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Lottery versus all-pay auction models of lobbying *

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Abstract. I first provide a complete characterization of the unique equilibrium of the lottery game by n lobbyists with asymmetric valuations, and then compare the lottery and the all-pay auction models of lobbying. I show that the exclusion principle discovered by Baye, Kovenock and de Vries (1993) for all-pay auction does not apply to lottery. I also show that the perverse effect that an exogenous cap may increase the total lobbying expenditure in a two-bidder all-pay auction discovered by Che and Gale (1998) does not apply to lottery.

1. Introduction

Lobbying, as a rent seeking endeavor, is an important feature of politics. In the United States, $1.42 billion was spent in 1998 by lobbyists trying to influence politicians, which means an average of $2.7 million for each of the 535 lawmakers (*New York Times*, July 29, 1999, page A14). Lobbyists often make implicit or explicit up-front payments, through channels such as campaign contributions, bribery, corporate jets etc., before a "prize" is awarded. The prizes eyed for by lobbyists, to name a few, can be a government contract, a monopoly privilege, a favorable legislation, a host right to summer Olympics, a host right to Democratic or Republican National Convention, a profitable government position etc.

The up-front payment feature underlines two commonly used formal models of lobbying in the literature: the *lottery* and *all-pay auction* models. Both are special cases of Tullock's (1975, 1980) model. The difference of the two models lies in the assumed relationship between the size of the lobbying expenditure and the probability of winning the prize. In the lottery model, the probability that a given lobbyist wins the prize is proportional to his expenditure relative to the total expenditure; while in the all-pay auction model the lobbyist with the highest expenditure wins with probability one. Rowley (1991, 1993) provides an excellent and comprehensive survey of the literature initiated by Tullock (1975, 1980).

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It is well known that there is no pure strategy equilibrium in all-pay auctions with complete information, but there may be a continuum of mixed strategy equilibria depending on the configurations of valuations (Baye, Kovenock and de Vries, 1996). Baye, Kovenock and de Vries (1993) provide a formula for the politician's expected revenue that applies to all equilibria. They then discover a very interesting *exclusion principle*: a revenue maximizing politician may find it in her best interest to exclude lobbyists with valuations above a threshold from participating in the all-pay auction.

Most of the papers on the lottery model of lobbying study either the case of $n = 2$ lobbyists (e.g., Nti, 1999) or $n > 2$ lobbyists with identical (e.g., Tullock, 1975, 1980; Baye, Kovenock and de Vries, 1994) or two different valuations (e.g., Ellingsen, 1991). In a notable exception, Hillman and Riley (1989) characterize an equilibrium in the lottery model with $n > 2$ lobbyists with arbitrary configurations of asymmetric valuations. In this paper, I construct a sequence of auxiliary problems in which the lobbyists are allowed to bet negative amounts, and then exploit the tight connection between the auxiliary problems and the original problem to establish that the equilibrium identified by Hillman and Riley (1989) is actually the unique equilibrium of the lottery model.

Using the characterization of the unique equilibrium of the lottery model, I first show that the exclusion principle identified by Baye, Kovenock and de Vries (1993) for the all-pay auction does not apply to lottery, that is, a revenue maximizing politician will not have a strict incentive to exclude lobbyists from participating in the lottery. However, her revenue will remain the same if she excludes low-valuation lobbyists who would have bid zero were they not excluded. Second, I compare the two modes of prize allocation both from the politician's and from the lobbyists' perspective. I show that allocating the prize via lottery generates a higher revenue for the politician, or equivalently, the rent dissipation is higher, when the valuations of the prize among lobbyists are heterogeneous enough. I also show that when the second highest valuation is more than about 62% of the highest valuations, the total surplus for the lobbyists is higher under lottery than under all-pay auction, regardless of the valuations of the other lobbyists. However, it is in general not clear which mode of prize allocation is socially more effcient.

I then proceed to analyze the lottery model with exogenous caps on lobbying spending. I provide a characterization of the equilibrium. This equilibrium characterization incorporates the previous equilibrium characterization without caps as a special case. This contributes to the existing literature, which has only recently compared the lottery and the all-pay auction models of lobbying with caps when lobbyists' valuations are identical (Che and Gale, 1997). In an interesting recent paper, Che and Gale (1998) analyze a two-bidder all-pay auction model with asymmetric valuations, and find that the exogenous cap on lobbying spending may have the perverse effect of increasing the total lobbying expenditure. I show that if the lobbying is arranged as a lottery, then in the two-bidder case, the imposition of an exogenous cap on lobbying spending will *never* increase the total lobbying expenditure.

Since Tullock (1975, 1980), the lobbying process has been mostly modelled as either a lottery or an all-pay auction. Whether the lottery model or the all-pay auction model is better suited to study the lobbying process remains an open empirical question. The theoretical analysis presented in this paper does identify different testable predictions of the two models: First, if lobbying process is a lottery, then those excluded lobbyists, if any, must have lower valuations than those included; while in sharp contrast, if lobbying is an all-pay auction then those excluded lobbyists, if any, may have higher valuations than those included. Second, the implications of an exogenous cap on the total lobbying expenditure are different for the lottery and the all-pay auction models. These provide a useful basis to identify which model is more empirically relevant. The analysis in this paper also helps us understand when lobbying is more likely to be arranged as a lottery or as an all-pay auction, if the politician has any discretion in choosing the form of the lobbying process.

The remainder of the paper is structured as follows. Section 2 describes the lottery and all-pay auction models of lobbying; Section 3 characterizes the unique equilibrium of the lottery model in the absence of caps on lobbying spending; Section 4 summarizes the main results on the all-pay auction model from Baye, Kovenock and de Vries (1993, 1996); Section 5 compares lottery and all-pay auction from three dimensions: the exclusion principle, the politician's revenue and the lobbyists' and social surplus; Section 6 characterizes the equilibrium of the lottery game with exogenous caps on lobbying spending; finally, Section 7 concludes.

