Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Economic Theory 126 (2006) 1–30
JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

Multidimensional private value auctions☆

Hanming Fang[a], Stephen Morris[b,*]

[a]*Department of Economics, Yale University, P.O. Box 208264, New Haven, CT 06520-8264, USA*
[b]*Department of Economics, Yale University, P.O. Box 208281, New Haven, CT 06520-8281, USA*

Received 5 June 2003; final version received 23 November 2004
Available online 11 February 2005

Abstract

We consider parametric examples of symmetric two-bidder private value auctions in which each bidder observes her own private valuation as well as noisy signals about her opponent's private valuation. We show that, in such environments, the revenue equivalence between the first and second price auctions (SPAs) breaks down and there is no definite revenue ranking; while the SPA is always efficient allocatively, the first price auction (FPA) may be inefficient; equilibria may fail to exist for the FPA. We also show that auction mechanisms provide different incentives for bidders to acquire costly information about opponents' valuation.
© 2004 Elsevier Inc. All rights reserved.

JEL classification: C70; D44; D82

Keywords: Multidimensional auctions; Revenue equivalence; Allocative efficiency; Information acquisition

1. Introduction

The paradigm of symmetric independent private value (IPV) auctions assumes that each bidder's valuation of an object is independently drawn from an identical distribution [30,35]. Each bidder observes her own valuation, and has no information about her opponent's

☆ We are grateful to Estelle Cantillon, David J. Cooper, Sergio Parreiras, Michael Peters, Andrew Postlewaite, seminar participants at Case Western Reserve, Ohio State, UCLA and 2004 North American Econometric Society Meetings at San Diego, two anonymous referees and especially an Associate Editor for helpful suggestions and comments. All errors are our own.

* Corresponding author. Fax: +1 203 432 6167.
E-mail addresses: hanming.fang@yale.edu (H. Fang), stephen.morris@yale.edu (S. Morris).

0022-0531/$ - see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jet.2004.11.002

valuation except for the distribution from which it is drawn. An important implication of this assumption is that each bidder's belief about her opponent's valuation is independent of her own or her opponents' valuations, and it is common knowledge.

This does not seem realistic. In actual auctions, bidders may possess or may have incentives to acquire information about their opponents' valuations. Such information in most cases is *privately observed* and *noisy*. This phenomenon arises in both private and common value auction environments. For example, in highway construction procurement auctions, the capacity constraints of the bidders are an important determinant of their costs [18]. While the actual cost of firm i is its private information, *other* firms may still obtain signals about firm i's cost based on their noisy observations of how much firm i is stretched in its capacity. Another example is timber auctions. Timber firms in the US Forest Service timber auctions can cruise the tract and form estimates of its characteristics [3]. A firm can obtain some noisy information about its opponents' estimates via insider rumors and industrial espionage. Our model captures two key features of both examples. First, a bidder's belief about her opponents' types is impacted by their actual types. Second, each bidder is aware that her opponents may have some signals about her type, but does not actually know the signals observed by her opponents; thus each bidder is uncertain about her opponents' belief regarding her type.

In this paper, we assume that bidders have noisy information about opponents' valuations and explore its consequences under the first-price (FPA) and second-price (SPA) auction mechanisms. Specifically, we consider parametric examples of two-bidder private value auctions in which each bidder's private valuation of the object is independently drawn from an identical distribution, and each bidder observes a noisy signal about her opponent's valuation. Thus, each bidder has a two-dimensional type that includes her own valuation (the *valuation* type) and the signal about her opponent's valuation (the *information* type). A bidder's information about her opponents' valuation is not known by her opponent. In such multidimensional auction environments, we show the following results. First, revenue equivalence between the SPA and the FPA in the standard one-dimensional symmetric IPV environments breaks down. However, our examples demonstrate that there is no general revenue ranking between the FPA and the SPA. Second, the equilibrium allocation of the object could be inefficient in the FPA but is always efficient in the SPA. Moreover, the revenue and allocative efficiency may not coincide: on the one hand, an inefficient FPA may generate a higher expected revenue for the seller; on the other hand, the seller's expected revenue could be higher in the SPA even when the object is efficiently allocated in both auctions. The inefficiency in the FPA will typically be non-monotonic in the accuracy of information, since with either complete information or no information, efficiency is obtained in the FPAs. Third, while the SPA always admits equilibrium in weakly dominant strategies, the FPA may not have any equilibrium. Finally, we show that different auction mechanisms provide different incentives for bidders to acquire costly information about opponents' private valuations. We illustrate all these results in simple examples that we can solve explicitly. However, we also argue in each case why the key features of the examples will occur more generally.

It is important to distinguish our environment from affiliated private value (APV) auctions [38,28]. In APV model, a bidder's belief about her opponents' valuations monotonically (in a stochastic sense) depends on her own private valuation, but does not depend on her opponents' actual valuation. That is, a bidder's private valuation at the same time provides information about opponents' valuations. Thus in APV, a bidder's belief about her opponents' valuations is no longer common knowledge; but there is still a one-to-one mapping between a bidder's belief and her own value. Our model introduces a simple but genuine *separation* between a bidder's private valuation and her signal about opponents' valuations—a bidder's belief is impacted by her opponents' realized values—thus breaking the one-to-one mapping between belief and value under the APV. It is in this sense that our model is "multidimensional." It is always possible to encode multidimensional types in a single-dimensional variable, so, as always, the content of the multidimensional signals depends on additional assumptions made about the signal space. [1]

Our examples add to the list of departures from the standard symmetric IPV auction environments in which the revenue equivalence between the FPA and SPA fails. Wilson [38] and Milgrom and Weber [28] showed that when bidders' valuations are symmetric and affiliated, the seller's expected revenue is (weakly) higher in the SPA than in the FPA, even though both auctions are allocatively efficient. [2] Maskin and Riley [23] consider private value auctions in which bidders are ex ante asymmetric in the sense that different bidders' valuations are drawn from different distributions. They show that the revenue ranking between the first and the SPAs is ambiguous even though the SPA is at least as efficient as the FPA (see also [1,7]). In their model, bidders' types are one-dimensional and the asymmetry among bidders is common knowledge. Holt [14] and Matthews [26] show that, when bidders are symmetrically risk averse, the seller's expected revenue in the FPA is higher than that in the SPA. Che and Gale [8] compare the standard auctions with financially constrained bidders, and show that the seller's expected revenue in the FPA is higher than that in the SPA. The bidders in their paper are privately informed of both their valuation of the object and their financial capacity, and thus have multidimensional types. However, both the bidders' valuation and financial capacity are assumed to be independently drawn from identical distributions, hence a bidder's belief about her opponents' type remains common knowledge.

A recent paper by Kim and Che [19] considers private value auction environments in which subgroups of bidders may *perfectly* observe the valuations of others within the group but have no information about bidders outside of their own subgroup. They show that the FPA is allocatively inefficient with positive probability and the seller's expected revenue is lower in the FPA than that in the SPA. A very nice feature of their model is that in equilibrium the competition is effectively among group leaders—bidders who have the highest valuations in their respectively subgroups—with the additional constraint that each group leader bids at least the second highest valuation in her subgroup. Their environment generates both ex ante and ex post asymmetries among group leaders. Ex ante, the leader in a larger group

[1] Multidimensional types can always be encoded in a single-dimensional variable using the inverse Peano function [36, p. 36] and other methods. The difficulty of such a one-dimensional representation of an intrinsically multidimensional problem, however, is that we could not impose reasonable restrictions on the information structure, such as types being drawn from a continuous distribution. Similar issues concerning the representation of multidimensional information with single-dimensional messages have been discussed in the mechanism design literature (see [29]). We are grateful to the Associate Editor for bringing this issue to our attention.

[2] Landsberger et al. [20] showed that in asymmetric APVs auction, the FPA may generated higher expected revenue than the SPA.

has stochastically higher valuation than the leader in a smaller group; and ex post, leaders may face different degrees of within-group competition. The ex post asymmetric within-group competition in particular underlies the inefficiency of the FPA. The specific forms of asymmetry in their model lead to their revenue ranking of the SPA over the FPA, rather than the ambiguous ranking under more general asymmetries as considered in [23]. Our model differs from Kim and Che [19] mainly in the information structure. In our model, a bidder's information regarding her opponents' valuation is *noisy* and *private*; while in theirs, the information about rivals' types are *perfect* and *public* within a subgroup. As a result, the reasons underlying the possible allocative inefficiency of the FPA are slightly different. In their paper, the ex post asymmetry in the within-group competition faced by group leaders is the key reason: a relatively low-valued group leader facing a close competitor in her subgroup may be driven to bid more than a relatively high-valued group leader facing a distant competitor. Because the information within a subgroup is perfect and public in their model, such within-group asymmetry is publicly known. In our model, the FPA inefficiency is also related to bidder asymmetry. The contrast is that in our model the asymmetry among agents are never publicly known and are only probabilistically perceived based on bidders' noisy signals about their opponents.

We restrict the seller to two possible mechanisms for allocating the object: FPA and SPA. With more general mechanisms, in our setting, sellers could fully extract the surplus, exploiting the correlation between bidders' multidimensional types, using the type of argument employed in [9]. [3] Such mechanisms rely on very strong common knowledge assumptions among the seller and the bidders and would not work on more realistic type spaces (see [31,5]). For this reason, we restrict attention to simple mechanisms. Our work is an attempt to make a first step at relaxing the standard (but unfortunate) assumption in auction theory of identifying players' beliefs with their payoff types. [4] An alternative way of allowing richer beliefs into standard IPV auctions is to introduce strategic uncertainty by relaxing the solution concept from equilibrium to rationalizability. This avenue has been pursued by Battigalli and Siniscalchi [4] and Dekel and Wolinsky [10], for first price auctions, while maintaining the assumption of no private information about others' values.

The remainder of the paper is structured as follows. Section 2 presents the parametric environment we examine; Section 3 shows the revenue non-equivalence between the FPA and SPA in our auction environment; Section 4 shows the possible inefficiency of the FPA; Section 5 shows that there may exist no equilibrium in the FPA; Section 6 provides examples that reverse the revenue ranking between the FPA and the SPA and illustrate the incentives of information acquisition under different auction mechanisms; and Section 7 concludes.

[3] The literature on general mechanisms with multidimensional types focusses on efficiency questions. Jehiel and Moldovanu [17] show that, generically, there are no efficient auction mechanisms when bidders have independent multidimensional signals and interdependent valuations. McLean and Postlewaite [27] study situations in which bidders' valuations consist of both common and idiosyncratic components. Bidders privately observe their idiosyncratic component of the valuation, and some signal regarding the common component. They show that a modification of the Vickrey auction is efficient under quite general conditions in their settings.

[4] See [12] for the same relaxation in the context of models of bargaining under incomplete information.

2. The model

Two bidders, $i = 1, 2$, compete for an object. Bidders' valuations of the object are private and independently drawn from identical distributions. We assume that bidders' valuation of the object takes on three possible values $\{V_l, V_m, V_h\}$ where $V_l < V_m < V_h$.[5] The *ex ante* probability of bidder i's valuation v_i taking on value V_k is denoted by $p_k \in [0, 1)$ where $k \in \{l, m, h\}$. Of course $\sum_{k\in\{l,m,h\}} p_k = 1$. To ease exposition, we will refer to bidder 1 as "she" and bidder 2 as "he", and refer to a generic bidder as "she" when no confusion shall arise.

