilde{k}(K)$ as the integer part of $\frac{1}{2}\ln\left(\frac{K+1}{2}\right) + 1$. Since $\frac{1}{2}\ln\left(\frac{K+1}{2}\right) - k^*$ is negative for $k^* \geq \widetilde{k}(K)$ this implies that type $\mathbf{l}|\theta_{k^*}^i = m$ is worse off from participating in the Groves mechanism than from the status quo outcome $d^0 = \left(d_2^0, ..., d_K^K\right)$ for every $k^* \geq \widetilde{k}(K)$. For brevity, denote this subset of types with a strict incentive to veto the Groves mechanism by Θ_V^i. That is

$$\Theta_V^i = \left\{ \theta^i | \theta_{k^*}^i = m \text{ for some } k^* \geq k^*(K) \text{ and } \theta_k^i = l \text{ for all } k \neq k^* \right\} \tag{24}$$

We note that, for any $k^* \geq k^*(K)$

$$\begin{aligned}
\Pr\left[\theta^i \in \Theta_V^i | \theta_{k^*}^i = m\right] &= \prod_{k \neq k^*} \Pr\left[\theta_k^i = l\right] > \Pr\left[\theta^i = \mathbf{l}\right] = \prod_{k=2}^{K} \Pr\left[\theta_k^i = l\right] = \prod_{k=2}^{K} \frac{k(k+2)}{(k+1)^2} \qquad (25) \\
&= \left(\frac{2 \times 4}{3^2}\right) \times \left(\frac{3 \times 5}{4^2}\right) \times ... \times \frac{(K-1)(K+1)}{K^2} \times \frac{K(K+2)}{(K+1)^2} \\
&= \frac{2}{3}\left[\frac{K+2}{K+1}\right] > \frac{2}{3}
\end{aligned}$$

In words, conditional on $\theta_{k^*}^i = m$, the probability that all other coordinates are ls is obviously (slightly) larger than the unconditional probability that the type $\mathbf{l}$ is realized. For $k \geq \widetilde{k}(K)$ we can therefore calculate an upper bound on the expected surplus conditional on $\theta_k^i = l$ and conditional on that agent i is aware that any agent $\theta^i \in \Theta_V^i$ will veto the mechanism.

$$\begin{aligned}
\mathrm{E}\left[s_k(\theta_k)\,|l,\text{veto by }\theta^i\in\Theta_V^i\right] &= \Pr\left[\theta_k^i=m\right]\left[1-\Pr\left[\theta^i\in\Theta_V^i|\theta_k^i=m\right]\right]s_k(lm)+\Pr\left[\theta_k^i=h\right]s_k(lh) \qquad (26)\\
&= \mathrm{E}\left[s_k(\theta_k)\,|l\right]-\Pr\left[\theta_k^i=m\right]\Pr\left[\theta^i\in\Theta_V^i|\theta_k^i=m\right]s_k(lm)\\
&= \underbrace{\frac{k+1}{2}+\frac{\varepsilon}{2(k+1)}}_{=\mathrm{E}[s_k(\theta_k)|l]\text{ by }(8)}-\underbrace{\frac{1}{2(k+1)^2}}_{=\Pr[\theta_k^i=m]}\Pr\left[\theta^i\in\Theta_V^i|\theta_k^i=m\right]\underbrace{\left[\frac{1}{4}\frac{(k+1)^2}{k}+\varepsilon(k+1)\right]}_{=s_k(lm)\text{ by }(2)}\\
/(25)/\quad &< \frac{k+1}{2}+\frac{\varepsilon}{2(k+1)}-\frac{1}{2(k+1)^2}\frac{2}{3}\left[\frac{1}{4}\frac{(k+1)^2}{k}+\varepsilon(k+1)\right]\\
&= \frac{k+1}{2}+\frac{\varepsilon}{2(k+1)}-\frac{1}{12k}-\frac{\varepsilon}{3(k+1)}=\frac{k+1}{2}+\frac{\varepsilon}{6(k+1)}-\frac{1}{12k}
\end{aligned}$$

The interim expected payoff of type $\mathbf{l}=(l,...,l)$ conditional on vetoes from types in Θ_V^i is thus

$$\begin{aligned}
U_K^i\left(\mathbf{l}|\text{veto by }\theta^i\in\Theta_V^i\right) &= \sum_{k=2}^{\mathrm{I}\widetilde{k}(K)-1}\mathrm{E}\left[s_k(\theta_k)|\,l\right]+\sum_{k=\widetilde{k}(K)}^{K}\mathrm{E}\left[s_k(\theta_k)\,|l,\text{veto by }\theta^i\in\Theta_V^i\right]-\frac{1}{2}\sum_{k=2}^{K}\mathrm{E}\left[s_k(\theta_k)\right] \qquad (27)\\
&= \frac{\varepsilon}{2}\sum_{k=2}^{\mathrm{I}\widetilde{k}(K)-1}\frac{1}{k+1}+\frac{\varepsilon}{6}\sum_{k=\widetilde{k}(K)}^{K}\frac{1}{k+1}-\frac{1}{12}\sum_{k=\widetilde{k}(K)}^{K}\frac{1}{k}
\end{aligned}$$