2. The Lottery and all-pay auction models

Both the lottery and the all-pay auction models are special cases of Tullock's model (1975, 1980). There are $n > 2$ potential lobbyists who want to receive a prize. The value of the prize to lobbyist i is $v_i > 0$, which is common knowledge. Without loss of generality, I order the lobbyists according to their valuations of the prize so that $v_1 \geq v_2 \geq \ldots \geq v_n > 0$. A politician must determine which lobbyist receives the prize. The politician does not care which lobbyist wins the prize but does care about the revenue she receives from the lobbying process. The politician's objective is to select a set of lobbyists, called "finalists" as in Baye, Kovenock and de Vries (1993), to participate in the lobbying. Given a set of n finalists, if the bids submitted by the n finalists are $\mathbf{b} = (b_1, \ldots, b_n)$, I follow Tullock (1980) and assume that the probability of lobbyist i winning the prize, τ_i, is given by

$$\tau_i(\mathbf{b}) = \begin{cases} b_i^r / \sum_{j=1}^n b_j^r \text{ if } \sum_{j=1}^n b_j^r > 0 \\ 1/n \text{ otherwise,} \end{cases} \tag{1}$$

where $r > 0$. The special case when $r = 1$ is often called the *lottery* model (see, e.g. Che and Gale, 1997). In the lottery, lobbyist i's expected payoff, denoted by u_i^L, is given by:

$$u_i^L(\mathbf{b}) = b_i v_i / \left(\sum_{j=1}^n b_j \right) - b_i.$$

This special case when $r = \infty$ is often called the *all-pay auction* model. In the all-pay auction, lobbyist i's expected pay off, denoted by u_i^{APA}, is given by:

$$u_i^{APA}(\mathbf{b}) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \forall i \neq j \\ \frac{v_i}{M} - b_i & \text{if i ties } M-1 \text{ others for high bid} \\ -b_i & \text{if } b_i < b_j \text{ for some } j \neq i. \end{cases}$$

3. The unique equilibrium of the lottery model

Hillman and Riley (1989, Section 6) provided an equilibrium of the lottery model. In this section, I show that the equilibrium they identified is in fact unique.

Suppose that the set of finalists consists of all the n potential lobbyists. An equilibrium of the lottery game among the n lobbyists is a profile of bribes $(b_1^*, \ldots, b_n^*) \in \mathscr{R}_+^n$ such that for each $i = 1, \ldots, n$, b_i^* maximizes lobbyist i's expected payoff given $b_{-i}^* \equiv (b_1^*, \ldots, b_{i-1}^*, b_{i+1}^*, \ldots, b_n^*)$.

First observe that, in any equilibrium of the lottery game the total bets must be strictly positive. That is, $\sum_{j=1}^n b_j^* > 0$ in any equilibrium. Otherwise, each lobbyist wins the prize with probability $1/n$. But then any lobbyist, say j, can deviate by bidding an arbitrarily small $\varepsilon > 0$ and increase her payoff from $(1/n)v_j$ to $v_j - \varepsilon$. A contradiction to equilibrium. With this observation, the objective of an arbitrary lobbyist i, given the bids by other lobbyists b_{-i}, is to solve the following problem (P):

$$\max_{b_i \geq 0} \left\{ \frac{b_i}{\sum_{j \neq i} b_j + b_i} v_i - b_i \right\}. \tag{P}$$

Since the objective function in (P) is globally concave, the first order condition is necessary and sufficient:

$$b_i\left[\frac{v_iB^P(n)-b_iv_i}{B^P(n)^2}-1\right]=0,\ \text{and}\ \frac{v_iV^P(n)-b_iv_i}{B^P(n)^2}=1 \text{ if } b_i>0$$

where $B^P(n)=\sum_{j=1}^n b_j$ is the sum of n lobbyists' bids.

To completely characterize the equilibrium of the above lottery model, I consider the following sequence of auxiliary problems (A–m): for each $m = 2,\ldots,n$, the group of lobbyists $\{1,\ldots,m\}$ participate in a modified "lottery" game that allows them to bid negative amount. That is, each lobbyist $i \in \{,\ldots,m\}$, taking as given $\{b_j, j=1,\ldots,i-1,i+1,\ldots,m\}$, solves:

$$\max_{b_i}\left\{\frac{b_i}{\sum_{j=1}^m b_j}v_i-b_i\right\}. \tag{A–m}$$

Note that the difference between (A–m) and (P) is that in (A–m) lobbyist i's bid is not restricted to be non-negative.

For every $m = 2,\ldots,n$, the necessary and sufficient condition for lobbyist $i \in \{1,\ldots,m\}$ is

$$\frac{v_iB^A(m)-b_iv_i}{B^A(m)^2}=1, \tag{2}$$

where $B^A(m)=\sum_{j=1}^m b_j$ is the sum of m lobbyists' equilibrium bids in the auxiliary lottery game. Simple manipulation of the above first order condition yields that

$$B^A(m)=\frac{m-1}{\sum_{j=1}^m \frac{1}{v_j}}.$$

This formula is also obtained in Hillman and Riley (1989) and Ellingsen (1991). This paper connects the sequence of the auxiliary problems with the equilibrium of the original lottery game to establish uniqueness of the equilibrium.

Since problem (A–m) is an unconstrained concave programming problem, it is easy to see that for each $m = 2,\ldots,n$, the unique equilibrium bids in the auxiliary lottery game (A–m) are given by

$$b_i^A(m)=B^A(m)-\frac{B^A(m)^2}{v_i}\ \text{for } i=1,\ldots,m. \tag{3}$$

Define

$$n^*=\min\{m: v_{m+1}\le B^A(m), m=2,\ldots,n-1\}\cup\{n\}. \tag{4}$$

Lemma 1. If $2 \leq m \leq n^*$, then $b_i^A(m) > 0$ for $i \in \{1 \ldots, m\}$.

Proof. Consider the problem (A–m). By the definition of n^*, and since $m \leq n^*$ we know that for any $i \in \{1, \ldots, m\}$, $v_i \geq v_m > B^A(m-1)$. Now I show that $v_m > B^A(m-1)$ implies that $v_m > B^A(m)$. Suppose instead $v_m \leq B^A(m)$, then

$$\frac{1}{v_m} \geq \frac{1}{B^A(m)} = \frac{\sum_{j=1}^{m-1} \frac{1}{v_j} + \frac{1}{v_m}}{m-1}$$
$$\Rightarrow \frac{1}{v_m} \geq \frac{\sum_{j=1}^{m-1} \frac{1}{v_j}}{m-2} = \frac{1}{B^A(m-1)},$$

a contradiction. Hence $v_i > B^A(m)$ for all $i \in \{1, \ldots, m\}$. Then $b_i^A(m) > 0$ follows from (3). □

The main result of this section is the following theorem and its proof is relegated to the Appendix.