As in standard private value auction models, bidder i observes her private valuation $v_i \in \{V_l, V_m, V_h\}$. The novel feature of this paper is as follows: we assume that each bidder also observes a noisy signal about her opponent's valuation. For tractability, we assume that the noisy signal takes on two possible qualitative categories $\{L, H\}$. Bidder i's signal $s_i \in \{L, H\}$ about j's valuation v_j is generated as follows. For $k \in \{l, m, h\}$, and $i, j \in \{1, 2\}, i \neq j$,

$$\Pr\left(s_i = L|v_j = V_k\right) = q_k, \quad \Pr\left(s_i = H|v_j = V_k\right) = 1 - q_k, \tag{1}$$

where $q_k \in [0, 1]$. We assume that $q_l \geqslant q_m \geqslant q_h$. Note that when $q_l = q_m = q_h$, the signals are completely uninformative about the opponent's valuation.[6] We assume that bidders' signals s_1 and s_2 are independent. To summarize, each bidder has a two-dimensional type $(v_i, s_i) \in \{V_l, V_m, V_h\} \times \{L, H\}$ where v_i is called bidder i's *valuation* type and s_i her *information* type. The primitives of our model are a tuple of nine parameters as follows:

$$\mathcal{E} = \left\{ \langle V_k, p_k, q_k \rangle_{k\in\{l,m,h\}} : V_l < V_m < V_h,\, p_k \in [0, 1], \right.$$
$$\left. \sum_k p_k = 1, q_k \in [0, 1], q_l \geqslant q_m \geqslant q_h \right\}.$$

Any element $e \in \mathcal{E}$ is called an *auction environment*.

We first compare the seller's expected revenue and the allocative efficiency of the standard auctions. Since we are in a two-person private value environment, Dutch and English auctions are strategically equivalent to the FPA and the SPA, respectively. Thus we will only analyze the FPA and the SPA: Bidders simultaneously submit bids; the high bidder wins the object. In the event of a tie, we assume that the bidder with higher valuation wins the object if the bidders' valuations are different; and the tie-breaking can be arbitrary if the bidders' valuations are the same.[7]

[5] Wang [37] and Campbell and Levin [6] studied common value auctions with discrete valuations.

[6] Because completely uninformative signals are the same as no signals at all, this special case corresponds to the standard one-dimensional IPV model.

[7] It is now well known that tie-breaking rules are important in guaranteeing equilibrium existence in FPAs. This tie-breaking rule is endogenous yet incentive compatible in the sense that bidders with tying bids in equilibrium would truthfully reveal their values if asked. See [16] for more general discussions of endogenous sharing rules. Kim and Che [19] and Maskin and Riley [24] used a similar assumption.

As usual, we will analyze the auctions as a Bayesian game of incomplete information between two bidders in which the type space for each bidder is $T \equiv \{V_l, V_m, V_h\} \times \{L, H\}$. Bidder i's generic type is $t_i = (v_i, s_i) \in T$. Given her information type s_i, bidder i updates her belief about j's valuation type v_j according to Bayes' rule as follows. For $s_i \in \{L, H\}$, and $k \in \{l, m, h\}$,

$$\Pr\left(v_j = V_k | s_i = L\right) = \frac{p_k q_k}{\sum_{k' \in \{l,m,h\}} p_{k'} q_{k'}},$$
$$\Pr\left(v_j = V_k | s_i = H\right) = \frac{p_k (1 - q_k)}{\sum_{k' \in \{l,m,h\}} p_{k'} (1 - q_{k'})}. \tag{2}$$

Analogously, given her valuation type v_i, bidder i updates her belief about j's information type s_j according to the signal technology specified by (1). For any $(t_1, t_2) = ((v_1, s_1), (v_2, s_2)) \in T^2$, the joint probability mass is

$$\Pr(t_1, t_2) = \Pr(v_1) \Pr(s_1 | v_2) \times \Pr(v_2) \Pr(s_2 | v_1)$$

and the conditional probability is

$$\Pr\left(t_i | t_j\right) = \Pr\left(v_i | s_j\right) \Pr\left(s_i | v_j\right) \quad \text{where } i \neq j. \tag{3}$$

3. Seller's expected revenue

We first show that the celebrated revenue equivalence result for the standard one-dimensional IPV auctions breaks down in our multidimensional setting. To demonstrate this result in the simplest possible fashion, we consider a special case of the above model:

- $p_m = 0, p_l \in (0, 1), p_h \in (0, 1)$. That is, the bidders' valuations are only of two possible types, $\{V_l, V_h\}$.
- $q_l = 1 - q_h = q \in [1/2, 1]$. That is, signal L is equally indicative of value V_l as signal H is of value V_h. The parameter q measures the accuracy of the signal: when $q = 1/2$, the signals are completely uninformative; and when $q = 1$, the signals are perfectly informative.

3.1. Second-price auction

In the SPA, it is routine to show that the unique equilibrium in weakly dominant strategies in this multidimensional setting is for a bidder of type (v_i, s_i) to bid her private value v_i regardless of her information type. That is, the equilibrium bidding strategy of bidder i in the SPA, denoted by B_i^{SPA}, is

$$B_i^{\text{SPA}}(v_i, s_i) = v_i \quad \text{for all } (v_i, s_i) \in \{V_l, V_h\} \times \{L, H\}. \tag{4}$$

In fact, this equilibrium characterization for the SPA is completely general to any private value auction environment and does not depend on the number of bidders, discrete valuation and signal types. We thus conclude that the multidimensional SPA is efficient; and the seller's expected revenue is independent of accuracy of the signals, hence equal to that in the standard environment where bidders only observe their own valuations.

3.2. First-price auction

The unique equilibrium of the FPA for this special case is characterized as follows:

Proposition 1. *If $p_m = 0$, $q_l = 1 - q_h = q \in [1/2, 1]$, then the unique equilibrium of the FPA is symmetric and is described as follows: for $i = 1, 2$,*

1. $B_i^{\text{FPA}}(V_l, s) = V_l$ *for* $s \in \{L, H\}$.
2. *Type-(V_h, L) bidder i mixes over $\left[V_l, \bar{b}_{(V_h,L)}\right]$ according to* CDF $G_{(V_h,L)}(\cdot)$ *specified by*

$$G_{(V_h,L)}(b) = \frac{p_l q (b - V_l)}{p_h (1-q)^2 (V_h - b)}, \tag{5}$$

where

$$\bar{b}_{(V_h,L)} = \frac{p_h (1-q)^2 V_h + p_l q V_l}{p_h (1-q)^2 + p_l q}. \tag{6}$$

3. *Type-(V_h, H) bidder i mixes over $\left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$ according to* CDF $G_{(V_h,H)}(\cdot)$ *specified by*

$$G_{(V_h,H)}(b) = \frac{(p_l + p_h q)(1-q)\left(b - \bar{b}_{(V_h,L)}\right)}{p_h q^2 (V_h - b)}, \tag{7}$$

where

$$\bar{b}_{(V_h,H)} = \frac{p_h q^2 V_h + (p_l + p_h q)(1-q)\,\bar{b}_{(V_h,L)}}{p_h q^2 + (p_l + p_h q)(1-q)}. \tag{8}$$

The proof is relegated to the appendix. Fig. 1 illustrates the equilibrium. The reason that a bidder with valuation V_l will bid V_l regardless of her signal about her opponent's value is similar to that in Bertrand competition between two firms with identical costs. Type-(V_h, L) bidder will play a mixed strategy over a support $\left[V_l, \bar{b}_{(V_h,L)}\right]$. The CDF $G_{(V_h,L)}(\cdot)$ is chosen to ensure that each bid in the support generates the same constant expected surplus. Type-(V_h, H) bidder will mix over a higher support $\left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$ because she perceives her opponent to more likely have valuation V_h, thus bidding more aggressively.

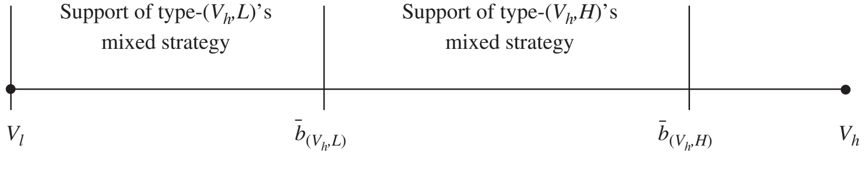

Fig. 1. A graphic illustration of the equilibrium of the FPA in Proposition 1.

3.3. Revenue non-equivalence

Now we compare the seller's expected revenue from the SPA and the FPA. Because bidders bid their own private valuations in the SPA, the seller receives V_h if and only if both bidders have valuation type V_h (an event that occurs with probability p_h^2) and the seller receives V_l otherwise. Hence the seller's expected revenue from the SPA, denoted by R^{SPA}, is

$$R^{\text{SPA}} = \left(1 - p_h^2\right) V_l + p_h^2 V_h. \tag{9}$$

Since in the SPA a bidder obtains positive surplus $(V_h - V_l)$ only when her valuation is V_h and her opponent's valuation is V_l, an event that occurs with probability $p_h p_l$, each bidder's *ex ante* expected surplus from the SPA, denoted by M^{SPA}, is

$$M^{\text{SPA}} = p_h p_l \left(V_h - V_l\right).$$

In the unique equilibrium of the FPA characterized in Proposition 1, the object is always efficiently allocated. Thus, the expected social welfare is $p_l^2 V_l + \left(1 - p_l^2\right) V_h$. In equilibrium, bidders with valuation type V_l obtains zero expected surplus; and type-(V_h, L) and type-(V_h, H) bidders respectively obtain expected surplus $K_{(V_h,L)}$ and $K_{(V_h,H)}$ as described by (A.2) and (A.5) in the appendix. The *ex ante* probabilities that bidder i is of type (V_h, L) and (V_h, H) are, respectively, $\Pr\left[t_i = (V_h, L)\right] = p_h\left[p_h(1-q) + p_l q\right]$ and $\Pr\left[t_i = (V_h, H)\right] = p_h\left[p_h q + p_l(1-q)\right]$. Thus, the *ex ante* expected surplus of each bidder from the FPA, denoted by M^{FPA}, is

$$\begin{aligned} M^{\text{FPA}}(q) &= \Pr\left[t_i = (V_h, L)\right] K_{(V_h,L)} + \Pr\left[t_i = (V_h, H)\right] K_{(V_h,H)} \\ &= \frac{p_h q(1-q) + p_l q}{p_h (1-q)^2 + p_l q}\, p_h p_l \left(V_h - V_l\right). \end{aligned}$$

We have the following observations. First, M^{FPA} depends on q and M^{SPA} is independent of q. The intuition is simply that bidders strategically use their information about opponent's valuation only in the FPA. Second, $M^{\text{FPA}}(q) > M^{\text{SPA}}$ for all $q \in (1/2, 1)$ and $M^{\text{FPA}}(1/2) = M^{\text{FPA}}(1) = M^{\text{SPA}}$. That is, a bidder's expected surplus is strictly higher in the FPA than that in the SPA except for the completely uninformative and completely informative signal cases. When $q = 1/2$, the signals are completely uninformative, and bidders would simply disregard their information type. We can see from Lemma A.1 that the probability densities of $G_{(V_h,L)}$ and $G_{(V_h,H)}$ can be smoothly pasted at $\bar{b}_{(V_h,L)}$ when $q = 1/2$, which implies that effectively, when $q = 1/2$, bidders of valuation type V_h are simply playing a mixed strategy on the whole support of $\left[V_l, \bar{b}_{(V_h,H)}\right]$. When $q = 1$, the FPA becomes a complete information auction, and it is well known that it is revenue equivalent to the SPA.

The seller's expected revenue in the FPA, denoted by R^{FPA}, is simply the difference between the expected social welfare and the sum of the bidders' expected surplus. That is,

$$R^{\text{FPA}}(q) = \left[p_l^2 V_l + \left(1 - p_l^2\right) V_h\right] - 2M^{\text{FPA}}(q)$$

$$= \left(1 - p_h^2\right) V_l + p_h^2 V_h - \frac{2p_h^2 p_l \left(2q - 1\right) \left(1 - q\right)}{p_h \left(1 - q\right)^2 + p_l q} \left(V_h - V_l\right)$$

$$= R^{\text{SPA}} - \frac{2p_h^2 p_l \left(2q - 1\right) \left(1 - q\right)}{p_h \left(1 - q\right)^2 + p_l q} \left(V_h - V_l\right). \tag{10}$$

The following proposition summarizes the comparison between $R^{\text{FPA}}(q)$ and R^{SPA}:

Proposition 2 (*Revenue non-equivalence*). *Let $p_m = 0$ and $q_l = 1 - q_h = q$. For any $q \in (1/2, 1)$, $R^{\text{FPA}}(q) < R^{\text{SPA}}$; and $R^{\text{FPA}}(1/2) = R^{\text{FPA}}(1) = R^{\text{SPA}}$; moreover, $R^{\text{FPA}}(q)$ has a unique minimizer.*

That $R^{\text{FPA}}(q)$ has a unique minimizer in q follows from simple algebra. Fig. 2 depicts the seller's expected revenues as a function of $q \in [1/2, 1]$ from the two auction mechanisms for an example where $p_h = 0.75$, $V_l = 0$, $V_h = 1$.