where,

$$\begin{aligned}
\sum_{k=2}^{\mathrm{I}\widetilde{k}(K)-1}\frac{1}{k+1} &= \sum_{k=2}^{\mathrm{I}\widetilde{k}(K)-1}\left[\int_k^{k+1}\frac{1}{k+1}dz\right]<\sum_{k=2}^{\mathrm{I}\widetilde{k}(K)-1}\left[\int_k^{k+1}\frac{1}{z}dz\right]=\int_2^{\mathrm{I}\widetilde{k}(K)}\frac{1}{z}dz=\ln\left(\frac{\mathrm{I}\widetilde{k}(K)}{2}\right)\\
\mathrm{I}\sum_{k=\widetilde{k}(K)}^{K}\frac{1}{k+1} &= \sum_{k=\widetilde{k}(K)}^{K}\left[\int_k^{k+1}\frac{1}{k+1}dz\right]<\sum_{k=\widetilde{k}(K)}^{K}\left[\int_k^{k+1}\frac{1}{z}dz\right]=\int_{\widetilde{k}(K)}^{\mathrm{I}K+1}\frac{1}{z}dz=\ln\left(\frac{\mathrm{I}K+1}{\widetilde{k}(K)}\right)\\
\sum_{k=\widetilde{k}(K)}^{K}\frac{1}{k} &= \sum_{k=\widetilde{k}(K)}^{K}\left[\int_k^{k+1}\frac{1}{k}dz\right]>\sum_{k=\widetilde{k}(K)}^{K}\left[\int_k^{k+1}\frac{1}{z}dz\right]=\ln\left(\frac{\mathrm{I}K+1}{\widetilde{k}(K)}\right) \qquad (28)
\end{aligned}$$

Hence,

$$\begin{aligned}
U_K^i\left(\mathbf{l}|\text{veto by }\theta^i\in\Theta_V^i\right) &= \frac{\varepsilon}{2}\sum_{k=2}^{\mathrm{I}\widetilde{k}(K)-1}\frac{1}{k+1}+\frac{\varepsilon}{6}\sum_{k=\widetilde{k}(K)}^{K}\frac{1}{k+1}-\frac{1}{12}\sum_{k=\widetilde{k}(K)}^{K}\frac{1}{k} \qquad (29)\\
&< \frac{\varepsilon}{2}\ln\left(\frac{\mathrm{I}\widetilde{k}(K)}{2}\right)-\frac{1-2\varepsilon}{6}\ln\left(\frac{\mathrm{I}K+1}{\widetilde{k}(K)}\right)\\
&= \ln\widetilde{k}(K)\left[\frac{1+\varepsilon}{6}\right]-\frac{\varepsilon}{2}\ln 2-\ln(K+1)\left[\frac{1-2\varepsilon}{6}\right] \qquad (30)
\end{aligned}$$

But, $\widetilde{k}(K)\le\frac{1}{2}\ln\left(\frac{K+1}{2}\right)+1$, so

$$\ln(K+1)\ge 2\widetilde{k}(K)+\ln 2-2$$

Since $\lim_{x\to\infty}\frac{\ln x}{x}=0$ and since $\lim_{K\to\infty}\widetilde{k}(K)=\infty$ it follows that the term with $\ln(K+1)$ eventually dominates in expression (29). Consequently, whenever $\varepsilon<\frac{1}{2}$ there exists $\overline{K}$ such that $U^i\left(\mathbf{l}|\text{veto by }\theta^i\in\Theta_V^i\right)<0$ for any $K\geq\overline{K}$.[3] In (25) we calculated the probability that $\theta^i=\mathbf{l}$ to be

$$\Pr\left[\theta^i=\mathbf{l}\right]=\frac{2}{3}\left[\frac{K+2}{K+1}\right].$$

We conclude that;

Claim 4 *Suppose that type* $\mathbf{l}=(l,....l)$ *expects that all types in* Θ_V^i *will veto play of the Groves mechanism. Then, there exists some* $\overline{K}$ *such that type* $\mathbf{l}=(l,....l)$ *will have a strict incentive to veto the Groves mechanism given any economy* $K\geq\overline{K}$. *Hence, the mechanism is vetoed by each agent with a probability of at least* $\frac{2}{3}$ *for every* $K\geq\overline{K}$.

The only difference between the Groves mechanism amended with a veto game and the mechanism actually considered in the proof of Proposition 2 is that the latter mechanism has a slightly (on a per problem basis) larger lump sum payment than the underlying budget balancing Groves mechanism. It follows that the conclusion immediately extends to the mechanism under consideration.

The reader may note that any type such that $\theta_k^i\in\{l,m\}$ for all k will have an incentive to cast a veto. The probability that there is one k such that $\theta_k^i\in\{m,h\}$ is less than $\frac{1}{3}$, and, conditional on this event occuring, the probability that the first such draw is m is $\frac{1}{2}$. Moreover, conditional on the first draw different from l being m, the probability that the rest of the sequence is all l is $\frac{2}{3}$, so we can immediately add $\frac{1}{3}\frac{1}{2}\frac{2}{3}=\frac{1}{9}$ to the lower bound on the probability for a veto.

Obviously we can do even better by taking into consideration that the larger k is for the first draw of m, the larger is the probability that the remaining sequence contain only ls. However, qualitatiely, the point with the example is that an outcome that is highly inefficient occurs when the possibility of a veto is introduced, and for that it is sufficient to observe that the (high probability) type $\mathbf{l}$ will cast a veto.

[3] We already have the restriction $\varepsilon\leq\frac{1}{24}$ in order for the surplus in (2) to be valid for every $k\geq2$.