Theorem 1. (Existence, characterization and uniqueness of equilibrium) For every $n \geq 2$, and every configuration of valuations $v_1 \geq \ldots \geq v_n > 0$, the unique equilibrium of the lottery game is characterized as follows:

1. for $i = 1, \ldots, n^*$, $b_i^* = b_i^A(n^*) > 0$;
2. for $i = n^* + 1, \ldots, n$, $b_i^* = 0$.

Theorem 1 completes the analysis in Hillman and Riley (1989, Section 6) in ruling out other pure strategy and mixed strategy equilibria by exploiting the tight connection between the auxiliary problem (A–m) and the original problem (P).

From Theorem 1, I can calculate, for any configuration of the lobbyists' valuations, the politician's revenue from lobbying, denoted by $R^L(n)$:

$$R^L(n) = \frac{n^* - 1}{\sum_{i=1}^{n^*} \frac{1}{v_i}} \tag{5}$$

where n^* is determined by (4).

4. All-pay auction

The all-pay auction model with complete information is thoroughly analyzed by Baye, Kovenock and de Vries (1993, 1996). One of their main results is the following theorem:

Theorem 2 (Baye, Kovenock and de Vries, 1993) Suppose $v_1 \geq \ldots \geq v_n > 0$ and the prize is allocated through a (first price) all-pay auction. Then:

1. The set of finalists that will be selected by a revenue maximizing politician is given by $\{k^*, k^* + 1, \ldots, n\}$ where k^* is determined by

$$\left(1 + \frac{v_{k^*+1}}{v_{k^*}}\right)\frac{v_{k^*} + 1}{2} \geq \left(1 + \frac{v_{i+1}}{v_i}\right)\frac{v_{i+1}}{2} \text{ for all } i = 1, \ldots, n-1. \quad (6)$$

2. The expected revenue of the politician from the all-pay auction participated by the finalists $\{k^*, k^* + 1, \ldots, n\}$, denoted by $R^{APA}(n)$, is given by:

$$R^{APA}(n) = \left(1 + \frac{v_{k^*+1}}{v_{k^*}}\right)\frac{v_{k^*+1}}{2}. \quad (7)$$

From (6), Baye, Kovenock and de Vries (1993) discovered an interesting *exclusion principle* for the all-pay auction model of lobbying, namely, a revenue maximizing politician may have a perverse incentive to exclude lobbyists with the *highest* valuations from the finalist set. The intuition is the following: if some lobbyists have valuations much higher than others, including them in the finalist set can make the playing fields too uneven for others in the subsequent all-pay auction and discourage others from submitting high bids. Formula (7) gives the politician's expected revenue when she optimally selects the finalist set of lobbyists.

5. Lottery versus all-pay auction

In this section, I compare lottery and all-pay auction on three dimensions. First, I show that the exclusion principle of the all-pay auction identified by Baye, Kovenock and de Vries (1993) does not extend to lottery; Second, I compare the revenue for the politician and identify cases under which the politician will strictly prefer arranging the lobbying process as a lottery game; Third, I compare lottery and all-pay auction from the perspectives of the lobbyists' total surplus and the social surplus.

5.1. *The exclusion principle*

Here I show that if lobbying is arranged as a lottery, then a revenue maximizing politician will not have strict incentive to exclude lobbyists from the finalist list.

Proposition 1. Suppose that there is a set of potential lobbyists with valuations $v_1 \geq v_2 \geq \ldots \geq v_n > 0$. Then the politician can not gain from constructing an agenda that excludes some lobbyists from the lottery game.

Proof. Let n^* be as defined in (4), and denote $\Omega^+ = \{1, \ldots, n^*\}$, $\Omega^0 = \{n^* + 1, \ldots, n\}$ and let $\Omega = \{1, \ldots, n\}$. The politician can not increase her revenue by excluding a subset $\hat{\Omega} \subseteq \Omega^0$, since the same argument as in the proof of Theorem 1 can be used to establish that the unique equilibrium of the reduced lottery game still gives the same revenue as (5). Now let us consider the more difficult case of excluding a subset $\hat{\Omega} \subseteq \Omega^+$. Suppose for now that only one member $k \in \Omega^+$ is excluded. This will change the set of positive bidders in the unique equilibrium involving $\Omega \backslash \{k\}$. By the characterization in Theorem 1, all the lobbyists in $\Omega^+ \backslash \{k\}$ will still bid positive amount. Suppose that for some $t \geq 1$, lobbyists $n^* + 1, \ldots, n^* + t$ in Ω^0 also bid a positive amount in the equilibrium of the reduced lottery game played by $\Omega \backslash \{k\}$. The politician's revenue from the lobby group $\Omega \backslash \{k\}$, denoted by $R^L_{\Omega \backslash \{k\}}$ is:

$$R^L_{\Omega \backslash \{k\}} = \frac{n^* + t - 2}{\sum_{i=1}^{n^*} \frac{1}{v_i} - \frac{1}{v_k} + \sum_{j=n^*+1}^{n^*+t} \frac{1}{v_j}},$$

then

$$\begin{aligned} R^L(n) - R^L_{\Omega \backslash \{k\}} &= \frac{n^*-1}{\sum_{i=1}^{n^*} \frac{1}{v_1}} - \frac{n^*+t-2}{\sum_{i=1}^{n^*} \frac{1}{v_i} - \frac{1}{v_k} + \sum_{j=n^*+1}^{n^*+t} \frac{1}{v_j}} \\ &= \frac{(n^*-1)\left\{\left[\sum_{j=n^*+1}^{n^*+t} \frac{1}{v_j} - \frac{1}{v_k}\right] - \frac{t-1}{n^*-1} \sum_{i=1}^{n^*} \frac{1}{v_i}\right\}}{\left(\sum_{i=1}^{n^*} \frac{1}{v_i}\right)\left(\sum_{i=1}^{n^*} \frac{1}{v_i} - \frac{1}{v_k} + \sum_{j=n^*+1}^{n^*+t} \frac{1}{v_j}\right)} \end{aligned}$$

Now note that for $j = n^* + 1, \ldots, n^* + t$, $v_j \leq B^A(n^*)$, hence

$$\frac{1}{v_j} \geq \frac{\sum_{i=1}^{n^*} \frac{1}{v_i}}{n^* - 1}.$$

Since $k \in \Omega^+$, $v_k \geq v_{n^*} > B^A(n^*)$, hence

$$-\frac{1}{v_k} > -\frac{\sum_{i=1}^{n^*} \frac{1}{v_i}}{n^* - 1}.$$

Therefore,

$$\left[\sum_{j=n^*+1}^{n^*+t} \frac{1}{v_j} - \frac{1}{v_k}\right] - \frac{t-1}{n^*-1} \sum_{i=1}^{n^*} \frac{1}{v_i} > 0,$$

which implies that $R^L(n) > R^L_{\Omega\setminus\{k\}}$ for any $k \in \Omega^+$.