The standard revenue equivalence theorem [35,30] crucially relies on the bidders' types being single-dimensional and on the valuations being drawn from continuous distributions. Both are important in the argument that bidders' expected payoffs are completely determined from the winning probabilities. One can show, however, that the revenue equivalence between the FPA and SPA still holds in a single dimensional private value auction environment with finite number of valuation types (see Appendix B for a proof). Therefore, the revenue non-equivalence we show in Proposition 2 is not due to the discreteness of the valuation space, rather it is due to the unique information structure.

We also note that in this two-valuation example (since $p_m = 0$), the SPA generates a higher expected revenue for the seller than the FPA despite the fact that both auction mechanisms are allocatively efficient. This is similar to the one-dimensional APV auctions: the objects are allocated efficiently in both the first and second-price APV auctions, but the SPA generates weakly higher expected revenue for the seller [28]. However, in a simple one-dimensional correlated-value analog of our discrete-value example, the FPA and the SPA are actually revenue equivalent. To see this, suppose that bidders' valuations are correlated as follows: $\Pr\left(v_i = V_l | v_j = V_l\right) = \Pr\left(v_i = V_h | v_j = V_h\right) = \rho \in [1/2, 1]$. Such symmetric correlation requires that, ex ante, v_i takes on values V_l and V_h with probability $1/2$, i.e., $\rho_l = \rho_h = 1/2$. Each bidder privately observes her valuation, and *no* additional information

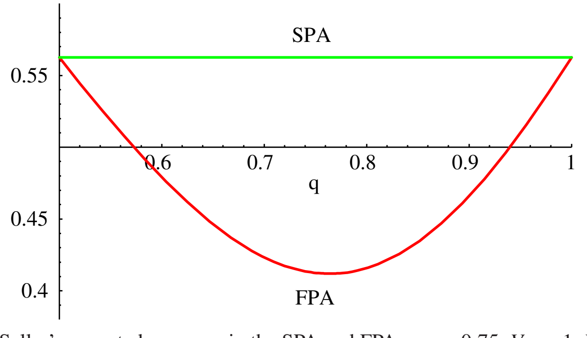

Fig. 2. Seller's expected revenues in the SPA and FPA: $p_h = 0.75$, $V_h = 1$, $V_l = 0$.

about her opponent's value. The seller's expected revenue under the SPA is

$$\frac{\rho V_h}{2} + \left(1 - \frac{\rho}{2}\right) V_l,$$

since with probability $\rho/2$ both bidders receive V_h and hence bid V_h; and with complementary probability at least one bidder receives V_l and the resulting second highest bid is V_l. It can be shown, analogous to the proof of Proposition 1 in the appendix, that the unique symmetric equilibrium of the FPA is as follows: type-V_l bidder bids her valuation V_l, while a type-V_h bidder bids according to a mixed strategy on the support $\left[V_l, \rho V_h + (1-\rho) V_l\right]$ with CDF $G^{CV}(\cdot)$ given by

$$G^{CV}(b) = \frac{(1-\rho)(b - V_l)}{\rho (V_h - b)} \quad \text{if } b \in \left[V_l, \rho V_h + (1-\rho) V_l\right].$$

Type-V_h bidder's expected surplus under the mixed strategy is simply $(1-\rho)(V_h - V_l)$. The seller's expected revenue under the FPA can be calculated as follows. The expected social surplus under the FPA is

$$\frac{\rho}{2} V_l + \left(1 - \frac{\rho}{2}\right) V_h$$

since the object is always efficiently allocated. The bidders' expected surplus is 0 for type-V_l bidder; and $(1-\rho)(V_h - V_l)$ for type-V_h bidder. Since ex ante, bidders are of type V_l and V_h with equal probability, the seller's expected revenue under the FPA is

$$\frac{\rho}{2} V_l + \left(1 - \frac{\rho}{2}\right) V_h - (1-\rho)(V_h - V_l) = \left(1 - \frac{\rho}{2}\right) V_l + \frac{\rho}{2} V_h.$$

Thus a one-dimensional correlated-value analog of our two-value example would yield revenue equivalence between the FPA and the SPA regardless of the degree of correlation. Note that this result is not robust, as [28] showed that in continuous affiliated value environments, the expected revenue from the SPA is in general at least weakly higher than that from the FPA.

4. Efficiency

While the SPA is always allocatively efficient in equilibrium, we argue in this section that the allocative efficiency of the FPA in Section 3 is an artifact of the two-valuation example. When we make all three valuations occur with positive probability, i.e., $p_k > 0$ for $k \in \{l, m, h\}$, the unique symmetric equilibrium of the FPA may be allocatively inefficient. Allocative inefficiency may arise in the FPA if a type-(V_m, H) bidder infers that her opponent is mostly likely of valuation type V_h and hence bid more aggressively than a type-(V_h, L) bidder, who perceives her opponent to be weak and is willing to sacrifice the probability of winning in exchange for a bigger surplus when winning against an opponent with valuation V_l. The subtle point is that this intuition works only if the following conditions are met: (1) Type-(V_m, H) bidder's posterior belief about her opponent puts a small weight on (V_h, L), and big weight on (V_h, H). This requires that q_m be sufficiently small and $\Pr\left(v_j = V_h | s_i = H\right)$ be sufficiently large; (2). Type-(V_h, L) bidder's posterior belief about her opponent puts a big weight on V_l. This requires that q_l be sufficiently large; (3). V_h cannot be too large relative to V_m since otherwise, type-(V_h, L) bidder is not willing to lower her probability of winning by bidding conservatively.

Arguments similar to those in Section 3 can be used to establish that, first, in any symmetric equilibrium of the FPA, bidders with valuation V_l must bid V_l in pure strategy regardless of their information type; second, other types of bidders must bid in mixed strategies with contiguous and non-overlapping supports; third, the support of type-(V_m, L) bidder's mixed strategy must be lower than that of type-(V_m, H); the support of type-(V_h, L) bidder's mixed strategy must be lower than that of type-(V_h, H); the support of type-(V_m, H) bidder's mixed strategy must be lower than that of type-(V_h, H), and the support of type-(V_m, L) bidder's mixed strategy must be lower than that of type-(V_h, L). Thus the symmetric equilibrium of the FPA in this section takes only two possible forms depending on the order of the mixed strategy supports of type-(V_m, H) and type-(V_h, L) bidders. A symmetric equilibrium is *efficient* if the equilibrium mixed strategy support of type-(V_m, H) bidder is lower than that of type-(V_h, L) bidder; and it is *inefficient* if the equilibrium mixed strategy support of type-(V_h, L) bidder is lower than that of type-(V_m, H) bidder. Our first interesting result is.

Proposition 3 (*Efficient and inefficient equilibria cannot coexist in the FPA*). *Any auction environment $e \in \mathcal{E}$ cannot simultaneously have both an efficient and an inefficient symmetric equilibrium in the FPA.*

We now show that, in contrast to the SPA, the FPA may be allocatively inefficient. The following result is proved by constructing an explicit auction environment with inefficient equilibrium. Proposition 3 then guarantees that it does not admit any efficient symmetric equilibrium. That there is an open set of auction environments in $\mathcal{E}$ with inefficient equilibrium in the FPA follows from continuity.

Proposition 4 (*Inefficiency of FPA*). *There exists an open set of auction environments in $\mathcal{E}$ in which the unique symmetric equilibrium of the FPA is inefficient.*

Our model has an interesting implication regarding the impact of more information on efficiency. The probability of the object being inefficiently allocated in an inefficient equilibrium is the probability that the two bidders' types are (V_m, H) and (V_h, L) respectively, which is given by

$$\begin{aligned} &2\Pr\left\{t_i = (V_m, H), t_j = (V_h, L), i \neq j\right\} \\ &\quad = 2 p_m p_h (1 - q_h) q_m. \end{aligned}$$

Recall that $1 - q_h$ is the probability of bidder i obtaining $s_i = H$ when her opponent's valuation $v_j = V_h$. Thus the higher $1 - q_h$ is, the more informative the signal H is about V_h, and also of course, the more informative the signal L is about V_l. Thus, the probability of inefficient allocation in the FPA may be increasing in $1 - q_h$ *locally* in the set of auction environments with inefficient equilibrium. We want to emphasize, however, that this is only a local result: as signal becomes more informative, the inefficient equilibrium may cease to exist, and efficiency may be restored.

Finally, it can also be verified that the seller's expected revenue in the inefficient equilibrium in the FPA is again smaller than that in the SPA except for some knife-edge cases with measure zero.

The key observations of this section would clearly continue to hold in more general settings. For example, suppose that bidders' private values were independently drawn from a continuous distribution and each bidder observed a continuous signal, correlated with the value of the other bidder. In a continuous setting, efficiency would require that each bidder's strategy depend only on his valuation and not his signal. This would be impossible if the signal was informative. To the extent that equilibrium exists in the FPA in such environments, the probability of inefficiency would therefore always be non-monotonic, since we have inefficiency with intermediate informativeness of signals, but we have efficiency with either no information about others' values (efficiency is a well known property of the FPA with symmetric distributions and IPVs) or full information about others' values (there is efficiency in the FPA with complete information).

5. Equilibrium existence

Up to now, we have assumed that bidders' information about the opponent's private valuation is of the same accuracy. In this section, we show that the existence of equilibrium in the FPA is contingent on this assumption in our model. For this purpose, we consider again the example we used in Section 3, with the exception that the accuracy of bidder i's signal regarding bidder j's valuation is $q_i \in [1/2, 1)$ and we let $q_1 > q_2$. Recall that $q_i = \Pr\left(s_i = L | v_j = V_l\right) = \Pr\left(s_i = H | v_j = V_h\right)$. Our main result in this section can be stated as follows.

Proposition 5. *If $p_m = 0$ and $1 \geqslant q_1 > q_2 \geqslant 1/2$, then the FPA does not admit any equilibrium for generic values of $\langle V_l, V_h, p_l, q_1, q_2 \rangle$.*

Examples of non-existence of equilibrium with multidimensional types are also presented in [15] in the context of auctions with both private and common value components, and the example of this section has a similar flavor.

The non-existence problem is surely not an artifact of the discrete type assumption. Consider again the case where bidders' private values were independently drawn from a continuous distribution and each bidder observed a continuous signal, correlated with the value of the other bidder. Even with strong assumptions of the signals (e.g., the monotone likelihood ratio property), if bidder 1 knows that bidder 2 is following a strategy that is monotonic in his valuation and his signal of bidder 1's valuation, bidder 1 will, in some cases, not have a best response that is monotonic (in the same sense). To see why, suppose that bidder 1 has a bimodal distribution on bidder 2's valuation, and thus on bidder 2's bid. Suppose that improvements in bidder 1's signal translate up her beliefs about bidder 2's bids. For low values of the signal, it will be optimal for bidder 1 to bid such that she wins against both modal bids. However, as her signal improves, there will be a point where she will give up on winning against 2's high modal bid and her bid will jump down to just above the 2's low modal bid. Thus her bid will jump downwards as her signal improves. It is hard to think of a primitive assumption on the signal structure that will prevent this type of non-monotonicity. This lack of monotonicity implies that the existence arguments such as those of [2,34] will not help in this problem. [8]

6. Discussion: revenue and information acquisition

In this section, we first present two examples of multidimensional private value auctions in which the revenue ranking of the FPA and the SPA are reversed; and then discuss the incentives of information acquisition.