By induction, one can show that the more general case of excluding more than one lobbyists from Ω^+, possibly together with some lobbyists in Ω^0, will also decrease the politician's revenue. □

Proposition 1 tells us that the exclusion principle identified in Baye, Kovenock and de Vries (1993) for the case of all-pay auction does not apply to lottery. The complication in the proof arises because the politician's revenue from lobbying R^L as given in (5) is not proportional to the harmonic mean of the valuations of all the finalist lobbyists, but only proportional to the harmonic mean of the active lobbyists who bid positive amount in equilibrium. So I need to keep track of the possible change of the active lobbyists when the finalist list is changed. Proposition 1 does not imply that the lottery model is inconsistent with the occasionally observed practice of politician "narrowing down a set of finalists", which is one major motivation of Baye, Kovenock and de Vries (1993). It is rather obvious that the politician's revenue from lobbying remains the same if lobbyists in Ω^0 are excluded. So one can rationalize the practice of "narrowing down a set of finalists" as the politician's effort to be perceived as picking "seriously" the right winner, even though she only cares about revenue. The sharp difference between the all-pay auction and the lottery models is that in the former, the excluded lobbyists may have the highest valuations of the object; while in the latter, they must have the lowest valuations. Furthermore, in the all-pay auction model of lottery, the politician's expected revenue could be maximized if she picks only *two* lobbyists with valuations v_{k^*} and v_{k^*+1} and exclude all the other lobbyists. The lottery model allows the finalists list to be longer than two. So if the politician has a slight preference for a shorter list of lobbyists conditional on the same revenue, the two models of lobbying have sharply different predictions of the number of finalists.

5.2. *The politician's revenue*

Suppose that a politician knows that a set of n potential lobbyists with valuations $v_1 \geq \ldots \geq v_n > 0$ are eyeing for a prize. When will she prefer to arrange the lobbying process as a lottery rather than an all-pay auction? Or equivalently, when will the rent dissipation under lottery be higher than that under all-pay auction?

Proposition 2. Let the valuations of potential lobbyists be $v_1 \geq \ldots \geq v_n > 0$. If $v_{k^*+1} < (\sqrt{2} - 1)v_{k^*}$ where k^* is defined in (6), then $R^L(n) > R^{APA}(n)$, i.e., a revenue maximizing politician strictly prefers lottery to all-pay auction.

Proof. Let $1 \leq k^* \leq n - 1$ satisfy (6).

$$R^L(n) \geq B^A(2) = \frac{1}{\frac{1}{v_1} + \frac{1}{v_2}} \geq \frac{1}{\frac{1}{v_{k^*} + \frac{1}{v_{k^*+1}}}} = \frac{v_{k^*} v_{k^*} + 1}{v_{k^*} + v_{k^*+1}}; \qquad (8)$$

while from formula (7),

$$R^{APA}(n) = \left(1 + \frac{v_{k^*+1}}{v_{k^*}}\right) \frac{v_{k^*+1}}{2}.$$

Then,

$$\begin{aligned} R^L(n) - R^{APA}(n) &\geq \frac{v_{k^*} v_{k^*+1}}{v_{k^*} + v_{k^*+1}} - \left(1 + \frac{v_{k^*+1}}{v_{k^*}}\right) \frac{v_{k^*+1}}{2} \\ &= \frac{2v_{k^*}^2 v_{k^*+1} - (v_{k^*} + v_{k^*+1})^2 v_{k^*+1}}{2(v_{k^*} + v_{k^*+1}) v_{k^*}} \\ &= -\frac{v_{k^*+1}(v_{k^*+1}^2 + 2v_{k^*} v_{k^*+1} - v_{k^*}^2)}{2(v_{k^*} + v_{k^*+1}) v_{k^*}}. \end{aligned}$$

If $v_{k^*+1} < (\sqrt{2} - 1)v_{k^*}$, then $v_{k^*+1}^2 + 2v_{k^*} v_{k^*+1} - v_{k^*}^2 < 0$, which implies that $R^L(n) > R^{APA}(n)$. □

In words, if the top two valuations of the finalist lobbyists that the politician would have optimally selected under the all-pay auction have a large enough gap, namely, if $v_{k^*+1} < (\sqrt{2} - 1)v_{k^*}$, then allocating the prize by means of lottery will generate a higher revenue for the politician. The intuition is as follows: The fierceness of the rent seeking competition in the all-pay auction game is largely determined by the gap in the top two valuations among the finalists. As this gap becomes more substantial, the competition becomes less fierce and the politician gains less in expected revenue. In contrast, the politician's revenue from the lottery is less dependent on the top valuations. Furthermore, since expression (7) for the all-pay auction is the *expected* revenue, while for the lottery the expression (5) is a *certain* revenue, the politician will strictly prefer lottery even when $R^L(n) = R^{APA}(n)$ if she is risk averse.

Proposition 2 provides a sufficient condition for the politician to prefer lottery to all-pay auction. If any one of the two weak inequalities in (8) is strict, then the condition may be too strong. The first weak inequality will be strict if $n^* > 2$, and the second will be if $k^* > 1$. The condition $v_{k^*+1} < (\sqrt{2} - 1)v_{k^*}$ is a necessary and sufficient condition for the politician to prefer lottery iff $n^* = 2$ and $k^* = 1$. The following example illustrates:

Example 1. Suppose $n = 3$ and $v_1 = 100$, $v_2 = 40$, and $v_3 = 20$. Then $k^* = 1$ for the all-pay auction since $(1 + v_2/v_1)v_2/2 = 28 > (1 + v_3/v_2)v_3/2 = 15$; and $n^* = 2$ since $v_3 < B^A(2)$. Note that $v_2 < (\sqrt{2} - 1)v_1$ holds, and indeed Proposition 2 is confirmed since $R^L(3) = B^A(2) = 28.57 > R^{APA}(3) = 28$.

Suppose, instead, that $v_1 = 100$, $v_2 = 50$ and $v_3 = 20$. Then again it can be verified that $n^* = 2$ and $k^* = 1$. Note that $v_2 > (\sqrt{2} - 1)v_1$. Indeed, we find that $R^L(3) = 33.33 < R^{APA}(3) = 37.5$.

On the other hand, if $n^* = 2$ and $k^* = 1$ do not hold, then the following example demonstrates that $v_{k^*+1} < (\sqrt{2} - 1)v_{k^*}$ is not necessary for lottery to be preferred:

Example 2. Suppose $n = 3$ and $v_1 = 200$, $v_2 = 50$, and $v_3 = 40$. Then in the all-pay auction, $k^* = 2$ since $(1 + v_3/v_2)v_3/2 = 36 > (1 + v_2/v_1)v_2/2 = 31.25$. Hence $R^{APA}(3) = 36$. Note that $R^L(3) = B^A(2) = 40 > R^{APA}(3)$ even though $v_{k^*+1} > (\sqrt{2} - 1)v_{k^*}$.

In cases when $n^* = 2$ and $k^* = 1$, Proposition 2 tells us that $v_2 < (\sqrt{2} - 1)v_1$ must hold for the politician to prefer the lottery. We should recognize that this is a strong condition: the second highest valuation must be less than about 41.4 percent of the highest. To the extent that the lobbyists' valuations of the prize are more or less homogeneous, all-pay auctions are likely preferred by the politician.

5.3. *The lobbyists' and the social surplus*

It is interesting to know whether lottery or all-pay auction allocates the prize more efficiently. Since the social surplus is simply the sum of the politician's revenue and the lobbyists' total surplus, it seems natural to compare the lobbyists' total expected surplus from the two modes of prize allocation. From the proof of Theorem 1 in Baye, Kovenockand de Vries (1993) we know that if lobbyists $\{k^*, k^*+1, \ldots, n\}$ are selected by the politician as the finalist, then in any equilibria of the subsequent all-pay auction, only the lobbyist k^* will obtain a (possibly) positive surplus equal to $v_{k^*} - v_{k^*+1}$. Hence the lobbyists' total surplus under all-pay auction, denoted by $S^{APA}(n)$, will be given by:

$$S^{APA}(n) = v_{k^*} -_{k^*+1} . \tag{9}$$

If the lobbying is arranged as a lottery, then the lobbyists' total surplus, denoted by $S^L(n)$, is given by:

$$S^L(n) = \sum_{i=1}^{n^*} \frac{b_i^A(n^*)}{B^A(n^*)} v_i - R^L(n^*)$$

$$= \sum_{i=1}^{n^*}[v_i - B^A(n^*)] - R^L(n^*) \tag{10}$$

$$= n^*[A(\{v_1, \ldots, v_{n^*}\}) - H(\{v_1, \ldots, v_{n^*}\})] + \left(\sum_{i=1}^{n^*} \frac{1}{v_i}\right)^{-1},$$

where $A(\{v_1, \ldots, v_{n^*}\}) = \sum_{i=1}^{n^*} v_i/n^*$ and $H(\{v_1, \ldots, v_n^*\}) = n^*/(\sum_{i=1}^{n^*} 1/v_i)$ are respectively the arithmetic and harmonic mean of $v_1, \ldots, v_n^*$. The second equality in (10) uses the first order condition (2). From the well known result that arithmetic mean is always noless than the harmonic mean (see, e.g., Hardy, Littlewood and Pólya 1934), we know that $S^L(n) > 0$. The following proposition provides a sufficient condition for the lobbyists' total surplus under the lottery to be higher than that under the all-pay auction.

Proposition 3. If $v_2 > (\sqrt{5} - 1)v_1/2$, then $S^{APA} < S^L$.

Proof. Suppose k^* satisfy (6). Then we have:

$$\left(1 + \frac{v_{k^*+1}}{v_{k^*}}\right)\frac{v_{k^*+1}}{2} \geq \left(1 + \frac{v_2}{v_1}\right)\frac{v_2}{2}.$$

Since $v_2 \geq v_{k^*+1}$, it be that:

$$1 \geq \frac{v_{k^*+1}}{v_{k^*}} \geq \frac{v_2}{v_1}.$$

Hence

$$S^{APA} = v_{k^*} - v_{k^*+1} = v_{k^*+1}\left(\frac{v_{k^*}}{v_{k^*+1}} - 1\right) \leq v_2\left(\frac{v_1}{v_2} - 1\right) = v_1 - v_2. \tag{11}$$

Using the formula (10), we have:

$$S^{APA} - S^L \leq (v_1 - v_2) - \left(\sum_{i=1}^{n^*} \frac{1}{v_i}\right)^{-1}.$$

Since $n^* \geq 2$, we have

$$\begin{aligned} S^{APA} - S^L &\leq (v_1 - v_2) - \tfrac{v_1 v_2}{v_1 + v_2} \\ &= \tfrac{v_1^2 - v_2^2 - v_1 v_2}{v_1 + v_2}. \end{aligned}$$

It can then be verified that when $v_2 > (\sqrt{5} - 1)v_1/2$, $S^{APA} - S^{L} < 0$. □

This is a somewhat surprising result: it states that whenever the second highest valuation is more than about 62% of the highest valuation, then the lobbyists' total surplus from lobbying is higher under lottery than that under all-pay auction, regardless of the valuations of other lobbyists. Proposition 3 holds regardless of the valuations of the other lobbyists because under all-pay auction a small gap between the top two valuations provides a *lower* bound on the competitiveness in the all-pay auction game, even when the top two lobbyists are not themselves selected by the politician as finalists. Whenever lobbyists 1 and 2 are excluded from the finalist list under the all-pay auction, it must have been that by excluding them (or one of them) the rent seeking competition among the remaining lobbyists could be made even more fierce. When the rent seeking competition is fierce, the lobbyists' surplus are competed away in the all-pay auction. Thus, as shown by inequality (11), $(v_1 - v_2)$ provides an upper bound of the lobbyists total surplus under the all-pay auction. Again, Proposition 3 provides a sufficient, but not necessary, condition for a lottery to generate a higher lobbyists' total surplus than an all-pay auction. The following example illustrates:

Example 3. Suppose $v_1 = 300$, $v_2 = 180$, $v_3 = 120$, and $v_4 = 100$. It can be verified that under the all-pay auction, the politician will choose $k^* = 2$. Hence $S^{APA}(4) = v_2 - v_3 = 60$. Under lottery, one can verify that $n^* = 3$. Thus $S^{L}(4) = \sum_{i=1}^{3} v_i - 4 \times 2/(\sum_{i=1}^{3} 1/v_i) = 135.48 > S^{APA}(4)$.