6.1. Revenue

We have observed that the SPA is efficient and the FPA is not in general. This may suggest the possibility that the SPA will generate more revenue (as suggested by the three valuation example of Section 4). We have also seen an example where the seller's revenue in the SPA is higher than in the FPA, even though the there is efficient allocation of the object under both auctions (the two valuation example in Section 3). What can be said in general about the revenue ranking?

There is an easy way to see that a general revenue ranking is not possible. For some special information structures, each bidder will know what private signal the other bidder has observed. This will be true if each bidder observes a partition of the other bidder's valuations. Now even though we start with a model that is completely symmetric across bidders, conditional on the observed signals, bidders are playing in an IPVs environment with asymmetric distributions. But from the work of Maskin and Riley [23], we already know that revenue ranking may go either way. We can use this insight to construct the following example where revenue in the FPA is higher than that in the SPA. Presumably, this revenue ranking would continue to hold in nearby models where private signals were not common knowledge among the players.

Example 1. Consider a private value auction with two bidders, $i = 1, 2$. Suppose that v_1 and v_2 are independent and both drawn from Uniform $[0, 1]$. Bidders also observe a noisy signal about their opponent's valuation. Suppose that the signal is generated as follows: for $i \neq j$,

$$s_i = \begin{cases} L & \text{if } v_j \in \left[0, \frac{1}{2}\right], \\ H & \text{if } v_j \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

That is, a bidder observes a signal that tells her if her opponent's value is higher or lower than $1/2$; and this information structure is common knowledge.

[8] It is possible that equilibrium existence may be restored by introducing communication and more complicated endogenous tie-breaking rules a la Jackson et al. [16]. However, it is not at all clear what would be the endogenous tie-breaking rule that would be compatible with equilibrium in this asymmetric environment.

In the equilibrium of the SPA for the auction environment described in Example 1, each bidder will bid their own private valuation. In the FPA, however, we have to consider three cases: (i) $v_i \in [0, 1/2]$ for $i = 1, 2$; (ii) $v_i \in (1/2, 1]$ for $i = 1, 2$; and (iii) $v_i \in [0, 1/2]$ and $v_j \in (1/2, 1]$ where $i \neq j$. In case (i), both bidders effectively compete in an auction environment in which it is common knowledge that the valuations are both drawn from Uniform [0, 1/2] distributions. In case (ii), both bidders effectively compete in an auction environment in which it is common knowledge that the valuations are both drawn from Uniform [1/2, 1] distributions. In case (iii), however, the bidders are asymmetric in their valuation distributions and it is common knowledge. Clearly the FPA and the SPA are revenue equivalent in case (i) and (ii) events. In case (iii) events, however, the bidder asymmetry breaks the revenue equivalence. It can be easily verified that case (iii) events satisfy the conditions for a theorem of Maskin and Riley [23, Proposition 4.3] which shows that the FPA would generate a higher expected revenue for the seller than the SPA under this type of asymmetry. Thus, we reach the conclusion that overall in this example, the seller's expected revenue is higher in the FPA than that in the SPA. It can also be numerically verified that in case (iii) events, the FPA may be allocatively inefficient. Thus we have an example that the FPA generates higher expected revenue for the seller than the SPA despite its possible allocative inefficiency relative to the SPA.

6.2. Information acquisition

So far we have assumed that bidder's information about her opponent's valuation or valuation distribution is provided by nature without incurring any cost. In reality, of course, such information is be costly to acquire. [9] Now we argue that if bidders have to costly acquire such information, then different auction mechanisms provide vastly different incentives for such information acquisition. This, together with the difference in revenue and allocative efficiency between the FPA and the SPA we documented earlier, provides yet another reason for the auction designer to prefer one auction mechanism over another even in private value auction environments.

In the SPA, bidders do not strategically use information about their opponents' valuation, thus there is no incentives at all to acquire such information if it is costly. This observation is completely general for any private value auction environments. The lack of incentives to acquire information about one's opponents in the SPA is related to the fact that bidding one's private valuation is an *ex post* equilibrium in the SPA.

In the FPA, however, information about the opponent's valuation does have strategic consequences in the bidding, thus bidders do have incentives to acquire such information if the cost is sufficiently small. We illustrate such incentives using an extension of Example 1 above. Suppose that bidder i can, at a cost $c_i > 0$, purchase a signal about her opponent's valuation that reveals whether her opponent's valuation is below or above 1/2. Assume that a bidder's signal purchase decision is observable to her opponent. Suppose that the timing of the game is as follows: first, bidders decide whether to purchase such a signal technology at cost c_i; second, nature draws private valuations from Uniform [0, 1] for each bidder;

[9] Most of the existing literature in information acquisition in auctions are concerned with common value auctions (for example [25,32]).

third, a bidder observes whether her opponent's private valuation is below or above $1/2$ if and only if she purchased the signal technology; and finally, bidders compete for the object in the FPA.

The equilibrium bidding strategies in the FPA depend on the signal purchase decisions:

- If neither bidder purchased the signal technology in stage 1, then both bidders will play the symmetric FPA in which the opponent's valuation is drawn from Uniform [0, 1] distribution. Thus bidders bid $v/2$ and the expected surplus for each bidder is given by $\left(\int_0^1 v^2\,dv\right)/2 = 1/6 \approx 0.16667$.
- If both bidders purchase the signal technology, then bidder i's ex ante expected surplus from the subsequent FPA is calculated as follows. (1) With probability $1/4$, both bidder valuations will be below $1/2$. In this case, bidders will bid $v/2$ in equilibrium and the expected surplus for each bidder is $\left(\int_0^{1/2} v^2\,dv\right)/2 = 1/48$. (2) With probability $1/4$, both valuations will be above $1/2$. In this case, bidders will bid $v/2$ again in equilibrium and the expected surplus for each bidder is $\left(\int_{1/2}^1 v^2\,dv\right)/2 = 7/48$. (3) With probability $1/4$, bidder i's valuation is below $1/2$ and bidder j's valuation is above $1/2$, where $j \neq i$. (4) With probability $1/4$, bidder i's valuation is above $1/2$ and bidder j's valuation is below $1/2$. The equilibria of the FPA in the events of case (3) and (4) cannot be analytically solved, but numerical calculation shows that bidder i's expected surplus in case (3) and (4) are 0.01848 and 0.34808 respectively. [10] Thus bidder i's *ex ante* expected surplus if both bidders purchased the signal technology is

$$\frac{1/48 + 7/48 + 0.01848 + 0.34808}{4} \approx 0.13331.$$

- If bidder i does not purchases the signal technology but bidder j does, then bidder i and j's *ex ante* expected surplus from the subsequent FPA can be calculated as follows. (1) With probability $1/2$, bidder i's valuation is below $1/2$. In this case, bidder i's belief about j's valuation is Uniform [0, 1] while bidder j's belief about i's valuation is Uniform [0, 1/2] and this is common knowledge. Hence, bidder i's expected surplus is that of a "weak" bidder (in the terminology of Maskin and Riley [23]) with Uniform [0, 1/2] valuation distribution against a "strong" bidder with Uniform [0, 1] distribution in the FPA, which can be analytically calculated to be approximately 0.0242334. [11] Likewise, bidder j (the "strong" bidder in this case)'s ex ante expected payoff is approximately 0.253449. (2) With probability $1/2$, bidder i's valuation is above $1/2$. In this case, bidder i's belief

[10] John Riley and Estelle Cantillon graciously provided various versions of BIDCOMP2 fortran codes that are used in calculating the bidders' ex ante expected payoffs in the asymmetric auctions.

[11] The unique equilibrium of a two-bidder asymmetric FPA with valuation distributions Uniform $[0, h_1]$ and Uniform $[0, h_2]$ respectively where $h_1 > 0, h_2 > 0$ and $h_1 \neq h_2$ are given by

$$b_1(v) = \frac{\sqrt{1 + mv^2} - 1}{mv}, \qquad b_2(v) = \frac{1 - \sqrt{1 - mv^2}}{mv},$$

where $m = \left(h_1^2 - h_2^2\right) / (h_1 h_2)^2$ is a constant. Appendix B in [11] provides an elementary derivation of the above equilibrium. See also [13,33].

Table 1
The expected payoff matrix

Bidder $i\backslash$Bidder j	No purchase	Purchase
No Purchase	0.16667, 0.16667	0.12135, $0.17878 - c_j$
Purchase	$0.17878 - c_i$, 0.12135	$0.13331 - c_i$, $0.13331 - c_j$

about bidder j's valuation is Uniform [0, 1] while bidder $j's$ belief about i's valuation is Uniform [1/2, 1] and this is common knowledge. Hence, bidder i's expected surplus is that of a "strong" bidder with Uniform [1/2, 1] valuation distribution against a "weak" bidder with Uniform [0, 1] distribution in the FPA, which can be numerically calculated to be 0.218465. Likewise, bidder j (the "weak" bidder in this case)'s expected surplus is approximately 0.104104. Thus bidder i (the non-purchaser)'s *ex ante* expected payoff is approximately

$$\frac{0.0242334 + 0.218465}{2} = 0.12135$$

and bidder j (the purchaser)'s *ex ante* expected payoff is approximately

$$\frac{0.253449 + 0.104104}{2} = 0.17878.$$

Table 1 lists the ex ante expected payoff matrix for the two bidders taking into account the information acquisition cost c_i and c_j. When c_i and c_j are sufficiently small, the unique equilibrium in the information acquisition stage is that both bidders purchase the signals. Both bidders are made worse off through two channels. First, they incur the information acquisition cost; second, in the subsequent FPA, they will be engaged in more fierce competition and the seller will be able to extract a higher revenue. The social welfare is also decreased for two reasons. First, the information acquisition cost is dissipative; second, the object will be allocated inefficiently with positive probability.

This example also illustrates the possibility that a decrease in the cost of information acquisition may increase allocative inefficiency in the FPAs. Imagine that initially the information acquisition cost c_i are sufficiently high that in equilibrium neither bidder purchases the signal technology. Thus we know that the subsequent FPA is allocatively efficient. However, as c_i is sufficiently low, both bidders will purchase information in equilibrium and the subsequent FPA is allocatively inefficient with positive probability.

7. Conclusion

This paper presents examples of two-bidder private value auctions in which each bidder observes her own private valuation as well as noisy signals about her opponent's private valuation. This departs from the one-dimensional symmetric IPV paradigm and provides a simple but genuine separation between a bidder's private valuation and her signal about opponents' valuations, unlike the one-dimensional APV model. We partially characterize the equilibrium of the FPA when each bidder's signal about her opponent's valuation is drawn from the same distribution, and show that the revenue-equivalence between standard auctions fails. Our examples demonstrate that, first, the revenue ranking between the FPA and the SPA is ambiguous; second, the equilibrium allocation of the object could be inefficient in the FPA but is always efficient in the SPA, but the revenue and allocative efficiency may not coincide: an inefficient FPA may generate a higher expected revenue for the seller; but it is also possible that the seller's expected revenue is higher in the SPA even when the object is efficiently allocated in both auctions. We also show that the equilibrium existence of the FPA may be problematic in multidimensional type environments. Finally, we show that different auction mechanisms provide different incentives for bidders to acquire cost information about opponents' private valuations. We also provide examples that the allocative inefficiency in the FPA may increase as the signal becomes more informative; and the allocative inefficiency may increase in the FPA as the information acquisition costs are decreased. While the results in our paper are derived in examples, we have explained how the underlying intuitions are general.