It is also interesting to note that in example 3 the politician's expected revenue is also higher under the lottery than under the all-pay auction since $R^{L}(4) = 116.13 > R^{APA}(4) = 100$. Let us define the *social surplus* as the expected valuation of the lobbyist to whom the prize is allocated, which is of course equal to the sum of the total lobbyists' surplus and the politician's expected revenue. In example 3, the social surplus under lottery, denoted by N^{L}, is $N^{L}(4) = R^{L}(4) + S^{L}(4) = 116.13 + 135.48 = 251.61$; and that under the all-pay auction, denoted by N^{APA}, is $N^{APA}(4) = R^{APA}(4) + S^{APA}(4) = 100 + 60 = 160$. Hence in example 3 the social surplus is higher under the lottery than under the all-pay auction. This implies that the prize is allocated more efficiently under the lottery than under the all-pay auction, albeit probabilistically.

To provide general conditions under which the social surplus is higher under the lottery turns out to be diffcult. The general formula for $N^{L}(n)$ is given by:

$$N^L(n) = \sum_{i=1}^{n^*} \frac{b_i^A(n^*)}{B^A(n^*)} v_i = \sum_{i=1}^{n^*} [v_i - B^A(n^*)] \tag{12}$$

$$= n^*[A(\{v_1, \ldots, v_{n^*}\}) - H(\{v_1, \ldots, v_{n^*}\})] + n^* \left(\sum_{i=1}^{n^*} \frac{1}{v_i} \right)^{-1}.$$

and the general formula for $N^{APA}(n)$, is given by:

$$N^{APA}(n) = (v_{k^*} - v_{k^*+1}) + \left(1 + \frac{v_{k^*+1}}{v_{k^*}}\right) \frac{v_{k^*+1}}{2}, \tag{13}$$

where the first term is the expected surplus of the lobbyists and the second term is the expected revenue of the politician.

The main diffculty in comparing the social surplus under the two modes of prize allocation can be seen from the incompatibility of conditions in Propositions 2 and 3: The former states that whenever $v_{k^*+1} < (\sqrt{2} - 1)v_{k^*}$, the politician's expected revenue is higher under lottery; the latter states that whenever $v_2 > (\sqrt{5} - 1)v_1/2$, the lobbyists' total surplus is higher under lottery. We would have a sufficient condition for the social surplus to be higher under lottery if the two conditions were compatible. But they are not compatible: whenever $v_{k^*+1} < (\sqrt{2}-1)v_{k^*}$ holds, it must be that $v_2 < (\sqrt{5}-1)v_1/2$. To see this, suppose otherwise, then we have:

$$\begin{aligned} \left(1 + \tfrac{v_{k^*+1}}{v_{k^*}}\right) \tfrac{v_{k^*+1}}{2} &< \left[1 + (\sqrt{2} - 1)\right] \tfrac{v_{k^*+1}}{2} \\ &< \left(1 + \tfrac{\sqrt{5}-1}{2}\right) \tfrac{v_2}{2} \\ &< \left(1 + \tfrac{v_2}{v_1}\right) \tfrac{v_2}{2}, \end{aligned}$$

which contradicts the definition of k^*.

6. Caps on lobbying spending

In this section I study the effects of caps on lobbying spending when the lobbying process is arranged as a lottery. I then show that for the two-lobbyist case, the perverse effect that lobbying caps may increase the total lobbying expenditure identified by Che and Gale (1998) cannot arise if the lobbying process is arranged as a lottery.

Suppose that there are n lobbyists with valuations $v_1 \geq v_2 \ldots \geq v_n > 0$, and assume an exogenously specified lobbying spending cap $z > 0$. To distinguish from the previous sections, I denote i's bid in the presence of the cap by c_i. Now lobbyist i, given the bids by other lobbyists c_i, solves the following problem (C):

$$\max_{0 \le c_i \le z} \left\{ \frac{c_i}{\sum_{j \ne i} c_j + c_i} v_i - c_i \right\}. \tag{C}$$

Note that his bid is constrained to be non-negative and no more than the cap.

To characterize the equilibrium of the lottery game in the presence of a cap, I first prove the following useful lemma.

Lemma 2. In any equilibrium of the lottery with caps if lobbyists i is constrained by the cap, i.e., if i bids z, then any lobbyist $j < i$ must also bid z.

Proof. Suppose that in the candidate equilibrium, the total bids by lobbyists other than i and j are $C^*_{-i,j}$, and the bids by i and j are respectively c_i^* and c_j^*. Since $c_j^* = z$, the first order condition for lobbyist i must satisfy

$$\frac{C^*_{-i,j} + c_j^*}{(C^*_{-i,j} + c_j^* + z)^2} v_i \ge 1. \tag{14}$$

Suppose to the contrary that lobbyist j is not constrained, that is, suppose $c_j^* < z$. Then it must be that

$$\frac{C^*_{-i,j} + z}{(C^*_{-i,j} + c_j^* + z)^2} v_j = 1. \tag{15}$$

However, (14) and (15) can not hold together if $v_j \ge v_i$ and $z > c_j^*$. Therefore $c_j^* = z$. □

Lemma 2 establishes that there exists a critical lobbyist, denoted by $\overline{n}$, such that $c_i^* = z$ for all $i = 1, \ldots, \overline{n}$, and $c_i^* < z$ for all $i = \overline{n} + 1, \ldots, n$. Of course, if $\overline{n} = 0$, then the cap is not binding, and the strategy profile we identified in Theorem 1 remains an equilibrium of the lottery game with the cap. In order to fully characterize the equilibrium of the lottery with the cap, I need to characterize $\overline{n}$.