Appendix A.. Proofs

Proof of Proposition 1. Proposition 1 follows from the following intermediate lemmas:

Lemma A.1. *In any equilibrium of the FPA, type-(V_l, s) bidders bid V_l in pure strategies for $s \in \{L, H\}$. That is, for $i = 1, 2$,*

$$B_i^{\text{FPA}}(V_l, s) = V_l \quad \textit{for } s \in \{L, H\}.$$

Proof. We first argue that bidders with valuation V_l must bid in pure strategies in equilibrium. Suppose that type-(V_l, H) bidders plays a mixed strategy equilibrium on support $[\underline{b}, \bar{b}]$ with $\underline{b} < \bar{b}$. (The lower limit of the interval may be open, but this is not important for the argument.) Clearly $\bar{b} \leqslant V_l$. Since the bid $(\underline{b} + \bar{b})/2$ wins positive probability, it yields a positive surplus for type-(V_l, H) bidder. However, bids close to $\underline{b}$ will win with probability almost zero, hence the expected surplus will approach zero. A contradiction for the indifference condition required for the mixed strategy. Hence type-(V_l, H) bidders must bid in pure strategies. Identical arguments show that type-(V_l, L) bidders must also bid in pure strategy. Now we argue that, if type-(V_l, L) and (V_l, H) bidders must bid their valuation V_l in pure strategy. To see this, suppose that type-(V_l, L) and (V_l, H) bidder 2 bids less than V_l. Then bidder 1 of these types can deviate by bidding ε more than bidder 2, which will be a profitable deviation if ε is made arbitrarily close to zero. A contradiction. □

Lemma A.2. *Together with the strategies specified in Lemma A.1 for bidders with valuation V_l, the following constitute a symmetric equilibrium*:

1. *Type-(V_h, L) bidders play a mixed strategy on $[V_l, \bar{b}_{(V_h,L)}]$ according to* CDF $G_{(V_h,L)}(\cdot)$ *given by* (5) *where $\bar{b}_{(V_h,L)}$ is given by* (6).

2. *Type-*(V_h, H) *bidders play a mixed strategy on* $\left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$ *according to* CDF $G_{(V_h,H)}(\cdot)$ *given by* (7) *where* $\bar{b}_{(V_h,H)}$ *is given by* (8).

Proof. Suppose that bidder 2 bids according to the postulated strategies.

First, consider type-(V_h, L) bidder 1. Her expected payoff from submitting a bid $b \in \left[V_l, \bar{b}_{(V_h,L)}\right]$ is

$$(V_h - b)\left\{\frac{p_l q}{p_l q + p_h(1-q)} + \frac{p_h(1-q)^2}{p_l q + p_h(1-q)} G_{(V_h,L)}(b)\right\}, \tag{A.1}$$

where

- The term $p_l q / [p_l q + p_h(1-q)]$ is the probability that bidder 2 has a valuation type V_l conditional on bidder 1's own information type L [recall formula (2)]. By Lemma A.1, bidder 2 with valuation type V_l bids V_l with probability one. Thus bidder 1 wins with probability 1 against such an opponent with any bid in the interval $\left[V_l, \bar{b}_{(V_h,L)}\right]$ (note that the tie-breaking rule is applied at the bid V_l).
- The term $p_h(1-q)^2 / [p_l q + p_h(1-q)]$ is the probability that bidder 2 is of type (V_h, L) conditional on bidder 1's own type (V_h, L) [recall formula (3)]. Since type-(V_h, L) bidder 2 is postulated to bid in mixed strategies according to $G_{(V_h,L)}(\cdot)$, bidder 1's bid of b wins against such an opponent with probability $G_{(V_h,L)}(b)$.

Plugging $G_{(V_h,L)}(\cdot)$ as described by (5) into (A.1) yields a positive constant, denoted by $K_{(V_h,L)}$, given by

$$K_{(V_h,L)} = \frac{p_l q}{p_h(1-q) + p_l q}(V_h - V_l), \tag{A.2}$$

which is type-(V_h, L) bidder's expected surplus. Therefore type-(V_h, L) bidder 1 indeed is indifferent between any bids in the interval $\left[V_l, \bar{b}_{(V_h,L)}\right]$ provided that bidder 2 follows the postulated strategy.

Now we check that type-(V_h, L) bidder 1 does not have incentive to deviate to other bids. First, she clearly does not have incentive to deviate to bids lower than or equal to V_l, since it would have yielded her a zero surplus instead of a positive $K_{(V_h,L)}$. Now suppose that she deviates to $\bar{b}_{(V_h,L)} < b \leqslant \bar{b}_{(V_h,H)}$, her expected payoff would be

$$(V_h - b)\left\{\frac{p_l q}{p_l q + p_h(1-q)} + \frac{p_h(1-q)^2}{p_l q + p_h(1-q)} + \frac{p_h(1-q)q}{p_l q + p_h(1-q)} G_{(V_h,H)}(b)\right\}, \tag{A.3}$$

where the term $p_h(1-q)q / [p_h(1-q) + p_l q]$ the probability that bidder 2 is of type (V_h, H) conditional on bidder 1's own type (V_h, L); and $G_{(V_h,H)}(b)$ is the probability that a bid $b \in \left(\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$ wins against such an opponent. Plugging $G_{(V_h,H)}(\cdot)$

as described by (7) into (A.3), we obtain

$$\frac{p_l q+p_h(1-q)^2}{p_l q+p_h(1-q)}(V_h-b)+\frac{p_h(1-q)q}{p_l q+p_h(1-q)}\frac{(p_l+p_h q)(1-q)\left(b-\bar{b}_{(V_h,L)}\right)}{p_h q^2}$$
$$=\frac{\left\{\left[p_l q+p_h(1-q)^2\right]V_h-(p_l+p_h q)(1-q)^2\,\bar{b}_{(V_h,L)}\right\}+p_l\left[(1-q)^2-q^2\right]b/q}{p_l q+p_h(1-q)},$$

which is non-increasing in b since $q\geqslant 1/2$. Hence type-(V_h, L) bidder 1 does not have incentive to deviate to bids in the interval $\left(\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$; which also implies that her expected payoff would be even smaller if she bids more than $\bar{b}_{(V_h,H)}$.

Now consider type-(V_h, H) bidder 1. Given that bidder 2 plays according to the postulated strategies, her expected payoff from bidding $b\in\left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$ is given by

$$(V_H-b)\left\{\frac{p_l(1-q)}{p_h q+p_l(1-q)}+\frac{p_h q(1-q)}{p_h q+p_l(1-q)}+\frac{p_h q^2}{p_h q+p_l(1-q)}G_{(V_h,H)}(b)\right\},\qquad (A.4)$$

where

- The term $p_l\,(1-q)\,/\,[p_h q+p_l\,(1-q)]$ is the probability that bidder 2 has valuation V_l conditional on bidder 1's signal H; and the term $[p_h q\,(1-q)]\,/\,[p_h q+p_l\,(1-q)]$ is the probability that bidder 2 is of type (V_h, L). In both events, a bid $b\in\left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$ wins against such opponents with probability one under the postulated strategies by bidder 2.
- The term $p_h q^2/\,[p_h q+p_l\,(1-q)]$ is probability that bidder 2 is of type (V_h, H) conditional on bidder 1's own type (V_h, H). In this case, a bid $b\in\left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$ wins with probability $G_{(V_h,H)}\,(b)$.

Plugging $G_{(V_h,H)}\,(\cdot)$ as described by (7) into (A.4), we obtain a positive constant, denoted by $K_{(V_h,H)}$, given by

$$K_{(V_h,H)}=\frac{(p_l+p_h q)\,(1-q)}{p_h q+p_l\,(1-q)}\left(V_h-\bar{b}_{(V_h,L)}\right).\qquad (A.5)$$

Hence type-(V_h, H) bidder 1 is indeed indifferent between any bids in the interval $\left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_h,H)}\right]$.

Now we check that type-(V_h, H) bidder 1 does not have incentive to deviate to other bids. First, she does not have incentive to bid more than $\bar{b}_{(V_h,H)}$, since bidding $\bar{b}_{(V_h,H)}$ strictly dominates any higher bid given bidder 2′s strategies; second, she does not have incentive to bid less than or equal to V_l since such bids will yield a zero surplus. Now we show that she does not have incentive to bid in the interval $\left(V_l, \bar{b}_{(V_h,L)}\right)$. Her expected payoff from a bid $b\in\left(V_l, \bar{b}_{(V_h,L)}\right)$ is given by

$$(V_h-b)\left\{\frac{p_l\,(1-q)}{p_h q+p_l\,(1-q)}+\frac{p_h q\,(1-q)}{p_h q+p_l\,(1-q)}G_{(V_h,L)}\,(b)\right\}\qquad (A.6)$$

since such a bid loses to type-(V_h, H) opponent with probability one and win against a type-(V_h, L) opponent with probability $G_{(V_h,L)}\,(b)$. Plugging $G_{(V_h,L)}\,(\cdot)$ as described by

(5) into (A.6), we get

$$\frac{p_l\,(1-q)}{p_hq+p_l\,(1-q)}\,(V_h-b)+\frac{p_hq\,(1-q)}{p_hq+p_l\,(1-q)}\,\frac{p_lq\,(b-V_l)}{p_h\,(1-q)^2}$$
$$=\frac{\left[p_l\,(1-q)^2\,V_h-p_lq^2V_l\right]+p_l\left[q^2-(1-q)^2\right]b}{\left[p_hq+p_l\,(1-q)\right](1-q)},$$

which is non-decreasing in b since $q \geqslant 1/2$. Hence type-(V_h, H) bidder 1 does not have incentive to deviate to bids in the interval $\left(V_l, \bar{b}_{(V_h,L)}\right)$.

Finally, note that the expressions for $\bar{b}_{(V_h,L)}$ and $\bar{b}_{(V_h,H)}$ respectively satisfy $G_{(V_h,L)}\left(\bar{b}_{(V_h,L)}\right) = 1$ and $G_{(V_h,H)}\left(\bar{b}_{(V_h,H)}\right) = 1$. This concludes the proof that the postulated bidding strategies constitute a symmetric equilibrium. □

Lemma A.3. *The symmetric equilibrium described in Lemma A.2 is the unique symmetric equilibrium of the FPA.*

Proof. The argument proceeds in three steps.

Step 1: We show that in any symmetric equilibrium type-(V_h, L) and type-(V_h, H) bidders must bid in mixed strategies. For example, suppose to the contrary that, say, a type-(V_h, L) bidder 2 bids in pure strategy an amount $\tilde{b} < V_h$, then type-(V_h, L) bidder 1 can profitably deviate by bidding $\tilde{b}+\varepsilon$ where $\varepsilon > 0$ is arbitrarily small. Such a deviation will provide a discrete positive jump in type-(V_h, L) bidder 1's probability of winning, hence it is profitable. The argument for type-(V_h, H) bidders is analogous.

Step 2: We show that in any symmetric mixed strategy equilibrium, the supports of $G_{(V_h,L)}(\cdot)$ and $G_{(V_h,H)}(\cdot)$ are contiguous and non-overlapping. That the supports should be contiguous follows from the same ε-deviation argument as the one to rule out pure strategies. Now suppose that the supports of $G_{(V_h,L)}(\cdot)$ and $G_{(V_h,H)}(\cdot)$ overlap in an interval $[b_1, b_2]$ with $b_2 > b_1$. To be consistent with mixed strategies, it must be the case that, the expected surplus for both types from any bid $b \in [b_1, b_2]$ is constant. That is, for some constants $\tilde{K}_{(V_h,L)}$ and $\tilde{K}_{(V_h,H)}$,

$$(V_h-b)\left\{\frac{p_l\,(1-q)}{p_hq+p_l\,(1-q)}+\frac{p_hq^2G_{(V_h,H)}(b)}{p_hq+p_l\,(1-q)}\right.$$
$$\left.+\frac{p_hq\,(1-q)\,G_{(V_h,L)}(b)}{p_hq+p_l\,(1-q)}\right\}=\tilde{K}_{(V_h,H)}, \tag{A.7}$$

$$(V_h-b)\left\{\frac{p_lq}{p_h\,(1-q)+p_lq}+\frac{p_h\,(1-q)\,qG_{(V_h,H)}(b)}{p_h\,(1-q)+p_lq}\right.$$
$$\left.+\frac{p_h\,(1-q)^2\,G_{(V_h,L)}(b)}{p_h\,(1-q)+p_lq}\right\}=\tilde{K}_{(V_h,L)}. \tag{A.8}$$

Multiplying Eq. (A.7) by $(1-q)\left[p_hq+p_l\,(1-q)\right]$, and Eq. (A.8) by $q\left[p_h\,(1-q)+p_lq\right]$, and summing up, we obtain

$$\tilde{K}_{(V_h,H)}\,(1-q)\left[p_hq+p_l\,(1-q)\right]-\tilde{K}_{(V_h,L)}q\left[p_h\,(1-q)+p_lq\right]$$
$$=(V_h-b)\,p_l\left[(1-q)^2-q^2\right]. \tag{A.9}$$

Because the left-hand side of Eq. (A.9) is a constant, this equation holds only for a single value of b unless $q = 1/2$. Therefore, the supports of the symmetric equilibrium mixed strategies of type-(V_h, L) and (V_h, H) bidders must be non-overlapping. The same argument also shows that the supports of the symmetric equilibrium mixed strategies of type-(V_h, L) and type-(V_h, H) bidders cannot overlap at more than one point.

Step 3: We show that the support of type-(V_h, L) bidders' mixed strategy must be lower than that of type-(V_h, H) bidders. Suppose to the contrary. Let $\left[V_l, \tilde{b}\right]$ be the support of type-(V_h, H) bidder and $\left[\tilde{b}, \hat{b}\right]$ be the support of type-(V_h, L) bidder, for some $\hat{b} > \tilde{b}$. For type-(V_h, L) bidders to randomize on $\left[\tilde{b}, \hat{b}\right]$, it must be the case that

$$(V_h - b)\left\{\frac{p_l q}{p_h(1-q)+p_l q} + \frac{p_h(1-q)q}{p_h(1-q)+p_l q} + \frac{p_h(1-q)^2}{p_h(1-q)+p_l q}\tilde{G}_{(V_h,L)}(b)\right\}$$
$$= \left(V_h - \tilde{b}\right)\frac{p_l q + p_h q(1-q)}{p_h(1-q)+p_l q},$$

from which, after solving for $\tilde{G}_{(V_h,L)}(b)$, we obtain

$$\tilde{G}_{(V_h,L)}(b) = \frac{q(1-p_h q)\left(b-\tilde{b}\right)}{p_h(1-q)^2(V_h - b)}. \tag{A.10}$$

Suppose that type-(V_h, H) bidder 2 mixes over $\left[V_l, \tilde{b}\right]$, and type-$(V_h, L)$ bidder 2 mixes over $\left[\tilde{b}, \hat{b}\right]$ according to $\tilde{G}_{(V_h,L)}(\cdot)$ as described by (A.10). Then the expected surplus for type-(V_h, H) bidder 1 from bidding $b \in \left(\tilde{b}, \hat{b}\right)$ is given by

$$(V_h - b)\left\{\frac{p_l(1-q)}{p_h q + p_l(1-q)} + \frac{p_h q^2}{p_h q + p_l(1-q)} + \frac{p_h q(1-q)}{p_h q + p_l(1-q)}\tilde{G}_{(V_h,L)}(b)\right\}$$
$$= \frac{(1-q)\left[p_l(1-q)+p_h q^2\right]V_h - q^2(1-p_h q)\tilde{b} + p_l\left[q^2-(1-q)^2\right]b}{(1-q)\left[p_h q + p_l(1-q)\right]},$$

which is non-decreasing in b. Therefore, type-(V_h, H) bidder will have an incentive to bid higher than $\tilde{b}$ if her opponent follows the prescribed strategies, a contradiction.

Combining Steps 1–3 and Lemma A.1, we know that the equilibrium described in Lemma A.2 is the only symmetric equilibrium. □

Lemma A.4. *There is no asymmetric equilibrium.*

Proof. First, arguments similar to step 3 in the proof of Lemma A.3 can be used to show that in an asymmetric equilibrium, the support of type-(V_h, L) bidders must be lower than that of type-(V_h, H) bidders.

Now suppose that type-(V_h, L) bidder 1 and bidder 2 respectively play a mixed strategy on the support $\left[V_l, \tilde{b}_1\right]$ and $\left[V_l, \tilde{b}_2\right]$, and without loss of generality, suppose that $\tilde{b}_1 > \tilde{b}_2$. Since type-$(V_h, L)$ bidder 1 must be indifferent between any bids in $(V_l, \tilde{b}_2]$, type-(V_h, L)

bidder 2's mixed strategy, denoted by $\tilde{G}_{2(V_h,L)}$, must satisfy

$$(V_h - b)\left\{\frac{p_l q}{p_l q + p_h (1-q)} + \frac{p_h (1-q)}{p_l q + p_h (1-q)}\tilde{G}_{2(V_h,L)}(b)\right\} = (V_h - V_l)\frac{p_l q}{p_l q + p_h (1-q)}$$

from which we obtain that $\tilde{b}_2 = \bar{b}_{(V_h,L)}$ where $\bar{b}_{(V_h,L)}$ is specified by formula (6). Now since type-(V_h, L) bidder 2 is indifferent between any bids in $\left[V_l, \tilde{b}_2\right]$, type-$(V_h, L)$ bidder 1's mixed strategy CDF, denoted by $\tilde{G}_{1(V_h,L)}$, in the interval $\left[V_l, \tilde{b}_2\right]$ must satisfy

$$(V_h - b)\left\{\frac{p_l q}{p_l q + p_h (1-q)} + \frac{p_h (1-q)}{p_l q + p_h (1-q)}\tilde{G}_{1(V_h,L)}(b)\right\} = (V_h - V_l)\frac{p_l q}{p_l q + p_h (1-q)}$$

from which we obtain that

$$\tilde{G}_{1(V_h,L)}(b) = \frac{p_l q (b - V_l)}{p_h (1-q)^2 (V_h - b)}.$$

But then $\tilde{G}_{1(V_h,L)}\left(\tilde{b}_2\right) = \tilde{G}_{1(V_h,L)}\left(\bar{b}_{(V_h,L)}\right) = 1$. Hence $\tilde{b}_1 = \tilde{b}_2$, a contradiction. □

Proof of Proposition 3. Suppose to the contrary that there is an auction environment that admits both types of symmetric equilibrium in the FPA. First, since the support of type-(V_m, L) bidders must be lower than those of type-(V_m, H), (V_h, L), and (V_h, H) bidders in both equilibria, the upper limit of type-(V_m, L) bidders' mixed strategies in both equilibria must be the same, which we denote by $\bar{b}_{(V_m,L)}$.

Let $\bar{b}^{\text{eff}}_{(V_h,L)}$ be the upper limit of the mixed strategy support of type-(V_h, L) bidder in the efficient equilibrium and let $\bar{b}^{\text{ineff}}_{(V_m,H)}$ be the upper limit of the mixed strategy support of type-(V_m, H) bidder in the inefficient equilibrium. We then consider two possible cases:

Case 1: $\bar{b}^{\text{eff}}_{(V_h,L)} \geqslant \bar{b}^{\text{ineff}}_{(V_m,H)}$. This case is illustrated in Fig. 3. Since in the inefficient equilibrium type-(V_h, L) bidder is indifferent between any bids in $\left[\bar{b}_{(V_m,L)}, \bar{b}^{\text{ineff}}_{(V_h,L)}\right]$, her expected

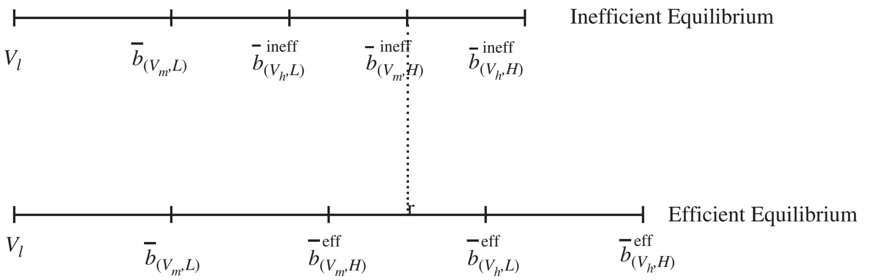

Fig. 3. Case 1 in the Proof of Proposition 3.

surplus in the inefficient equilibrium is the same as that when she bids $\bar{b}_{(V_m,L)}$ (recall our tie-breaking rule), which is simply

$$Z_1 = \left(V_h - \bar{b}_{(V_m,L)}\right)\left\{\frac{p_l q_l}{\sum_{k\in\{l,m,h\}} p_k q_k} + \frac{p_m q_m\,(1-q_h)}{\sum_{k\in\{l,m,h\}} p_k q_k}\right\}, \tag{A.11}$$

where the term in the bracket is the expected probability of winning against bidders with V_l valuation and type-(V_m, L) bidders. Given that her opponent follows the prescribed strategy in the inefficient equilibrium, her expected payoff from deviating to a bid of $\bar{b}^{\text{ineff}}_{(V_m,H)}$ is

$$Z_2 = \left(V_h - \bar{b}^{\text{ineff}}_{(V_m,H)}\right)\left\{\frac{p_l q_l}{\sum_{k\in\{l,m,h\}} p_k q_k} + \frac{p_m q_m}{\sum_{k\in\{l,m,h\}} p_k q_k} + \frac{p_h q_h\,(1-q_h)}{\sum_{k\in\{l,m,h\}} p_k q_k}\right\}, \tag{A.12}$$

where the term in the bracket is the expected probability of winning against bidders with V_l and V_m valuations and type-(V_h, L) bidders. By the requirement of the inefficient equilibrium, we have $Z_1 > Z_2$.[12] Moreover, since in this case $V_h > \bar{b}^{\text{eff}}_{(V_h,L)} \geqslant \bar{b}^{\text{ineff}}_{(V_m,H)}$ by assumption, we immediately have

$$Z_2 \geqslant \left(V_h - \bar{b}^{\text{eff}}_{(V_h,L)}\right)\left\{\frac{p_l q_l}{\sum_{k\in\{l,m,h\}} p_k q_k} + \frac{p_m q_m}{\sum_{k\in\{l,m,h\}} p_k q_k} + \frac{p_h q_h\,(1-q_h)}{\sum_{k\in\{l,m,h\}} p_k q_k}\right\}. \tag{A.13}$$

But the right-hand side of inequality (A.13) is exactly type-(V_h, L) bidder's expected surplus in the efficient equilibrium, which by the definition of the efficient equilibrium is required to be larger than Z_1 [as given by expression (A.11)]. This is so because Z_1 is also type-(V_h, L) bidder's expected surplus from deviating to a bid of $\bar{b}_{(V_m,L)}$ in the efficient equilibrium. Thus we have $Z_2 \geqslant Z_1$, which is a contradiction to our earlier conclusion that $Z_1 > Z_2$.

Case 2: $\bar{b}^{\text{eff}}_{(V_h,L)} < \bar{b}^{\text{ineff}}_{(V_m,H)}$. A contradiction can be derived for type-(V_m, H) bidder using arguments analogous to Case 1. □

Proof of Proposition 4. Consider an example of the model presented in Section 2 as follows:

- $V_l = 0$, $V_m = 1$ and $V_h = 2$. We can set $V_l = 0$ and $V_m = 1$ by normalization and scaling with no loss of generality. For the inefficient equilibrium described below to exist, V_h cannot be too high relative to V_m.
- $p_l = p_m = p_h = 1/3$. That is, ex ante bidders' valuations of the object take on the three values with equal probability. This assumption is purely for computational ease in Bayesian updating;

[12] The strict, rather than the usual weak, inequality, is valid because, tedious algebra shows that the deviation surplus function is strictly decreasing in the whole interval $\left[\bar{b}^{\text{ineff}}_{(V_h,L)}, \bar{b}^{\text{ineff}}_{(V_m,H)}\right]$ in order to be consistent with the inefficient equilibrium.