I now define a sequence of auxiliary problems $(A - (m, \overline{n}))$: for each $\overline{n} = 0, 1, \ldots, n$, each $m = \max\{\overline{n} + 1, 2\}, \ldots, n$, the group of lobbyists $\{\overline{n} + 1, \ldots, m\}$ participate in a "lottery" game that does not restrict their bids to be non-negative. The objective of bidder i in the auxiliary problem $(A - (m, \overline{n}))$ is given by:

$$\max_{c_i} \left\{ \frac{c_i}{\overline{n}z + \sum_{j=\overline{n}+1}^{m} c_j} v_i - c_i \right\}. \tag{$A - (m, \overline{n})$}$$

Denote $C^A(m, \overline{n}) = \overline{n}z + \sum_{j=\overline{n}+1}^{m} c_j$, then the necessary and sufficient first order condition for lobbyist i in problem $(A - (m, \overline{n}))$ is given by

$$\frac{1}{C^A(m, \overline{n})} - \frac{c_i}{C^A(m, \overline{n})^2} = \frac{1}{v_i}.$$

Summing over $i = \overline{n} + 1, \dots, m$, we get

$$\frac{(m - \overline{n})}{C^A(m, \overline{n})} - \frac{\sum_{j=\overline{n}+1}^{m} c_j}{C^A(m, \overline{n})^2} = \sum_{j=\overline{n}+1}^{m} \frac{1}{v_j}$$

which is equivalent to

$$\frac{(m - \overline{n} - 1)}{C^A(m, \overline{n})} + \frac{\overline{n}z}{C^A(m, \overline{n})^2} = \sum_{j=\overline{n}+1}^{m} \frac{1}{v_j}$$

Solving for $C^A(m, \overline{n})$, we obtain

$$C^A(m, \overline{n}) = \frac{(m - \overline{n} - 1) + \sqrt{(m - \overline{n} - 1)^2 + 4\overline{n}z \sum_{j=\overline{n}+1}^{m} v_j^{-1}}}{2 \sum_{j=\overline{n}+1}^{m} v_j^{-1}}. \quad (16)$$

The equilibrium bids of lobbyist i in auxiliary problem $(A - (m, \overline{n}))$ is given by

$$c_i^A(m, \overline{n}) = C^A(m, \overline{n}) - \frac{C^A(m, \overline{n})^2}{v_i}. \quad (17)$$

Define $n^*(\overline{n}) = \min\{m : v_{m+1} \leq C^A(m, \overline{n})\} \cup \{n\}$. It is clear that $n^*(0) = n^*$ as defined in (4). The definition of $\overline{n}$ as the critical lobbyist described in Lemma 2 requires that the following two conditions must be satisfied:

1. If $\overline{n} < n$, then the $(\overline{n} + 1)$-th lobbyist will optimally bid less than z:

$$c_{\overline{n}+1}^A(n^*(\overline{n}), \overline{n}) < z; \quad (18)$$

2. If $\overline{n} > 0$, then given the bids of others, the first order derivative of the $\overline{n}$-th lobbyist's objective function evaluated at z must be non-negative:

$$\frac{C^A(n^*(\overline{n}), \overline{n}) - z}{C^A(n^*(\overline{n}), \overline{n})^2} v_{\overline{n}} - 1 \geq 0. \quad (19)$$

The first condition says that the $(n + 1)$-th bidder will optimally choose to bid less than z given others' bids; the second condition says that $\overline{n}$-th lobbyists must find bidding z optimal given bids by others. The existence of $\overline{n}$ can be proved by checking if any value of $\overline{n} = 1, \ldots, n$ satisfies the above two conditions. If all positive value of $\overline{n}$ violate at least one of the above two conditions, then $\overline{n} = 0$ must satisfy both.

Theorem 3. Let $\overline{n}$ be a critical lobbyist that satisfies conditions (18) and (19). Then the following is an equilibrium of the lottery game with cap z:

1. for $i = 1, \ldots, \overline{n}$, $c_i^* = z$;
2. for $i = \overline{n}, \ldots, n^*(\overline{n})$, $c_i^* = c_i^A(n^*(\overline{n}), \overline{n})$;
3. for $i = n^*(\overline{n}) + 1, \ldots, n$, $c_i^* = 0$.

The proof of Theorem 3 follows quite closely that of Theorem 1, and it is thus omitted. It is interesting to note that the equilibrium characterization of the lottery game with a cap incorporates the lottery without a cap as a special case. From the expression (16) one can see that if there is no cap on spending, then $\overline{n} = 0$, which yields that $C^A(m, 0) = B^A(m)$. To the best of my knowledge, this is the first study on the lottery model of lobbying with spending caps.

For the case of two lobbyists with valuations $v_1 > v_2 > 0$, Che and Gale (1998) show that if the cap $z \in (v_2(v_1 + v_2)/4v_1, v_2/2)$, then the total lobbying expenditure (or the politician's expected revenue) in an all-pay auction game will be higher with the cap than without. I will show that if the lobbying process is arranged as a lottery, then the imposition of an exogenous cap on bids will never increase the total lobbying expenditure for the two-lobbyist case.

Proposition 4. If $n = 2$ and $v_1 > v_2 > 0$. Then for any value of the cap z, the total lobbying expenditure with the cap z is no more than that without the cap.

Proof. Consider three possible cases. (Case I). Both lobbyists are not constrained by the cap z. Obviously in this case the lobbying expenditures are the same with the cap as that without. (Case II). Both lobbyists are constrained by the cap. For this to be an equilibrium, lobbyist 2's first order condition must satisfy $zv_2/(4z^2) \geq 1$, which implies that the totalexpenditure with the cap 2z is no more than $v_2/2$. Recall that without the cap, the total lobbying expenditure is $R^L(2) = v_1v_2)$, which is no less than $v_2/2$. (Case

III).Lobbyist 1 is constrained by z, while 2 is not. In this case, the total lobbying expenditure is given by expression $C^A(2, 1)$ as defined in (16), i.e., $C^A(2, 1) = \sqrt{zv_2}$. For this to be consistent with equilibrium, $\bar{n} = 1$ must satisfy condition (18), that is,

$$\frac{\sqrt{zv_2} - z}{zv_2} v_1 - 1 \geq 0.$$

After some simple manipulations, the above inequality is equivalent to $\sqrt{zv_2} \leq 1/(v_1^{-1} + v_2^{-1})$, that is, $C^A(2, 1) \leq R^L(2)$. □

7. Conclusion

In this paper I first provide a complete characterization of the unique equilibrium in a lottery model of the lobbying process with n lobbyists with asymmetric valuations, and then compare the lottery and the all-pay auction models of lobbying. It is shown that a revenue maximizing politician will not have strict incentive to construct an agenda that excludes some lobbyists from participating in the lottery, in contrast to the exclusion principle discovered by Baye, Kovenock and de Vries (1993) for all-pay auction. I also show that whenever the valuations of lobbyists among the finalists selected by the politician under the all-pay auction game are quite different, namely, if $v_{k^*+1} < (\sqrt{2} - 1)v_{k^*}$, then lottery will generate a higher expected revenue for the politician. I also characterize the equilibrium of the lottery game with an exogenous cap on the lobbying spending, and prove that the perverse effect of an exogenous spending cap on the total lobbying expenditure for all-pay auction discovered by Che and Gale (1998) does not extend to lottery. The theoretical comparison of the lottery and the all-pay auction models of lobbying in this paper yield some testable implications. For example, the excluded lobbyists under all-pay auction may have the highest valuations, while those excluded under lottery, if any, must have the lowest valuations. In the two-lobbyist case, the total expenditure would never increase as a result of the lobbying cap under lottery, while it could increase under all-pay auction. These can serve as a useful basis to identify which model is more empirically relevant.

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Appendix: Proof of Theorem 1

I proceed by first showing that the candidate bids specified in Theorem 1 indeed constitute an equilibrium of the lottery game, and then show that there are no other equilibria.

Claim 1. $b_i^* = b_i^A(n^*)$ for $i = 1, \ldots, n^*$ and $b_i^* = 0$ for $i = n^* + 1, \ldots, n$ constitute an equilibrium.

Proof. Consider a lobbyist $i \in \{1, \ldots, n^*\}$. Suppose that all other lobbyists are betting according to the candidate equilibrium. Then it suffices to show that:

$$b_i^A(n^*) = \arg\max_{b_i \geq 0} \left\{ \frac{b_i}{B^A(n^*) - b_i^A(n^*) + b_i} v_i - b_i \right\}$$

Lemma 1 implies that $b_i^A(n^*) > 0$, and it is easy to see that $b_i^A(n^*)$ satisfies the necessary and sufficient first order condition.

Now consider lobbyist $i \in \{n^* + 1, \ldots, n\}$. Given the postulated bids by other lobbyists, i's problem is

$$\max_{b_i \geq 0} \left\{ \frac{b_i}{B^A(n^*) + b_i} v_i - b_i \right\}$$

The first order condition evaluated at zero is

$$\frac{B^A(n^*)[v_i - B^A(n^*)]}{B^A(n^*)^2} \leq \frac{V^A(n^*)[v_{n^*+1} - B^A(n^*)]}{B^A(n^*)^2} \leq 0$$

hence $b_i^* = 0$ is indeed a best response. □

Claim 2. There is no other pure strategy equilibrium.

Proof. Suppose that there is another pure strategy equilibrium $\{b_1^{**}, \ldots, b_n^{**}\}$. Define $\Omega^+ \equiv \{i : b_i^{**} > 0\}$, and $B^{**}(\Omega^+) \equiv \sum_{i \in \Omega^+} b_i^{**}$. From the first order conditions of lobbyists in the set Ω^+, we can obtain:

$$B^{**}(\Omega^+) = \frac{|\Omega^+|}{\sum_{i \in \Omega^+} \frac{1}{v_i}},$$

where $|\Omega^+|$ is the cardinality of the set Ω^+. Now we claim that for any $j = 1, \ldots, n$, if $v_j \geq v_i$ for some $i \in \Omega^+$, then $j \in \Omega^+$. To see this, first note that since $i \in \Omega^+$ lobbyist i's first order condition implies that $v_i > B^{**}(\Omega^+)$. Suppose to the contrary that $j \notin \Omega^+$, then j could profitably deviate by bidding a positive amount since his first order condition evaluated at zero is

$$\frac{B^{**}(\Omega^+)[v_j - B^{**}(\Omega^+)]}{B^{**}(\Omega^+)^2} > 0.$$

Hence it must be true that $\Omega^+ = \{1, \ldots, \hat{n}\}$ for some $2 \le \hat{n} \le n$. Now what is left to show is that $\hat{n} = n^*$. Suppose $\hat{n} < n^*$. Then lobbyist $\hat{n} < i \le n^*$ can profitably deviate by bidding positive amount since his first order condition evaluated at zero is strictly positive by the definition of n^*. If $\hat{n} > n^*$, then lobbyist $n^* < i \le \hat{n}$ can profitably deviate by bidding zero. To see this, we note that by the definition of n^*, $v_{n^*+1} \le B^A(n^*)$, which implies that

$$\frac{1}{v_{n^*+1}} \ge \frac{\sum_{j=1}^{n^*} \frac{1}{v_j}}{n-1}.$$

Suppose that $v_{n^*+1} > B^A(n^*+1)$, then

$$\frac{1}{v_{n^*+1}} < \frac{1}{B^A(n^*+1)} = \frac{\sum_{j=1}^{n^*} \frac{1}{v_j} + \frac{1}{v_{n^*+1}}}{n^*}$$
$$\Rightarrow \frac{1}{v_{n^*+1}} < \frac{\sum_{j=1}^{n^*} \frac{1}{v_j}}{n-1}$$

a contradiction. Hence $v_{n^*+1} \le B^A(n^*+1)$. By induction, $v_i \le B^A(i)$ for every $n^* < i \le \hat{n}$. Therefore the first order condition for $\hat{n}$ evaluated at zero is non-positive, hence $b_{\hat{n}}^{**} = 0$. A contradiction to $\hat{n} \in \Omega^+$. □

Claim 3. There is no mixed strategy equilibrium.

Proof. Suppose that there is an equilibrium in which a set of lobbyists, denoted by Ω^m, play non-degenerate mixed strategies. For each $i \in \Omega^m$, write i's mixed strategy (in c.d.f.) as $F_i(\cdot)$. Denote the set of lobbyists who in this candidate equilibrium still play a pure strategy as $\overline{\Omega}^m$. Now consider $k \in \Omega^m$. He solves the following problem, taking as given $\{F_i : i \in \Omega^m \backslash \{k\}\}$ and $\{b_j : j \in \overline{\Omega}^m\}$:

$$\max_{b_i \ge 0} \underbrace{\int \cdots \int}_{|\Omega^m|-1} \left[\frac{b_k}{\sum_{i \in \Omega^m \backslash \{k\}} b_i + \sum_{j \in \overline{\Omega}^m} b_j + b_k} v_k - b_k \right] d\Pi_{i \in \Omega^m} \backslash \{k\} F_i(b_i),$$

however since the objective function is strictly concave and hence there is a unique optimum, contradicting $i \in \Omega^m$. Hence $\Omega^m = \oslash$. □

The above three claims complete the proof of Theorem 1.