- $q_l = 0.9, q_m = 0.1, q_h = 0.05$. As we described in the beginning of this section, the intuition for inefficient equilibrium requires that q_l be big, q_m to be small, and q_h even smaller.

As in the example in Section 3, we know that bidders with valuation type V_l will bid V_l regardless of their information types. Now we show that the following mixed strategies for the other types of bidders constitute the unique symmetric equilibrium:

- Type-(V_m, L) bidders bid according to a mixed strategy on the support $\left[V_l, \bar{b}_{(V_m,L)}\right]$ with CDF $G_{(V_m,L)}(\cdot)$ where

$$\bar{b}_{(V_m,L)} = \frac{q_m^2 V_m + q_l V_l}{q_l + q_m^2},$$
$$G_{(V_m,L)}(b) = \frac{q_l}{q_m^2}\left(\frac{b - V_l}{V_m - b}\right).$$

- Type-(V_h, L) bidders bid according to a mixed strategy on the support $\left[\bar{b}_{(V_m,L)}, \bar{b}_{(V_h,L)}\right]$ with CDF $G_{(V_h,L)}(\cdot)$ where

$$\bar{b}_{(V_h,L)} = \frac{[q_l + q_m q_h]\,\bar{b}_{(V_m,L)} + q_h^2 V_h}{q_l + q_m q_h + q_h^2},$$
$$G_{(V_h,L)}(b) = \frac{q_l + q_m q_h}{q_h^2}\frac{b - \bar{b}_{(V_m,L)}}{V_h - b}.$$

- Type-(V_m, H) bidders bid according to a mixed strategy on the support $\left[\bar{b}_{(V_h,L)}, \bar{b}_{(V_m,H)}\right]$ with CDF $G_{(V_m,H)}(\cdot)$ where

$$\bar{b}_{(V_m,H)} = \frac{[1 - q_l + (1 - q_m)\,q_m + (1 - q_h)\,q_m]\,\bar{b}_{(V_h,L)} + (1 - q_m)^2 V_m}{1 - q_l + (1 - q_m)\,q_m + (1 - q_h)\,q_m + (1 - q_m)^2},$$
$$G_{(V_m,H)}(b) = \frac{1 - q_l + (1 - q_m)\,q_m + (1 - q_h)\,q_m}{(1 - q_m)^2}\frac{b - \bar{b}_{(V_h,L)}}{V_m - b}.$$

- Type-(V_h, H) bidders bid according to a mixed strategy on the support $\left[\bar{b}_{(V_m,H)}, \bar{b}_{(V_h,H)}\right]$ with CDF $G_{(V_h,H)}(\cdot)$ where

$$\bar{b}_{(V_h,H)} = \frac{(1 - q_h)^2 V_h + \left[3 - q_l - q_m - q_h - (1 - q_h)^2\right]\bar{b}_{(V_m,H)}}{3 - q_l - q_m - q_h},$$
$$G_{(V_h,H)}(b) = \left[\frac{3 - q_l - q_m - q_h - (1 - q_h)^2}{(1 - q_h)^2}\right]\frac{b - \bar{b}_{(V_m,H)}}{V_h - b}.$$

Under the above parameterization,

$$\bar{b}_{(V_h,H)} \approx 1.32531 > \bar{b}_{(V_m,H)} \approx 0.744012$$
$$> \bar{b}_{(V_h,L)} \approx 0.0164684 > \bar{b}_{(V_m,L)} \approx 0.010989 > V_l = 0.$$

To show that the above strategy profile constitutes an equilibrium, we need to demonstrate that, given that the opponent follows the postulated strategies, each type-(v, s) bidder, where $v \in \{V_m, V_h\}$ and $s \in \{L, H\}$, obtains a constant expected surplus from any bids in the support of the CDF $G_{(v,s)}(\cdot)$, which is in turn higher than the expected surplus from any other deviation bids. The details of the verifications are straightforward but arithmetically tedious, and can be found at Appendix A of Fang and Morris [11]. □

Proof of Proposition 5. Using standard ε-deviation arguments, we can show that (1) bidders with valuation V_l must bid V_l in pure strategy in any equilibrium; (2) each bidder of type-(V_h, L) and type-(V_h, H) must bid in mixed strategies with bids higher than V_l; (3) the highest bid that may be submitted by each bidder must be the same; (4) there is no gap in the bids submitted in equilibrium. We denote the mixed strategy CDF of type-(V_h, L) and type-(V_h, H) bidder i by $G_{i(V_h,L)}$ and $G_{i(V_h,H)}$ respectively, where $i = 1, 2$.

Next, we show that for each bidder i, the supports of $G_{i(V_h,L)}$ and $G_{i(V_h,H)}$ cannot overlap at more than one point. Without loss of generality, consider bidder 1. Let B_1 be the set of points in which the supports of $G_{i(V_h,L)}$ and $G_{i(V_h,H)}$ overlap. For any overlap bid $b \in B_1$, the following must be true:

$$(V_h - b)\left[\frac{p_l q_1}{p_l q_1 + p_h (1-q_1)} + \frac{p_h (1-q_1)(1-q_2)\, G_{2(V_h,L)}(b)}{p_l q_1 + p_h (1-q_1)} + \frac{p_h (1-q_1)\, q_2 G_{2(V_h,H)}(b)}{p_l q_1 + p_h (1-q_1)}\right] = \tilde{K}_{1(V_h,L)},$$

$$(V_h - b)\left[\frac{p_l (1-q_1)}{p_l (1-q_1) + p_h q_1} + \frac{p_h q_1 (1-q_2)\, G_{2(V_h,L)}(b)}{p_l (1-q_1) + p_h q_1} + \frac{p_h q_1 q_2 G_{2(V_h,H)}(b)}{p_l (1-q_1) + p_h q_1}\right] = \tilde{K}_{1(V_h,H)}.$$

Similar to the arguments in step 2 of the proof of Lemma A.3, the above system equations can hold for at most one value of b.

Therefore, we are left with four possible cases to consider depending on the order of the supports of type-(V_h, L) and (V_h, H) mixed strategies for each bidder. We will derive a contradiction for one of the cases, and the other cases can be dealt with analogously.

We consider the following case: The support of $G_{i(V_h,L)}$ is $\left[V_l, \bar{b}_{i(V_h,L)}\right]$ and the support of $G_{i(V_h,H)}$ is $[\bar{b}_{i(V_h,L)}, \bar{b}_{i(V_h,H)}]$. From discussions above, $\bar{b}_{1(V_h,H)} = \bar{b}_{2(V_h,H)} = \bar{b}_{(V_h,H)}$.

Step 1: Simple calculation shows that it must be the case that $\bar{b}_{1(V_h,L)} > \bar{b}_{2(V_h,L)}$.

Step 2: From the necessary indifference condition of type-(V_h, L) bidder 1 in the interval $\left[V_l, \bar{b}_{2(V_h,L)}\right]$, we can obtain $G_{2(V_h,L)}$:

$$\begin{aligned}
&(V_h - b)\left[\frac{p_l q_1}{p_l q_1 + p_h (1-q_1)} + \frac{p_h (1-q_1)(1-q_2)\, G_{2(V_h,L)}(b)}{p_l q_1 + p_h (1-q_1)}\right] \\
&\quad = (V_h - V_l)\,\frac{p_l q_1}{p_l q_1 + p_h (1-q_1)} \\
&\Rightarrow G_{2(V_h,L)}(b) = \frac{p_l q_1}{p_h (1-q_1)(1-q_2)}\,\frac{b - V_l}{V_h - b}, \\
&\bar{b}_{2(V_h,L)} = \frac{p_h (1-q_1)(1-q_2)\, V_h + p_l q_1 V_l}{p_h (1-q_1)(1-q_2) + p_l q_1}.
\end{aligned}$$

Step 3: The indifference condition for type-(V_h, L) bidder 2 requires that $G_{1(V_h,L)}(b)$ must satisfy, for $b \in \left[V_l, \bar{b}_{2(V_h,L)}\right]$,

$$(V_h - b)\left[\frac{p_l q_2}{p_l q_2 + p_h(1-q_2)} + \frac{p_h(1-q_2)(1-q_1)G_{1(V_h,L)}(b)}{p_l q_1 + p_h(1-q_2)}\right] = (V_h - V_l)\frac{p_l q_2}{p_l q_2 + p_h(1-q_2)}$$

from which we can obtain $G_{1(V_h,L)}(b)$ for $b \in (V_l, \bar{b}_{2(V_h,L)}]$ as

$$G_{1(V_h,L)}(b) = \frac{p_l q_2}{p_h(1-q_1)(1-q_2)}\frac{b - V_l}{V_h - b}.$$

Step 4: To obtain the $G_{1(V_h,L)}(b)$ for $b \in \left[\bar{b}_{2(V_h,L)}, \bar{b}_{1(V_h,L)}\right]$, we make use of the indifference condition of type-(V_h, H) bidder 2, which is given by

$$(V_h - b)\left[\frac{p_l(1-q_2)}{p_l(1-q_2) + p_h q_2} + \frac{p_h q_2(1-q_1)}{p_l(1-q_2) + p_h q_2}G_{1(V_h,L)}(b)\right] = \left(V_h - \bar{b}_{2(V_h,L)}\right)\left[\frac{p_l(1-q_2)}{p_l(1-q_2) + p_h q_2} + \frac{p_h q_2(1-q_1)}{p_l(1-q_2) + p_h q_2} \times G_{1(V_h,L)}\left(\bar{b}_{2(V_h,L)}\right)\right]$$

hence, for $b \in \left[\bar{b}_{2(V_h,L)}, \bar{b}_{1(V_h,L)}\right]$

$$G_{1(V_h,L)}(b) = \frac{p_l(1-q_2)\left[b - \bar{b}_{2(V_h,L)}\right] + p_h q_2(1-q_1)G_{1(V_h,L)}\left(\bar{b}_{2(V_h,L)}\right)\left[V_h - \bar{b}_{2(V_h,L)}\right]}{p_h q_2(1-q_1)(V_h - b)}.$$

Setting $G_{1(V_h,L)}(b) = 1$, we obtain

$$\bar{b}_{1(V_h,L)} = \frac{p_h q_2(1-q_1)\left\{\left[1 - G_{1(V_h,L)}(\bar{b}_{2(V_h,L)})\right]V_h + G_{1(V_h,L)}(\bar{b}_{2(V_h,L)})\bar{b}_{2(V_h,L)}\right\} + p_l(1-q_2)\bar{b}_{2(V_h,L)}}{p_h q_2(1-q_1) + p_l(1-q_2)}.$$

Step 5: The indifference condition of type-(V_h, L) bidder 1 for the bids in the interval $\left[\bar{b}_{2(V_h,L)}, \bar{b}_{1(V_h,L)}\right]$ requires that $G_{2(V_h,H)}(b)$ for $b \in \left[\bar{b}_{2(V_h,L)}, \bar{b}_{1(V_h,L)}\right]$ must satisfy

$$(V_h - b)\left[\frac{p_l q_1}{p_l q_1 + p_h(1-q_1)} + \frac{p_h(1-q_1)(1-q_2)}{p_l q_1 + p_h(1-q_1)} + \frac{p_h(1-q_1)q_2}{p_l q_1 + p_h(1-q_1)}G_{2(V_h,H)}(b)\right] = (V_h - V_l)\frac{p_l q_1}{p_l q_1 + p_h(1-q_1)}$$

thus,

$$G_{2(V_h,H)}(b) = \frac{b - V_l}{V_h - b}\frac{p_l q_1}{p_h(1-q_1)q_2} - \frac{p_h(1-q_1)(1-q_2)}{p_h(1-q_1)q_2}$$

from which can obtain $G_{2(V_h,H)}\left(\bar{b}_{1(V_h,L)}\right)$.

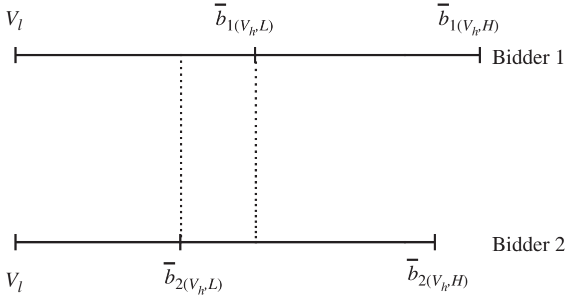

Fig. 4. A graphic illustration of the Proof of Proposition 5.

Step 6: The indifference condition of type-(V_h, H) bidder 2 requires that $G_{1(V_h,H)}(b)$ satisfy

$$(V_h-b)\left[\frac{p_l(1-q_2)}{p_l(1-q_2)+p_hq_2}+\frac{p_hq_2(1-q_1)}{p_l(1-q_2)+p_hq_2}+\frac{p_hq_2q_1}{p_l(1-q_2)+p_hq_2}G_{1(V_h,H)}(b)\right]$$
$$=\left(V_h-\bar{b}_{2(V_h,L)}\right)\left[\frac{p_l(1-q_2)}{p_l(1-q_2)+p_hq_2}+\frac{p_hq_2(1-q_1)}{p_l(1-q_2)+p_hq_2}G_{1(V_h,L)}\left(\bar{b}_{2(V_h,L)}\right)\right],$$

which implies a value for $\bar{b}_{1(V_h,H)}$.

Step 7: Likewise, the indifference condition of type-(V_h, H) bidder 1 requires that $G_{2(V_h,H)}(b)$ satisfy, for $b \in \left[\bar{b}_{1(V_h,L)}, \bar{b}_{2(V_h,H)}\right]$,

$$(V_h-b)\left[\frac{p_l(1-q_1)}{p_l(1-q_1)+p_hq_1}+\frac{p_hq_1}{p_l(1-q_1)+p_hq_1}G_{2(V_h,H)}(b)\right]$$
$$=(V_h-\bar{b}_{1(V_h,L)})\left[\frac{p_l(1-q_1)}{p_l(1-q_1)+p_hq_1}+\frac{p_hq_1}{p_l(1-q_1)+p_hq_1}G_{2(V_h,H)}(\bar{b}_{1(V_h,L)})\right],$$

which implies a value for $\bar{b}_{2(V_h,H)}$.

Step 8: For generic values of $\langle V_l, V_h, p_l, q_1, q_2\rangle$, $\bar{b}_{1(V_h,H)}$ and $\bar{b}_{2(V_h,H)}$ are not equal, which contradicts the equilibrium requirement by the standard ε-deviation argument. (see Fig. 4 for a graphic illustration of the above steps). □

Appendix B.

In this appendix, we provide a proof that the seller receives the same expected revenue from the FPA and the SPA in an IPV environment where the valuations are drawn from a discrete distribution. This extends Maskin and Riley [21] who showed the revenue equivalence for an environment with two bidders and two valuations. There are n agents bidding for a single object. The bidders' valuations are independent drawn from a discrete distribution with values $0 < V_1 < V_2 < \cdots < V_m$, and the probability of V_j is written as $p_j \in (0, 1)$. Assume that ties are broken as in Section 2.

Proposition B.1. *The seller's expected revenue is the same under the SPA and the FPA in the above environment.*

Proof. The weakly dominant strategy equilibrium in the SPA is for each bidders to bid her own valuation. Maskin and Riley [22] showed that the FPA will admit a unique symmetric equilibrium. In this equilibrium, type-V_1 bids V_1 with probability 1; type-V_2 bids in a mixed strategy on $\left[V_1, \bar{b}_2\right]$; and in general type-$V_k$ bids in a mixed strategy on $\left[\bar{b}_{k-1}, \bar{b}_k\right]$ where $\bar{b}_k > \bar{b}_{k-1}$.

Due to the monotonicity of the equilibrium bidding strategy, the expected probabilities of winning for a type-V_k bidder under both FPA and SPA are the same and they are $\sum_{j=1}^{k-1} p_j + p_k/n$. Since a type-$V_k$ bidder's expected payoff is given by her expected probability of winning $\times$ V_k minus her expected payment, her expected payment must be the same under the FPA and the SPA if her expected payoffs are the same, which we show below. Clearly, the expected payoffs for type-V_1 bidders are 0 under both the SPA and the FPA.

For $k = 2, \ldots, m$, a type-V_k bidder's equilibrium expected payoff in the SPA is given by

$$(p_1)^{n-1}(V_k - V_1) + \left[(p_1 + p_2)^{n-1} - p_1^{n-1}\right](V_k - V_2) + \cdots + \left[\left(\sum_{j=1}^{k-1} p_j\right)^{n-1} - \left(\sum_{j=1}^{k-2} p_j\right)^{n-1}\right](V_k - V_{k-1}), \tag{B.1}$$

where $(p_1)^{n-1}$ and $\left(\sum_{j=1}^{k-1} p_j\right)^{n-1} - \left(\sum_{j=1}^{k-2} p_j\right)^{n-1}$ are respectively the probability that the highest value among the $n-1$ opponents is V_1 and V_k.

In the unique symmetric equilibrium of the FPA, for $k \geqslant 2$, a type-V_k bidder will bid in a mixed strategy on a support $\left[\bar{b}_{k-1}, \bar{b}_k\right]$; thus her expected payoff is equal to that from bidding $\bar{b}_{k-1}$. Given the tie-breaking rule, a bid of $\bar{b}_{k-1}$ by a type-V_k bidder will win when all her $n-1$ opponents have valuation lower than V_k, which occurs with probability $\left(\sum_{j=1}^{k-1} p_j\right)^{n-1}$. Thus her expected payoff from bidding $\bar{b}_{k-1}$ is

$$\left(\sum_{j=1}^{k-1} p_j\right)^{n-1}\left(V_k - \bar{b}_{k-1}\right). \tag{B.2}$$

It can be verified, by induction, that $\bar{b}_{k-1}$, for $k \geqslant 2$, must satisfy

$$\left(\sum_{j=1}^{k-1} p_j\right)^{n-1}\bar{b}_{k-1} = (p_1)^{n-1} V_1 + \left[(p_1 + p_2)^{n-1} - (p_1)^{n-1}\right] V_2 + \cdots + \left[\left(\sum_{j=1}^{k-1} p_j\right)^{n-1} - \left(\sum_{j=1}^{k-2} p_j\right)^{n-1}\right] V_k. \tag{B.3}$$

Plugging (B.3) into (B.2), we immediately show that type-V_k bidder's expected payoff in the FPA is identical to that in the SPA given by expression (B.1). Thus all types of bidders' expected payment must be the same under the FPA and the SPA. Hence the seller's expected revenue must be the same. □

References

[1] L. Arozamena, E. Cantillon, Investment incentives in procurement auctions, Rev. Econ. Stud. 71 (2004) 1–18.
[2] S. Athey, Single crossing properties and the existence of pure strategy equilibria in games with incomplete information, Econometrica 69 (2001) 861–890.
[3] L. Baldwin, R. Marshall, J.F. Richard, Bidder collusion at forest service timber sales, J. Polit. Economy 105 (1997) 657–699.
[4] P. Battigalli, M. Siniscalchi, Rationalizable bidding in first-price auctions, Games Econ. Behav. 45 (2003) 38–72.
[5] D. Bergemann, S. Morris, Robust Mechanism Design, Cowles Foundation Discussion Paper No. 1421, 2003.
[6] C.M. Campbell, D. Levin, Can the seller benefit from an insider in common-value auctions?, J. Econ. Theory 91 (2000) 106–120.
[7] E. Cantillon, The effect of bidders' asymmetries on expected revenue in auctions, Cowles Foundation Discussion Paper 1279, 2000.
[8] Y.-K. Che, I. Gale, Standard auctions with financially constrained bidders, Rev. Econ. Stud. 65 (1998) 1–22.
[9] J. Cremer, R. McLean, Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdependent, Econometrica 53 (1985) 345–361.
[10] E. Dekel, A. Wolinsky, Rationalizable outcomes of large private-value first-price discrete auctions, Games Econ. Behav. 43 (2003) 175–188.
[11] H. Fang, S. Morris, Multidimensional Private Value Auctions, Cowles Foundation Discussion Paper No. 1423, 2003.
[12] Y. Feinberg, A. Skrzypacz, Uncertainty about uncertainty and delay in bargaining, Econometrica 73 (2005) 69–91.
[13] J.H. Griesmer, R.E. Levitan, M. Shubik, Toward a study of bidding processes: part IV: games with unknown costs, Naval Res. Logistics Quart. 14 (1967) 415–433.
[14] C. Holt, Competitive bidding for contracts under alternative auction procedures, J. Polit. Economy 88 (1980) 433–445.
[15] M. Jackson, The non-existence of equilibrium in auctions with two dimensional types, Mimeo, California Institute of Technology.
[16] M. Jackson, L.K. Simon, J.M. Swinkels, W.R. Zame, Communication and equilibrium in discontinuous games of incomplete information, Econometrica 70 (2002) 1711–1740.
[17] P. Jehiel, B. Moldovanu, Efficient design with interdependent valuations, Econometrica 69 (2001) 1237–1259.
[18] M. Jofre-Bonet, M. Pesendorfer, Estimation of a dynamic auction game, Econometrica 71 (2003) 1443–1489.
[19] J. Kim, Y.-K. Che, Asymmetric information about rivals' types in standard auctions, Games Econ. Behav. 46 (2004) 383–397.
[20] M. Landsberger, J. Rubinstein, E. Wolfstetter, S. Zamir, First-price auctions when the ranking of valuations is common knowledge, Rev. Econ. Design 6 (2001) 461–480.
[21] E. Maskin, J. Riley, Auction theory with private values, Amer. Econ. Rev. Papers Proc. 75 (1985) 150–155.
[22] E. Maskin, J. Riley, Uniqueness in sealed high bid auctions, Mimeo, UCLA, 1996.
[23] E. Maskin, J. Riley, Asymmetric auctions, Rev. Econ. Stud. 67 (2000) 413–438.
[24] E. Maskin, J. Riley, Equilibrium in sealed high bid auctions, Rev. Econ. Stud. 67 (2000) 439–454.
[25] S. Matthews, Information acquisition in discriminatory auctions, in: M. Boyer, R.E. Kihlstrom (Eds.), Bayesian Models in Economic Theory, North-Holland, Amsterdam, 1984.
[26] S. Mathews, Comparing auctions for risk averse buyers: a buyer's point of view, Econometrica 55 (1987) 633–646.
[27] R. McLean, A. Postlewaite, Informational size and efficient auctions, Rev. Econ. Stud. 71 (2004) 809–827.
[28] P. Milgrom, R. Weber, A theory of auctions and competitive bidding, Econometrica 70 (1982) 1089–1122.
[29] K. Mount, S. Reiter, The information size of message spaces, J. Econ. Theory 8 (1974) 161–192.
[30] R. Myerson, Optimal auction design, Math. Oper. Res. 6 (1981) 58–73.
[31] Z. Neeman, The relevance of private information in mechanism design, J. Econ. Theory 117 (2004) 55–77.
[32] N. Persico, Information acquisition in auctions, Econometrica 68 (2000) 135–148.

[33] M. Plum, Characterization and computation of Nash-equilibria for auctions with incomplete information, Int. J. Game Theory 20 (1992) 393–418.
[34] P. Reny, S. Zamir, On the existence of pure strategy monotone equilibria in asymmetric first-price auctions, Econometrica 72 (2004) 1105–1125.
[35] J. Riley, W.F. Samuelson, Optimal auctions, Amer. Econ. Rev. 71 (1981) 381–392.
[36] H. Sagan, Space-Filling Curves, Springer, Berlin, 1984.
[37] R. Wang, Common-value auctions with discrete private information, J. Econ. Theory 54 (1991) 429–447.
[38] R. Wilson, A bidding model of perfect competition, Rev. Econ. Stud. 44 (1977) 511–518.